Add and Subtract Integers

By the end of this section, you will be able to:

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add integers
- Subtract integers

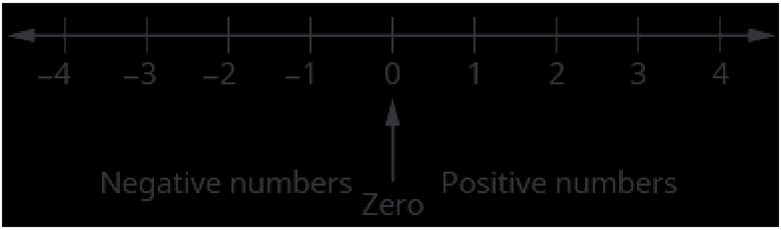
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

The number line shows the location of positive and negative numbers. The numbers on a number line increase in value going from left to right and decrease in value going from right to left. All the marked numbers are called *integers*. The opposite of 3 is -3.

Use Negatives and Opposites

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you

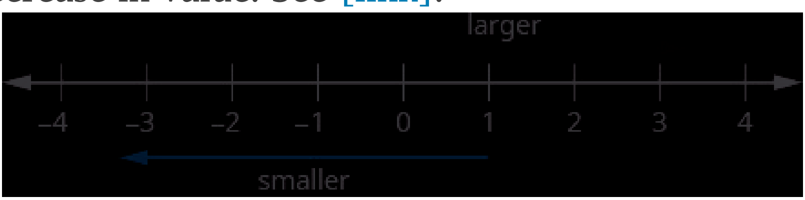
are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See [link].



The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See [link].



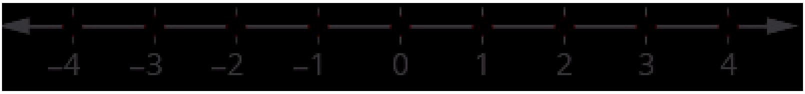
Doing the Manipulative Mathematics activity "Number Line-part 2" will help you develop a better understanding of integers.

Remember that we use the notation:

$a < b$ (read "a is less than b") when a is to the left of b on the number line.

$a > b$ (read "*a* is greater than *b*") when *a* is to the right of *b* on the number line.

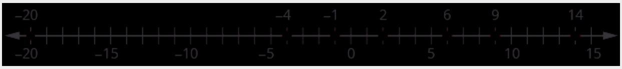
Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in [link] are called the integers. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$



Order each of the following pairs of numbers, using $<$ or $>$: ⓐ $14\_\_6$ ⓑ $-1\_\_9$ ⓒ $-1\_\_-4$ ⓓ $2\_\_-20$.

Solution

It may be helpful to refer to the number line shown.



ⓐ 14 is to the right of 6 on the number line.	$14\_\_6$ $14>6$
ⓑ -1 is to the left of 9 on the number line.	$-1\_\_9$ $-1<9$
ⓒ -1 is to the right of -4 on the number line.	$-1\_\_-4$ $-1>-4$
ⓓ 2 is to the right of -20 on the number line.	$2\_\_-20$ $2>-20$

Order each of the following pairs of numbers,

using $<$ or $>$: ⓐ $15\_\_7$ ⓑ $-2\_\_5$ ⓒ $-3\_\_$
-7
ⓓ $5\_\_-17$.

ⓐ $>$ ⓑ $<$ ⓒ $>$ ⓓ $>$

Order each of the following pairs of numbers, using $<$ or $>$: ⓐ $8\_\_13$ ⓑ $3\_\_-4$ ⓒ $-5\_\_$
-2
ⓓ $9\_\_-21$.

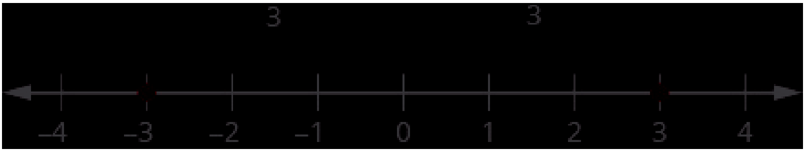
ⓐ $<$ ⓑ $>$ ⓒ $<$ ⓓ $>$

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2.

Opposite

The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

[link] illustrates the definition.



Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol “ $-$ ” used in three different ways.

$10-4$Between two numbers, it indicates the operation ofsubtraction.We read$10-4$as“10minus4.”-8In front of a number, it indicates anegativenumber.We read -8as “negative eight.”$-x$In front of a variable, it indicates theopposite.We read $-x$as “the opposite ofx.”$-(-2)$Here there are two“ $-$ ”signs. The one in the parentheses tells us the number isnegative2.The one outside the parentheses tells us to take theoppositeof -2.We read $-(-2)$as “the opposite of negative two.”

$10-4$	Between two numbers, it indicates the operation of *subtraction*. We read $10-4$ as "10

	minus 4."
-8	In front of a number, it indicates a *negative* number. We read -8 as "negative eight."
$-x$	In front of a variable, it indicates the *opposite*. We read $-x$ as "the opposite of x."
$-(-2)$	Here there are two "$-$" signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the *opposite* of -2. We read $-(-2)$ as "the opposite of negative two."

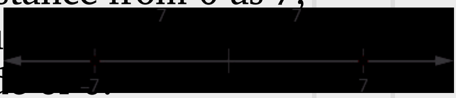
Opposite Notation

$-a$ means the opposite of the number *a*.
The notation $-a$ is read as "the opposite of *a*."

Find: ⓐ the opposite of 7 ⓑ the opposite of -10 ⓒ $-(-6)$.

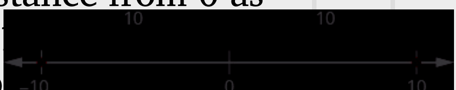
Solution

ⓐ -7 is the same distance from 0 as 7, bu[illegible] sid[illegible]



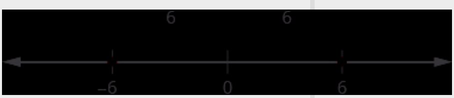
The opposite of 7 is -7.

ⓑ 10 is the same distance from 0 as $-$[illegible] op[illegible]



The opposite of -10 is 10.

ⓒ $-(-6)$



The opposite of $-(-6)$ is 6.

Find: ⓐ the opposite of 4 ⓑ the opposite of -3 ⓒ $-(-1)$.

ⓐ -4 ⓑ 3 ⓒ 1

Find: ⓐ the opposite of 8 ⓑ the opposite of -5 ⓒ $-(-5)$.

ⓐ -8 ⓑ 5 ⓒ 5

Our work with opposites gives us a way to define the integers.The whole numbers and their opposites are called the **integers**. The integers are the numbers $\ldots -3,-2,-1,0,1,2,3\ldots$

Integers

The whole numbers and their opposites are called the **integers**.

The integers are the numbers

$\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in [link].

Evaluate ⓐ $-x$, when $x = 8$ ⓑ $-x$, when $x = -8$.

Solution

ⓐ

[redacted]	$-x$
[redacted]	
Write the opposite of 8. [redacted]	

ⓑ

[redacted]

$-x$

[redacted]

Write the opposite of 8

$-8.$

Evaluate $-n$, when ⓐ $n=4$ ⓑ $n=-4$.
ⓐ -4 ⓑ 4

Evaluate $-m$, when ⓐ $m=11$ ⓑ $m=-11$.
ⓐ -11 ⓑ 11

The integers 5and are5 units away from 0.

Simplify: Expressions with Absolute Value

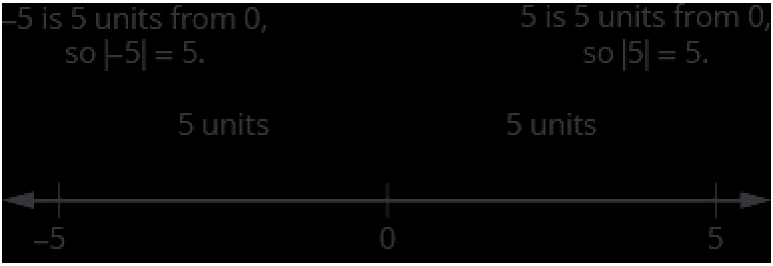
We saw that numbers such as 2and -2 are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Absolute Value
The **absolute value** of a number is its distance from 0 on the number line.
The absolute value of a number *n* is written as $|n|$.

For example,

- -5is5 units away from 0, so $|-5|=5$.
- 5is5 units away from 0, so $|5|=5$.

[link] illustrates this idea.



The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from 0to0 on the number line is zero units.

Property of Absolute Value
$|n| \geq 0$ for all numbers
Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, "absolute values are always non-negative." Non-negative means greater than or equal to zero.

Simplify: ⓐ $|3|$ ⓑ $|-44|$ ⓒ $|0|$.

Solution

The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

ⓐ $|3|$
3

ⓑ $|-44|$
44

ⓒ $|0|$
0

Simplify: ⓐ $|4|$ ⓑ $|-28|$ ⓒ $|0|$.

ⓐ 4 ⓑ 28 ⓒ 0

Simplify: ⓐ $|-13|$ ⓑ $|47|$ ⓒ $|0|$.

ⓐ 13 ⓑ 47 ⓒ 0

In the next example, we'll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

Fill in $<, >, \text{or} =$ for each of the following pairs of numbers:

ⓐ $|-5|\_\_-|-5|$ ⓑ $8\_\_-|-8|$ ⓒ $-9\_\_-|-9|$ ⓓ $-(-16)\_\_-|-16|$

Solution

ⓐ Simplify. Order.	$|-5|\_\_-|-5|$ $5\_\_-5$ $5>-5$ $|-5|>-|-5|$
ⓑ Simplify. Order.	$8\_\_-|-8|$ $8\_\_-8$ $8>-8$ $8>-|-8|$
ⓒ	$9\_\_-|-9|$ $-9\_\_-9$

Simplify. Order.		$-9=-9$ $-9=-\vert-9\vert$
ⓓ Simplify. Order.		$-(-16)\_\_-\vert-16\vert$ $16\_\_-16$ $16>-16$ $-(-16)>-\vert-16\vert$

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers: ⓐ $|-9|\_\_-|-9|$ ⓑ $2\_\_-|-2|$ ⓒ $-8\_\_|-8|$
ⓓ $-(-9)\_\_-|-9|$.

ⓐ $>$ ⓑ $>$ ⓒ $<$ ⓓ $>$

Fill in $<$, $>$, or $=$ for each of the following pairs of numbers: ⓐ $7\_\_-|-7|$ ⓑ $-(-10)\_\_-|-10|$
ⓒ $|-4|\_\_-|-4|$ ⓓ $-1\_\_|-1|$.

ⓐ $>$ ⓑ $>$ ⓒ $>$ ⓓ $<$

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols							
Parentheses	$()$	Braces	$\{\}$	Brackets	$[]$	Absolute value	$\| \|$

In the next example, we simplify the expressions inside absolute value bars first, just as we do with parentheses.

Simplify: $24-|19-3(6-2)|$.

Solution

	$24-\|19-3(6-2)\|$
Work inside	$24-\|19-3(4)\|$

parentheses first: subtract 2 from 6.	
Multiply 3(4).	$24-\lvert 19-12\rvert$
Subtract inside the absolute value bars.	$24-\lvert 7\rvert$
Take the absolute value.	$24-7$
Subtract.	17

Simplify: $19-\lvert 11-4(3-1)\rvert$.

16

Simplify: $9-\lvert 8-4(7-5)\rvert$.

9

Evaluate: ⓐ $|x|$ when $x = -35$ ⓑ $|-y|$ when $y = -20$ ⓒ $-|u|$ when $u = 12$ ⓓ $-|p|$ when $p = -14$.

Solution

ⓐ $|x|$ when $x = -35$

	$\|x\|$
[redacted]	[redacted]
Take the absolute value.	35

ⓑ $|-y|$ when $y = -20$

	$\|-y\|$

[redacted]	[redacted]
Simplify.	$\lvert 20 \rvert$
Take the absolute value.	20

ⓒ $-\lvert u \rvert$ when $u = 12$

	$-\lvert u \rvert$
[redacted]	[redacted]
Take the absolute value.	-12

ⓓ $-\lvert p \rvert$ when $p = -14$

	$-\lvert p \rvert$

[redacted]	[redacted]
Take the absolute value.	-14

Evaluate: ⓐ $|x|$ when $x = -17$ ⓑ $|-y|$ when $y = -39$ ⓒ $-|m|$ when $m = 22$ ⓓ $-|p|$ when $p = -11$.

ⓐ 17 ⓑ 39 ⓒ -22 ⓓ -11

Evaluate: ⓐ $|y|$ when $y = -23$ ⓑ $|-y|$ when $y = -21$ ⓒ $-|n|$ when $n = 37$ ⓓ $-|q|$ when $q = -49$.

ⓐ 23 ⓑ 21 ⓒ -37 ⓓ -49

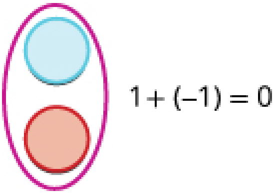
Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

Doing the Manipulative Mathematics activity "Addition of Signed Numbers" will help you develop a better understanding of adding integers."

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one color (blue) represent positive. The other color (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



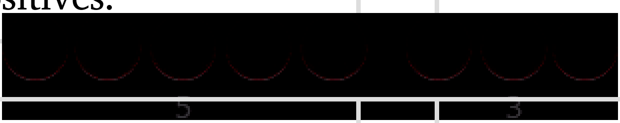
We will use the counters to show how to add the four addition facts using the numbers 5, -5 and 3, -3.

$5+3-5+(-3)-5+35+(-3)$

To add $5+3$, we realize that $5+3$ means the sum of 5 and 3.

We start with 5 positives.

And then we add 3 positives.



We now have 8 positives.
The sum of 5 and 3 is 8.

Now we will add $-5+(-3)$. Watch for similarities to the last example $5+3=8$.

To add $-5+(-3)$, we realize this means the sum of -5 and -3.

We start with 5 negatives.

And then we add 3 negatives.

We now have 8 negatives. The sum of -5 and -3 is -8.

In what ways were these first two examples similar?

- The first example adds 5 positives and 3 positives—both positives.
- The second example adds 5 negatives and 3 negatives—both negatives.

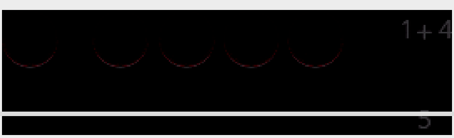
In each case we got 8—either 8 positives or 8 negatives.

When the signs were the same, the counters were all the same color, and so we added them.

Add: ⓐ $1+4$ ⓑ $-1+(-4)$.

Solution

ⓐ



1 positive plus 4 positives is 5 positives.

ⓑ

1 negative plus 4 negatives is 5 negatives.

Add: ⓐ $2+4$ ⓑ $-2+(-4)$.

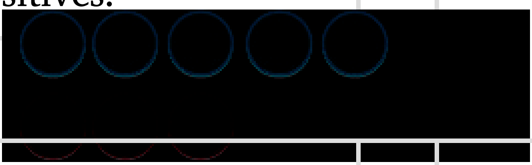
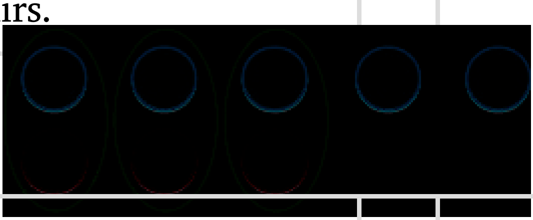
ⓐ 6 ⓑ -6

Add: ⓐ $2+5$ ⓑ $-2+(-5)$.

ⓐ 7 ⓑ -7

So what happens when the signs are different? Let's add $-5+3$. We realize this means the sum of -5 and 3. When the counters were the same color, we

put them in a row. When the counters are a different color, we line them up under each other.

	$-5 + 3$ means the sum of -5 and 3.
We start with 5 negatives. 	
And then we add 3 positives. 	
We remove any neutral pairs. 	
We have 2 negatives left. 	
The sum of -5 and 3 is -2.	$-5 + 3 = -2$

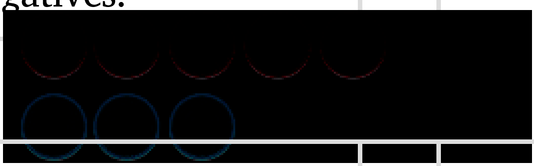
Notice that there were more negatives than positives, so the result was negative.

Let's now add the last combination, $5 + (-3)$.

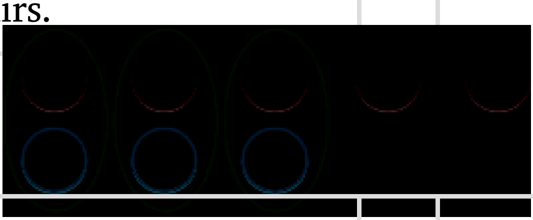
$5 + (-3)$ means the sum of 5 and -3.

We start with 5 positives.

And then we add 3 negatives.



We remove any neutral pairs.

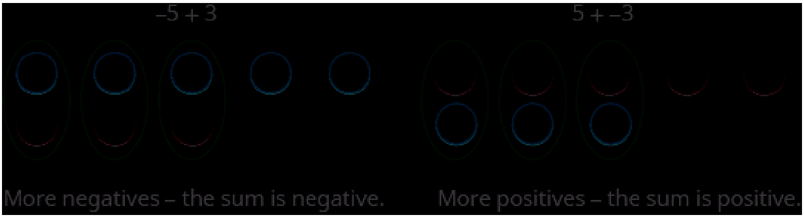


We have 2 positives left.

The sum of 5 and -3 is 2.

$5 + (-3) = 2$

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

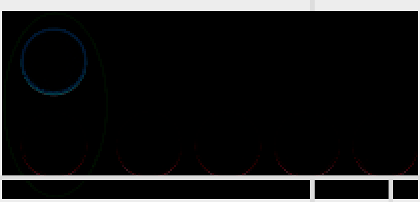


Add: ⓐ $-1+5$ ⓑ $1+(-5)$.

Solution

ⓐ

$-1 + 5$



There are more positives, so the sum is positive.	4

ⓑ

	$1+(-5)$

There are more negatives, so the sum is negative.	-4

Add: ⓐ $-2+4$ ⓑ $2+(-4)$.

ⓐ 2 ⓑ -2

Add: ⓐ $-2+5$ ⓑ $2+(-5)$.

ⓐ 3 ⓑ -3

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37+(-53)$, you really don’t want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53-37=16$, there are 16 more red counters.

Therefore, the sum of $37+(-53)$ is -16.
$37+(-53)=-16$

Let's try another one. We'll add $-74+(-27)$. Again, imagine 74 red counters and 27 more red counters, so we'd have 101 red counters. This means the sum is -101.
$-74+(-27)=-101$

Let's look again at the results of adding the different combinations of $5, -5$ and $3, -3$.

Addition of Positive and Negative Integers
$5+3$ $-5+(-3)$ 8 -8 both positive, sum positive both negative, sum negative
When the signs are the same, the counters would be all the same color, so add them.
$-5+3$ $5+(-3)$ -2 2 different signs, more negatives, sum negative different signs, more positives, sum positive
When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

Simplify: ⓐ $19+(-47)$ ⓑ $-14+(-36)$.

Solution

ⓐ Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives.

$19+(-47)$ Add. -28

ⓑ Since the signs are the same, we add. The answer will be negative because there are only negatives.

$-14+(-36)$ Add. -50

Simplify: ⓐ $-31+(-19)$ ⓑ $15+(-32)$.

ⓐ -50 ⓑ -17

Simplify: ⓐ $-42+(-28)$ ⓑ $25+(-61)$.

ⓐ -70 ⓑ -36

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

Simplify: $-5+3(-2+7)$.

Solution

	$-5+3(-2+7)$
Simplify inside the parentheses.	$-5+3(5)$
Multiply.	$-5+15$
Add left to right.	10

Simplify: $-2+5(-4+7)$.

13

Simplify: $-4+2(-3+5)$.

0

Subtract Integers

Doing the Manipulative Mathematics activity "Subtraction of Signed Numbers" will help you develop a better understanding of subtracting integers.

We will continue to use counters to model the

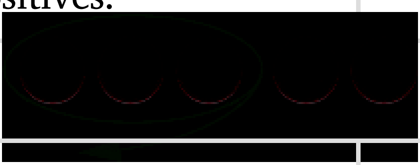
subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read “$5-3$” as “5 take away 3.” When you use counters, you can think of subtraction the same way!

We will model the four subtraction facts using the numbers 5 and 3.

$5-3-5-(-3)-5-35-(-3)$

To subtract $5-3$, we restate the problem as “5 take away 3.”

We start with 5 positives.	
	
We ‘take away’ 3 positives.	
	
We have 2 positives left.	
The difference of 5 and 3 is 2.	2

Now we will subtract $-5-(-3)$. Watch for similarities to the last example $5-3=2$.

To subtract $-5-(-3)$, we restate this as “–5 take away –3”

We start with 5 negatives.

We ‘take away’ 3 negatives.

We have 2 negatives left.

The difference of -5 and -3 is -2.

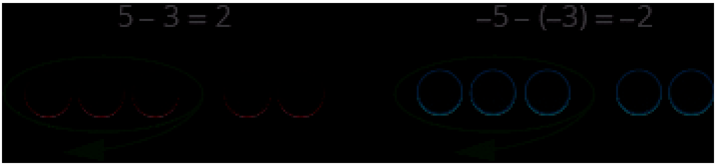
-2

Notice that these two examples are much alike: The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and

the “take away” model of subtraction was easy to apply.



Subtract: ⓐ $7-5$ ⓑ $-7-(-5)$.

Solution

ⓐ Take 5 positive from 7 positives and get 2 positives.	$7-5$ 2
ⓑ Take 5 negatives from 7 negatives and get 2 negatives.	$-7-(-5)$ -2

Subtract: ⓐ $6-4$ ⓑ $-6-(-4)$.
ⓐ 2 ⓑ -2

Subtract: ⓐ $7-4$ ⓑ $-7-(-4)$.
ⓐ 3 ⓑ -3

What happens when we have to subtract one positive and one negative number? We'll need to use both white and red counters as well as some neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

- To subtract $-5-3$, we restate it as -5 take away 3.

We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

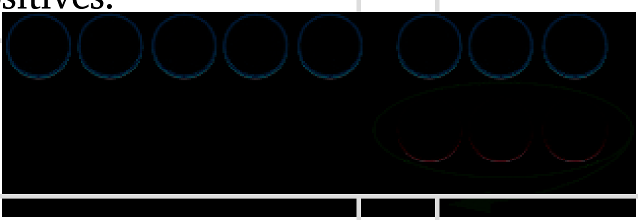
Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

$-5 - 3$ means -5 take away 3.

We start with 5 negatives.

We now add the neutrals needed to get 3 positives.

We remove the 3 positives.



We are left with 8 negatives.

The difference of -5 and 3 is -8.

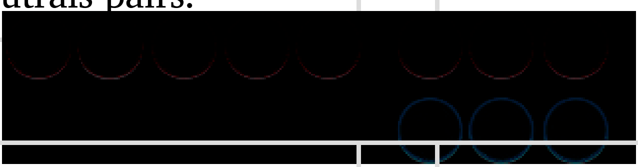
$-5-3=-8$

And now, the fourth case, $5-(-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

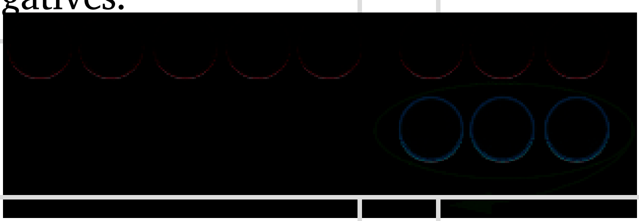
$5-(-3)$ means 5 take away -3.

We start with 5 positives.

We now add the needed neutrals pairs.



We remove the 3 negatives.



We are left with 8 positives.	

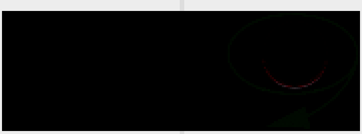
The difference of 5 and -3 is 8.	$5-(-3)=8$

Subtract: ⓐ $-3-1$ ⓑ $3-(-1)$.

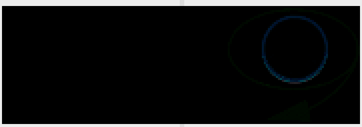
Solution

ⓐ

Take 1 positive from the one ad pair.	$-3-1$
	-4



ⓑ

Take 1 negative from the neutral pair.		$3-(-1)$
		4

Subtract: ⓐ $-6-4$ ⓑ $6-(-4)$.

ⓐ -10 ⓑ 10

Subtract: ⓐ $-7-4$ ⓑ $7-(-4)$.

ⓐ -11 ⓑ 11

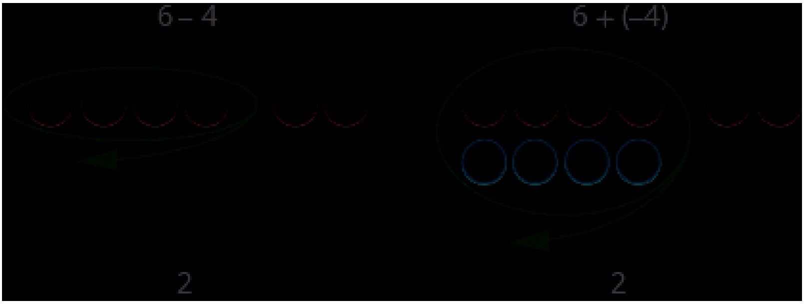
Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In [link], $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the **subtraction property**, written as follows:

Subtraction Property

$a-b=a+(-b)$

Subtracting a number is the same as adding its opposite.

Look at these two examples.



$6-4$ gives the same answer as $6+(-4)$.

Of course, when you have a subtraction problem that has only positive numbers, like $6-4$, you just do the subtraction. You already knew how to subtract $6-4$ long ago. But *knowing* that $6-4$ gives the same answer as $6+(-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6-4$ and $6+(-4)$ give the same results!

Simplify: ⓐ $13-8$ and $13+(-8)$ ⓑ $-17-9$ and $-17+(-9)$.

Solution

ⓐ Subtract.	$13-8$ 5	$13+(-8)$ 5
ⓑ Subtract.	$-17-9$ -26	$-17+(-9)$ -26

Simplify: ⓐ $21-13$ and $21+(-13)$ ⓑ $-11-7$ and $-11+(-7)$.

ⓐ 8 ⓑ -18

Simplify: ⓐ $15-7$ and $15+(-7)$ ⓑ $-14-8$ and $-14+(-8)$.

ⓐ 8 ⓑ -22

Look at what happens when we subtract a negative.

$8-(-5)$gives the same answer as$8+5$

Subtracting a negative number is like adding a positive!

You will often see this written as $a-(-b)=a+b$.

Does that work for other numbers, too? Let’s do the

following example and see.

Simplify: ⓐ $9-(-15)$ and $9+15$ ⓑ $-7-(-4)$ and $-7+4$.

Solution

ⓐ Subtract.	$9-(-15)$	24	$9+15$	24
ⓑ Subtract.	$-7-(-4)$	-3	$-7+4$	-3

Simplify: ⓐ $6-(-13)$ and $6+13$ ⓑ $-5-(-1)$ and $-5+1$.

ⓐ 19 ⓑ -4

Simplify: ⓐ $4-(-19)$ and $4+19$ ⓑ $-4-(-7)$ and $-4+7$.

ⓐ 23 ⓑ 3

Let's look again at the results of subtracting the different combinations of $5, -5$ and $3, -3$.

Subtraction of Integers
$5-3-5-(-3)2-25$positives take away3positives5negatives take away3negatives2positives2negatives
When there would be enough counters of the color to take away, subtract.
$-5-35-(-3)-885$negatives, want to take away3positives5positives, want to take away3negativesneed neutral pairsneed neutral pairs
When there would be not enough counters of the color to take away, add.

What happens when there are more than three

integers? We just use the order of operations as usual.

Simplify: $7-(-4-3)-9$.

Solution

	$7-(-4-3)-9$
Simplify inside the parentheses first.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract.	5

Simplify: $8-(-3-1)-9$.

3

Simplify: $12-(-9-6)-14$.

13

Access these online resources for additional instruction and practice with adding and subtracting integers. You will need to enable Java in your web browser to use the applications.

- Add Colored Chip
- Subtract Colored Chip

Key Concepts

- **Addition of Positive and Negative Integers**
$5+3 \quad -5+(-3)$ $8 \quad -8$ both positive, both negative, sum positive sum negative
$-5+3 \quad 5+(-3)$ $-2 \quad 2$ different signs, different signs, more negatives more positives sum negative sum positive
- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater

than or equal to zero!

- **Subtraction of Integers**
  5 – 3 – 5 – (– 3)2 – 25positives5negativestake away3positivestake away3negatives2 positives2 negatives – 5 – 35 – (– 3) – 885negatives, want to5positives, want tosubtract3positivessubtract3negativesneed neutral pairsneed neutral pairs
- **Subtraction Property:** Subtracting a number is the same as adding its opposite.

Practice Makes Perfect

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

ⓐ $9\_\_4$
ⓑ $-3\_\_6$
ⓒ $-8\_\_-2$
ⓓ $1\_\_-10$

ⓐ $>$ ⓑ $<$ ⓒ $<$ ⓓ $>$

ⓐ $-7\_\_3$
ⓑ $-10\_\_-5$
ⓒ $2\_\_-6$
ⓓ $8\_\_9$

In the following exercises, find the opposite of each number.

ⓐ 2 ⓑ -6

ⓐ -2 ⓑ 6

ⓐ 9ⓑ -4

In the following exercises, simplify.

$-(-4)$

4

$-(-8)$

$-(-15)$

15

$-(-11)$

In the following exercises, evaluate.

$-c$ when ⓐ $c=12$ ⓑ $c=-12$

ⓐ -12 ⓑ 12

$-d$ when
ⓐ $d=21$
ⓑ $d=-21$

Simplify Expressions with Absolute Value

In the following exercises, simplify.

ⓐ $|-32|$ ⓑ $|0|$ ⓒ $|16|$

ⓐ 32 ⓑ 0 ⓒ 16

ⓐ $|0|$
ⓑ $|-40|$ ⓒ $|22|$

In the following exercises, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

ⓐ $-6\_\_|-6|$ ⓑ $-|-3|\_\_-3$

ⓐ $<$ ⓑ $=$

ⓐ $|-5|\_\_-|-5|$ ⓑ $9\_\_-|-9|$

In the following exercises, simplify.

$-(-5) \text{ and } -|-5|$

$5, -5$

$-|-9| \text{ and } -(-9)$

$8|-7|$

56

$5|-5|$

$|15-7|-|14-6|$

0

$|17-8|-|13-4|$

$18-|2(8-3)|$

8

$18-|3(8-5)|$

In the following exercises, evaluate.

ⓐ $-|p|\text{when}p=19$
ⓑ $-|q|\text{when}q=-33$

ⓐ -19 ⓑ -33

ⓐ $-|a|\text{when}a=60$
ⓑ $-|b|\text{when}b=-12$

Add Integers

In the following exercises, simplify each expression.

$-21+(-59)$

-80

$-35+(-47)$

$48+(-16)$

32

$34+(-19)$

$-14+(-12)+4$

-22

$-17+(-18)+6$

$135+(-110)+83$

108

$-38+27+(-8)+12$

$19+2(-3+8)$

29

$24+3(-5+9)$

Subtract Integers

In the following exercises, simplify.

$8-2$

6

$-6-(-4)$

$-5-4$

-9

$-7-2$

$8-(-4)$

12

$7-(-3)$

ⓐ $44-28$ⓑ $44+(-28)$

ⓐ 16 ⓑ 16

ⓐ $35-16$ⓑ $35+(-16)$

ⓐ $27-(-18)$ⓑ $27+18$

ⓐ 45 ⓑ 45

ⓐ $46-(-37)$ⓑ $46+37$

In the following exercises, simplify each expression.

$15-(-12)$

27

$14-(-11)$

$48-87$

-39

$45-69$

$-17-42$

-59

$-19-46$

$-103-(-52)$

-51

$-105-(-68)$

$-45-(-54)$

-99

$-58-(-67)$

$8-3-7$

-2

$9-6-5$

$-5-4+7$

-2

$-3-8+4$

$-14-(-27)+9$

22

$64+(-17)-9$

$(2-7)-(3-8)$

5

$(1-8)-(2-9)$

$-(6-8)-(2-4)$

4

$-(4-5)-(7-8)$

$25-[10-(3-12)]$

6

$32-[5-(15-20)]$

$6 \cdot 3-4 \cdot 3-7 \cdot 2$

-8

$5 \cdot 7-8 \cdot 2-4 \cdot 9$

$5^2 - 6^2$

-11

$6^2 - 7^2$

Everyday Math

Elevation The highest elevation in the United States is Mount McKinley, Alaska, at 20,320 feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level.

Use integers to write the elevation of:

ⓐ Mount McKinley. ⓑ Death Valley.

ⓐ 20,320 feet ⓑ -282 feet

Extreme temperatures The highest recorded temperature on Earth was $58°$ Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature was $90°$below$0°$ Celsius, recorded in Antarctica in 1983.

Use integers to write the:

ⓐ highest recorded temperature.
ⓑ lowest recorded temperature.

State budgets In June, 2011, the state of Pennsylvania estimated it would have a budget surplus of $540 million. That same month, Texas estimated it would have a budget deficit of $27 billion.

Use integers to write the budget of:

ⓐ Pennsylvania.
ⓑ Texas.

ⓐ $540 million ⓑ −$27 billion

College enrollments Across the United States, community college enrollment grew by 1,400,000 students from Fall 2007 to Fall 2010. In California, community college enrollment declined by 110,171 students from Fall 2009 to Fall 2010.

Use integers to write the change in enrollment:

ⓐ in the U.S. from Fall 2007 to Fall 2010.

ⓑ in California from Fall 2009 to Fall 2010.

Stock Market The week of September 15, 2008 was one of the most volatile weeks ever for the US stock market. The closing numbers of the Dow Jones Industrial Average each day were:

Monday	−	504
Tuesday	+	142
Wednesday	−	449
Thursday	+	410
Friday	+	369

What was the overall change for the week? Was it positive or negative?

32, negative

Stock Market During the week of June 22, 2009, the closing numbers of the Dow Jones Industrial Average each day were:

Monday	-201
Tuesday	-16
Wednesday	-23
Thursday	$+172$
Friday	-34

What was the overall change for the week? Was it positive or negative?

Writing Exercises

Give an example of a negative number from your life experience.

Answers may vary

What are the three uses of the “$-$” sign in algebra? Explain how they differ.

Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.

Answers may vary

Give an example from your life experience of

adding two negative numbers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
use negatives and opposites of integers.			
simplify expressions with absolute value.			
add integers.			
subtract integers.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

absolute value
: The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.

integers
: The whole numbers and their opposites are called the integers: $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$

opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero: $-a$ means the opposite of the number. The notation $-a$ is read “the opposite of a.”

Multiply and Divide Integers

By the end of this section, you will be able to:

- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate English phrases to algebraic expressions
- Use integers in applications

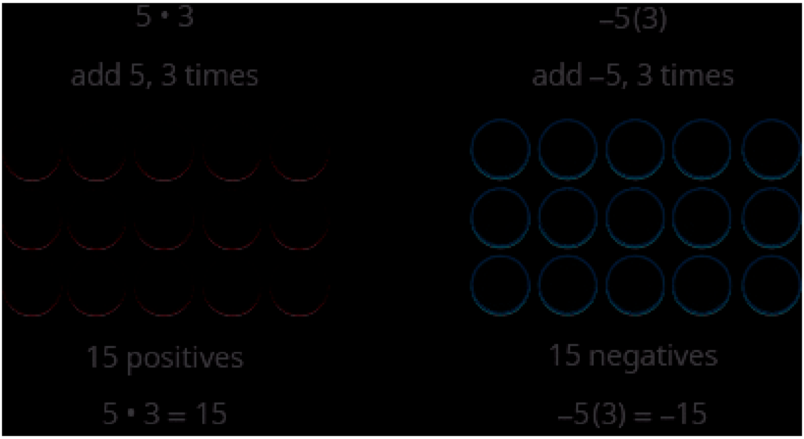
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let’s look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the

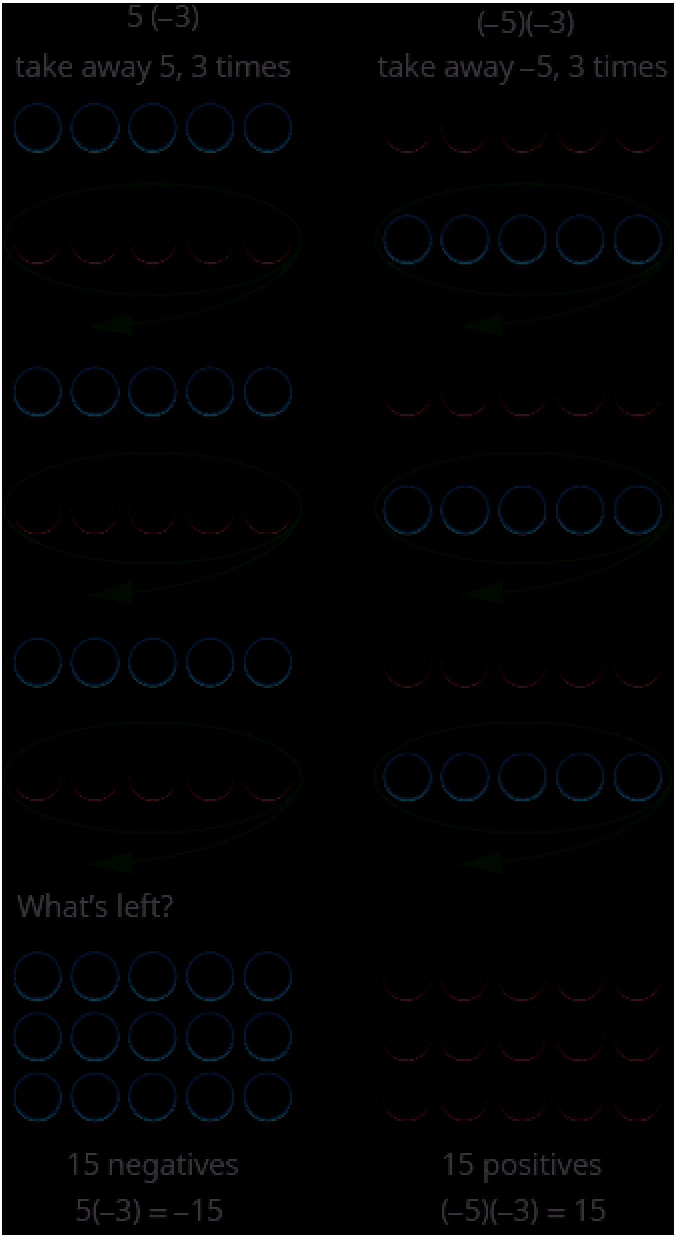
model just to help us discover the pattern.

We remember that a·b means add *a*, *b* times. Here, we are using the model just to help us discover the pattern.



The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as "taking away," it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.



In summary:

$$5 \cdot 3 = 15 \quad -5(3) = -15 \quad 5(-3) = -15 \quad (-5)(-3) = 15$$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below.

Multiplication of Signed Numbers

For multiplication of two signed numbers:

Same signs	**Product**	**Example**
Two positives	Positive	$7\cdot 4=28$
Two negatives	Positive	$-8(-6)=48$

Different signs	**Product**	**Example**
Positive · negative	Negative	$7(-9)=-63$
Negative · positive	Negative	$-5\cdot 10=-50$

Multiply: ⓐ $-9 \cdot 3$ ⓑ $-2(-5)$ ⓒ $4(-8)$ ⓓ $7 \cdot 6$.

Solution

ⓐ	$-9 \cdot 3$
Multiply, noting that the signs are different so the product is negative.	-27
ⓑ	$-2(-5)$
Multiply, noting that the signs are the same so the product is positive.	10
ⓒ	$4(-8)$
Multiply, with different signs.	-32
ⓓ	$7 \cdot 6$
Multiply, with same signs.	42

Multiply: ⓐ $-6 \cdot 8$ ⓑ $-4(-7)$ ⓒ $9(-7)$ ⓓ $5 \cdot 12$.

ⓐ -48 ⓑ 28 ⓒ -63 ⓓ 60

Multiply: ⓐ $-8 \cdot 7$ ⓑ $-6(-9)$ ⓒ $7(-4)$ ⓓ $3 \cdot 13$.

ⓐ -56 ⓑ 54 ⓒ -28 ⓓ 39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

$-1 \cdot 4 \quad -1(-3)$ Multiply. $-4 \quad 3$ -4 is the opposite of 4. 3 is the opposite of -3.

Each time we multiply a number by -1, we get its opposite!

Multiplication by -1

$-1a = -a$

Multiplying a number by -1 gives its opposite.

Multiply: ⓐ $-1 \cdot 7$ ⓑ $-1(-11)$.

Solution

ⓐ		
Multiply, noting that the signs are different so the product is negative.	$-1 \cdot 7$ -7	-7 is the opposite of 7.
ⓑ		
Multiply, noting that the signs are the same so the product is positive.	$-1(-11)$ 11	11 is the opposite of -11.

Multiply: ⓐ $-1 \cdot 9$ ⓑ $-1 \cdot (-17)$.

ⓐ -9 ⓑ 17

Multiply: ⓐ $-1 \cdot 8$ ⓑ $-1 \cdot (-16)$.

ⓐ -8 ⓑ 16

Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \cdot 3 = 15$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.

$5 \cdot 3 = 15 \text{so} 15 \div 3 = 5 - 5(3) = -15 \text{so} -15 \div 3 = -5(-5)(-3) = 15 \text{so} 15 \div (-3) = -55(-3) = -15 \text{so} -15 \div (-3) = 5$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Multiplication and Division of Signed Numbers

For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	Result
Two positives	Positive
Two negatives	Positive
If the signs are the same, the result is positive.	

Different signs	Result
Positive and negative	Negative
Negative and positive	Negative
If the signs are different, the result is negative.	

Divide: ⓐ $-27 \div 3$ ⓑ $-100 \div (-4)$.

Solution

ⓐ	$-27 \div 3$
Divide. With different signs, the quotient is negative.	-9
ⓑ	$-100 \div (-4)$
Divide. With signs that are the same, the quotient is positive.	25

Divide: ⓐ $-42 \div 6$ ⓑ $-117 \div (-3)$.

ⓐ -7 ⓑ 39

Divide: ⓐ $-63 \div 7$ ⓑ $-115 \div (-5)$.

ⓐ -9 ⓑ 23

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let’s try some examples. We’ll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

Simplify: $7(-2)+4(-7)-6$.

Solution

	$7(-2)+4(-7)-6$
Multiply first.	$-14+(-28)-6$
Add.	$-42-6$
Subtract.	-48

Simplify: $8(-3)+5(-7)-4$.

-63

Simplify: $9(-3)+7(-8)-1$.

-84

Simplify: ⓐ $(-2)^4$ ⓑ -2^4.

Solution

ⓐ	$(-2)^4$
Write in expanded form.	$(-2)(-2)(-2)(-2)$
Multiply.	$4(-2)(-2)$
Multiply.	$-8(-2)$
Multiply.	16
ⓑ	-2^4
Write in expanded form. We are asked to find the opposite of 2^4.	$-(2\cdot2\cdot2\cdot2)$
Multiply.	$-(4\cdot2\cdot2)$
Multiply.	$-(8\cdot2)$
Multiply.	-16

Notice the difference in parts ⓐ and ⓑ. In part ⓐ, the exponent means to raise what is in

the parentheses, the (-2) to the 4th power. In part ⓑ, the exponent means to raise just the 2 to the 4th power and then take the opposite.

Simplify: ⓐ $(-3)4$ ⓑ -34.

ⓐ 81 ⓑ -81

Simplify: ⓐ $(-7)2$ ⓑ -72.

ⓐ 49 ⓑ -49

The next example reminds us to simplify inside parentheses first.

Simplify: $12-3(9-12)$.

Solution

	$12 - 3(9 - 12)$	
Subtract in parentheses first.	$12 - 3(-3)$	
Multiply.	$12 - (-9)$	
Subtract.	21	

Simplify: $17 - 4(8 - 11)$.

29

Simplify: $16 - 6(7 - 13)$.

52

Simplify: $8(-9) \div (-2)3$.

Solution

	$8(-9) \div (-2)3$
Exponents first.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

Simplify: $12(-9) \div (-3)3$.

4

Simplify: $18(-4) \div (-2)3$.

9

Simplify: $-30 \div 2 + (-3)(-7)$.

Solution

	$-30 \div 2 + (-3)(-7)$
Multiply and divide left to right, so divide first.	$-15 + (-3)(-7)$
Multiply.	$-15 + 21$
Add.	6

Simplify: $-27 \div 3 + (-5)(-6)$.

21

Simplify: $-32 \div 4 + (-2)(-7)$.

6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

When $n = -5$, evaluate: ⓐ $n + 1$ ⓑ $-n + 1$.

Solution

ⓐ

[redacted]

[redacted]

Simplify. -4

ⓑ

[redacted]

[redacted]

Simplify.

[redacted]

Add. 6

When $n = -8$, evaluate ⓐ $n + 2$ ⓑ $-n + 2$.

ⓐ -6 ⓑ 10

When $y=-9$, evaluate ⓐ $y+8$ ⓑ $-y+8$.

ⓐ -1 ⓑ 17

Evaluate $(x+y)2$ when $x=-18$ and $y=24$.

Solution

[redacted]

[redacted]

Add inside parenthesis.	$(6)2$
Simplify.	36

Evaluate $(x+y)^2$ when $x=-15$ and $y=29$.

196

Evaluate $(x+y)^3$ when $x=-8$ and $y=10$.

8

Evaluate $20-z$ when ⓐ $z=12$ and ⓑ $z=-12$.

Solution

ⓐ

[redacted]

[redacted]

Subtract. 8

ⓑ

[redacted]

[redacted]

Subtract. 32

Evaluate: $17 - k$ when ⓐ $k = 19$ and ⓑ $k = -19$.

ⓐ -2 ⓑ 36

Evaluate: $-5-b$ when ⓐ $b=14$ and ⓑ $b=-14$.

ⓐ -19 ⓑ 9

Evaluate: $2x2+3x+8$ when $x=4$.

Solution

Substitute 4forx. Use parentheses to show multiplication.

[redacted]

Substitute.	
[redacted]	
Evaluate exponents.	
[redacted]	
Multiply.	
[redacted]	
Add.	52

Evaluate: $3x2-2x+6$ when $x=-3$.

39

Evaluate: $4x2-x-5$ when $x=-2$.

13

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

Translate and simplify: the sum of 8 and -12, increased by 3.

Solution

	the **sum** of 8 and -12, increased by 3.
Translate.	$[8+(-12)]+3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4)+3$
Add.	-1

Translate and simplify the sum of 9 and -16, increased by 4.

$(9+(-16))+4; -3$

Translate and simplify the sum of -8 and -12, increased by 7.

$(-8+(-12))+7; -13$

When we first introduced the operation symbols, we saw that the expression may be read in several ways. They are listed in the chart below.

$a - b$	a minus b the difference of a and b b subtracted from a b less than a

Be careful to get *a* and *b* in the right order!

Translate and then simplify ⓐ the difference of 13 and -21 ⓑ subtract 24 from -19.

Solution

ⓐ	
Translate.	the difference of 13 and -21 $13 - (-21)$
Simplify.	34
ⓑ	
Translate. Remember, "subtract b from a means $a - b$.	subtract 24 from -19 $-19 - 24$
Simplify.	43

Translate and simplify ⓐ the difference of 14 and -23 ⓑ subtract 21 from -17.

ⓐ $14-(-23);37$ ⓑ $-17-21;-38$

Translate and simplify ⓐ the difference of 11 and -19 ⓑ subtract 18 from -11.

ⓐ $11-(-19);30$ ⓑ $-11-18;-29$

Once again, our prior work translating English to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for multiplication is “product” and for division is “quotient.”

Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution

	the product of -2 and 14
Translate.	$(-2)(14)$
Simplify.	-28

Translate to an algebraic expression and simplify if possible: the product of -5 and 12.

$-5(12); -60$

Translate to an algebraic expression and simplify if possible: the product of 8 and -13.

$8(-13); -104$

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution

	the quotient of -56 and -7	
Translate.	$-56 \div (-7)$	
Simplify.	8	

Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9.

$-63 \div (-9); 7$

Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9.

$-72 \div (-9); 8$

Use Integers in Applications

We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

How to Apply a Strategy to Solve Applications with Integers

In the morning, the temperature in Urbana, Illinois was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference of the morning and afternoon temperatures?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the difference of the morning and afternoon temperatures
Step 3. Write a phrase the gives the information to find it.	the *difference of* 11 *and* –9
Step 4. Translate the phrase to an expression.	$11 - (-9)$
Step 5. Simplify the expression.	20
Step 6. Write a complete sentence that answers the question.	The difference in temperatures was 20 degrees.

In the morning, the temperature in Anchorage, Alaska was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

The difference in temperatures was 45 degrees.

The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

The difference in temperatures was 9 degrees.

Apply a Strategy to Solve Applications with Integers.

Read the problem. Make sure all the words and ideas are understood Identify what we are asked to find. Write a phrase that gives the information to find it. Translate the phrase to an expression. Simplify the expression. Answer the question with a complete sentence.

The Mustangs football team received three penalties in the third quarter. Each penalty gave them a loss of fifteen yards. What is the number of yards lost?

Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the number of yards lost
Step 3. Write a phrase that gives the information to find it.	three times a 15-yard penalty

Step 4. Translate the phrase to an expression.	$3(-15)$
Step 5. Simplify the expression.	-45
Step 6. Answer the question with a complete sentence.	The team lost 45 yards.

The Bears played poorly and had seven penalties in the game. Each penalty resulted in a loss of 15 yards. What is the number of yards lost due to penalties?

The Bears lost 105 yards.

Bill uses the ATM on campus because it is convenient. However, each time he uses it he is charged a $2 fee. Last month he used the ATM eight times. How much was his total fee for using the ATM?

A $16 fee was deducted from his checking account.

Key Concepts

- **Multiplication and Division of Two Signed Numbers**
  - Same signs—Product is positive
  - Different signs—Product is negative
- **Strategy for Applications**

  Identify what you are asked to find. Write a phrase that gives the information to find it. Translate the phrase to an expression. Simplify the expression. Answer the question with a complete sentence.

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply.

$-4\cdot 8$

-32

$-3\cdot 9$

$9(-7)$

-63

$13(-5)$

-1.6

-6

-1.3

$-1(-14)$

14

$-1(-19)$

Divide Integers

In the following exercises, divide.

$-24 \div 6$

-4

$35 \div (-7)$

$-52 \div (-4)$

13

$-84 \div (-6)$

$-180 \div 15$

-12

$-192 \div 12$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

$5(-6)+7(-2)-3$

-47

$8(-4)+5(-4)-6$

$(-2)6$

64

$(-3)5$

-42

-16

-62

$-3(-5)(6)$

90

$-4(-6)(3)$

$(8-11)(9-12)$

9

$(6-11)(8-13)$

$26-3(2-7)$

41

$23-2(4-6)$

$65 \div (-5) + (-28) \div (-7)$

-9

$52 \div (-4) + (-32) \div (-8)$

$9-2[3-8(-2)]$

-29

$11-3[7-4(-2)]$

$(-3)2-24\div(8-2)$

5

$(-4)2-32\div(12-4)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

$y+(-14)$ when ⓐ $y=-33$ ⓑ $y=30$

ⓐ -47 ⓑ 16

$x+(-21)$ whenⓐ $x=-27$ⓑ $x=44$

ⓐ $a+3$ when $a=-7$
ⓑ $-a+3$ when $a=-7$

ⓐ -4 ⓑ 10

ⓐ $d+(-9)$ when $d=-8$
ⓑ $-d+(-9)$ when $d=-8$

$m+n$ when
$m=-15, n=7$

-8

$p+q$ when
$p=-9, q=17$

$r+s$ when $r=-9, s=-7$

-16

$t+u$ when $t=-6, u=-5$

$(x+y)2$ when
$x=-3, y=14$

121

$(y+z)2$ when
$y=-3, z=15$

$-2x+17$ when

ⓐ $x=8$
ⓑ $x=-8$

ⓐ 1
ⓑ 33

$-5y+14$ when
ⓐ $y=9$
ⓑ $y=-9$

$10-3m$ when
ⓐ $m=5$
ⓑ $m=-5$

ⓐ -5 ⓑ 25

$18-4n$ when
ⓐ $n=3$

ⓑ $n = -3$

$2w2 - 3w + 7$ when $w = -2$

21

$3u2 - 4u + 5$ when $u = -3$

$9a - 2b - 8$ when $a = -6 and b = -3$

-56

$7m - 4n - 2$ when $m = -4 and n = -9$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

the sum of 3 and -15, increased by 7

$(3+(-15))+7; -5$
the sum of -8 and -9, increased by 23
the difference of 10 and -18
$10-(-18); 28$
subtract 11 from -25
the difference of -5 and -30
$-5-(-30); 25$
subtract -6 from -13
the product of -3 and 15
$-3 \cdot 15; -45$
the product of -4 and 16

the quotient of -60 and -20

$-60 \div (-20);3$

the quotient of -40 and -20

the quotient of -6 and the sum of a and b

$-6a+b$

the quotient of -7 and the sum of m and n

the product of -10 and the difference of pandq

$-10(p-q)$

the product of -13 and the difference of candd

Use Integers in Applications

In the following exercises, solve.

Temperature On January 15, the high temperature in Anaheim, California, was $84°$.

That same day, the high temperature in Embarrass, Minnesota was $-12°$. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

$96°$

Temperature On January 21, the high temperature in Palm Springs, California, was $89°$, and the high temperature in Whitefield, New Hampshire was $-31°$. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Football On the first down, the Chargers had the ball on their 25-yard line. They lost 6 yards on the first-down play, gained 10 yards on the second-down play, and lost 8 yards on the third-down play. What was the yard line at the end of the third-down play?

21

Football On first down, the Steelers had the ball on their 30-yard line. They gained 9 yards on the first-down play, lost 14 yards on the second-down play, and lost 2 yards on the

third-down play. What was the yard line at the end of the third-down play?

Checking Account Mayra has $124 in her checking account. She writes a check for $152. What is the new balance in her checking account?

−$28

Checking Account Selina has $165 in her checking account. She writes a check for $207. What is the new balance in her checking account?

Checking Account Diontre has a balance of −$38 in his checking account. He deposits $225 to the account. What is the new balance?

$187

Checking Account Reymonte has a balance of −$49 in his checking account. He deposits $281 to the account. What is the new balance?

Everyday Math

Stock market Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped \$12 per share. What was the total effect on Javier’s portfolio?

$-\$3600$

Weight loss In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

In your own words, state the rules for multiplying integers.

Answers may vary

In your own words, state the rules for dividing integers.

Why is $-2^4 \neq (-2)^4$?

Answers may vary

Why is $-43=(-4)3$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
multiply integers.			
divide integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Exponents and Scientific Notation

In this section students will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2{,}048 \times 1{,}536 \times 48 \times 24 \times 3{,}600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the

exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, *x*, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$x^3 \cdot x^4 = \underbrace{x \cdot x \cdot x}_{3\text{ factors}} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{4\text{ factors}} = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{7\text{ factors}} = x^7$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents.*

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8,

2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$a^m \cdot a^n = a^{m+n}$

Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

1. $t^5 \cdot t^3$
2. $(-3)^5 \cdot (-3)$
3. $x^2 \cdot x^5 \cdot x^3$

Use the product rule to simplify each expression.

1. $t^5 \cdot t^3 = t^{5+3} = t^8$
2. $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = ($

$-3)^{5+1} = (-3)^6$

3. $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

Write each of the following products with a single base. Do not simplify further.

1. $k^6 \cdot k^9$
2. $(2y)^4 \cdot (2y)$
3. $t^3 \cdot t^6 \cdot t^5$

---

1. k^{15}
2. $(2y)^5$
3. t^{14}

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m>n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\frac{y^9}{y^5} = \frac{y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y}{y\cdot y\cdot y\cdot y\cdot y} = \frac{y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y\cdot y}{y\cdot y\cdot y\cdot y\cdot y} = \frac{y\cdot y\cdot y\cdot y}{1} = y^4$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m>n$. Otherwise, the difference $m-n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

The Quotient Rule of Exponents
For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that
$$\frac{a^m}{a^n} = a^{m-n}$$

Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

1. $\frac{(-2)^{14}}{(-2)^9}$
2. $\frac{t^{23}}{t^{15}}$
3. $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

Use the quotient rule to simplify each expression.

1. $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$
2. $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$
3. $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

Write each of the following products with a single base. Do not simplify further.

1. s 75 s 68
2. (−3) 6 −3
3. (e f 2) 5 (e f 2) 3

---

1. s 7
2. (−3) 5
3. (e f 2) 2

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression (x 2) 3 . The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

(x2)3 = (x2)·(x2)·(x2)3 factors =
(x·x□2 factors)·(x·x
□2 factors)·(x·x□2 factors)3 factors = x·x·x·x·x·x =

x^6

The exponent of the answer is the product of the exponents: $(x^2)^3 = x^{2\cdot3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

Product Rule Power Rule $5^3 \cdot 5^4 = 5^{3+4} = 5^7$ but $(5^3)^4 = 5^{3\cdot4} = 5^{12}$ $x^5 \cdot x^2 = x^{5+2} = x^7$ but $(x^5)^2 = x^{5\cdot2} = x^{10}$ $(3a)^7 \cdot (3a)^{10} = (3a)^{7+10} = (3a)^{17}$ but $((3a)^7)^{10} = (3a)^{7\cdot10} = (3a)^{70}$

The Power Rule of Exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m\cdot n}$$

Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

1. (x 2) 7
2. ((2t) 5) 3
3. ((−3) 5) 11

Use the power rule to simplify each expression.

1. (x 2) 7 = x 2·7 = x 14
2. ((2t) 5) 3 = (2t) 5·3 = (2t) 15
3. ((−3) 5) 11 = (−3) 5·11 = (−3) 55

Write each of the following products with a single base. Do not simplify further.

1. ((3y) 8) 3
2. (t 5) 7
3. ((−g) 4) 4

---

1. (3y) 24

2. t^{35}
3. $(-g)^{16}$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m>n$ so that the difference $m-n$ would never be zero or negative. What would happen if $m=n$? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{t^8}{t^8} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for

now, we will consider the value to be undefined.

The Zero Exponent Rule of Exponents
For any nonzero real number a, the zero exponent rule of exponents states that
a 0 =1

Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

1. c 3 c 3
2. −3 x 5 x 5
3. (j 2 k) 4 (j 2 k)· (j 2 k) 3
4. 5 (r s 2) 2 (r s 2) 2

Use the zero exponent and other rules to simplify each expression.

1. c3c3 = c3−3 = c0 = 1
2. −3x5x5 = −3·x5x5 = −3·x5−5 = −3·x0 = −3·1 = −3
3.

(j2k)4(j2k)·(j2k)3 = (j2k)4(j2k)1 + 3
Use the product rule in the denominator.
= (j2k)4(j2k)4 Simplify. = (j2k)4 − 4
Use the quotient rule. = (j2k)0 Simplify.
= 1

4.

5(r s 2)2(rs2)2 = 5(rs2)2 − 2
Use the quotient rule. = 5(rs2)0 Simplify.
= 5·1 Use the zero exponent rule. = 5
Simplify.

Simplify each expression using the zero exponent rule of exponents.

1. t 7 t 7
2. (d e 2) 11 2 (d e 2) 11
3. w 4 · w 2 w 6
4. t 3 · t 4 t 2 · t 5

---

1. 1
2. 1 2
3. 1
4. 1

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m>n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m<n$—that is, where the difference $m-n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\frac{h^3}{h^5} = \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} = \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} = \frac{1}{h \cdot h} = \frac{1}{h^2}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{h^3}{h^5} = h^{3-5} = h^{-2}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n

is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

The Negative Rule of Exponents

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

1. $\frac{\theta^3}{\theta^{10}}$
2. $\frac{z^2 \cdot z}{z^4}$
3. $\frac{(-5t^3)^4}{(-5t^3)^8}$

---

1. $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$
2. $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$
3. $\frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)$

$4 - 8 = (-5 t 3) - 4 = 1 (-5 t 3) 4$

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

1. $(-3t) 2 (-3t) 8$
2. $f 47 f 49 \cdot f$
3. $2 k 4 5 k 7$

---

1. $1 (-3t) 6$
2. $1 f 3$
3. $2 5 k 3$

Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

1. $b 2 \cdot b -8$
2. $(-x) 5 \cdot (-x) -5$

3. $\frac{-7z}{(-7z)^5}$

1. $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$
2. $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$
3. $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

1. $t^{-11} \cdot t^6$
2. $\frac{25^{12}}{25^{13}}$

1. $t^{-5} = \frac{1}{t^5}$
2. $\frac{1}{25}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents,* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$(pq)^3 = \underbrace{(pq)\cdot(pq)\cdot(pq)}_{3\text{ factors}} = p\cdot q\cdot p\cdot q\cdot p\cdot q = \underbrace{p\cdot p\cdot p}_{3\text{ factors}}\cdot\underbrace{q\cdot q\cdot q}_{3\text{ factors}} = p^3\cdot q^3$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

The Power of a Product Rule of Exponents

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n$$

Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

1. $(ab^2)^3$
2. $(2t)^{15}$
3. $(-2w^3)^3$

4. $\frac{1}{(-7z)^4}$
5. $(e^{-2}f^2)^7$

Use the product and quotient rules and the new definitions to simplify each expression.

1. $(ab^2)^3 = (a)^3 \cdot (b^2)^3 = a^{1\cdot 3} \cdot b^{2\cdot 3} = a^3b^6$
2. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15}t^{15} = 32{,}768t^{15}$
3. $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3\cdot 3} = -8w^9$
4. $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \cdot (z)^4} = \frac{1}{2{,}401z^4}$
5. $(e^{-2}f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2\cdot 7} \cdot f^{2\cdot 7} = e^{-14}f^{14} = \frac{f^{14}}{e^{14}}$

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

1. $(g^2h^3)^5$
2. $(5t)^3$
3. $(-3y^5)^3$
4. $\frac{1}{(a^6b^7)^3}$
5. $(r^3s^{-2})^4$

1. g 10 h 15
2. 125 t 3
3. −27 y 15
4. 1 a 18 b 21
5. r 12 s 8

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule,* which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let’s look at the following example.

(e −2 f 2) 7 = f 14 e 14

Let’s rewrite the original problem differently and look at the result.

(e−2f2)7 = (f2 e2)7 = f14e14

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

(e −2 f 2) 7 = (f 2 e 2) 7 = (f 2) 7 (e 2) 7 = f 2·7 e 2·7 = f 14 e 14

The Power of a Quotient Rule of Exponents

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

(a b) n = a n b n

Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

1. (4 z 11) 3
2. (p q 3) 6
3. (−1 t 2) 27
4. (j 3 k −2) 4
5. (m −2 n −2) 3

1. (4 z 11) 3 = (4) 3 (z 11) 3 = 64 z 11⋅3 = 64 z 33
2. (p q 3) 6 = (p) 6 (q 3) 6 = p 1⋅6 q 3⋅6 = p 6 q 18
3. (−1 t 2) 27 = (−1) 27 (t 2) 27 = −1 t 2⋅27 = −1 t 54 = − 1 t 54
4. (j 3 k −2) 4 = (j 3 k 2) 4 = (j 3) 4 (k 2) 4 = j 3⋅4 k 2⋅4 = j 12 k 8
5. (m −2 n −2) 3 = (1 m 2 n 2) 3 = (1) 3 (m 2 n 2) 3 = 1 (m 2) 3 (n 2) 3 =

$$\frac{1}{m^{2\cdot3} \cdot n^{2\cdot3}} = \frac{1}{m^6 n^6}$$

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

1. $\left(\frac{b^5}{c}\right)^3$
2. $\left(\frac{5}{u^8}\right)^4$
3. $\left(\frac{-1}{w^3}\right)^{35}$
4. $\left(p^{-4} q^3\right)^8$
5. $\left(c^{-5} d^{-3}\right)^4$

---

1. $\frac{b^{15}}{c^3}$
2. $\frac{625}{u^{32}}$
3. $\frac{-1}{w^{105}}$
4. $\frac{q^{24}}{p^{32}}$
5. $\frac{1}{c^{20} d^{12}}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

1. $(6m^2n^{-1})^3$
2. $17^5\cdot 17^{-4}\cdot 17^{-3}$
3. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$
4. $(-2a^3b^{-1})(5a^{-2}b^2)$
5. $(x^{2\sqrt{2}})^4(x^{2\sqrt{2}})^{-4}$
6. $\frac{(3w^2)^5}{(6w^{-2})^2}$

---

1. $$\begin{aligned} (6m^2n^{-1})^3 &= (6)^3(m^2)^3(n^{-1})^3 && \text{The power of a product rule} \\ &= 6^3m^{2\cdot 3}n^{-1\cdot 3} && \text{The power rule} \\ &= 216m^6n^{-3} && \text{Simplify.} \\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule} \end{aligned}$$

2. $$\begin{aligned} 17^5\cdot 17^{-4}\cdot 17^{-3} &= 17^{5-4-3} && \text{The product rule} \\ &= 17^{-2} && \text{Simplify.} \\ &= \frac{1}{17^2}\ \text{ or }\ \frac{1}{289} \end{aligned}$$

The negative exponent rule

3.

(u −1 v v −1) 2 = (u −1 v) 2 (v −1) 2 The power of a quotient rule = u −2 v 2 v −2 The power of a product rule = u −2 v 2−(−2) The quotient rule = u −2 v 4 Simplify. = v 4 u 2

The negative exponent rule

4.

(−2 a 3 b −1)(5 a −2 b 2) = −2·5· a 3 · a −2 · b −1 · b 2

Commutative and associative laws of multiplication

= −10· a 3−2 · b −1+2

The product rule = −10ab Simplify.

5.

(x 2 2) 4 (x 2 2) −4 = (x 2 2) 4−4

The product rule = (x 2 2) 0 Simplify.

= 1 The zero exponent rule

6.

(3 w 2) 5 (6 w −2) 2 = (3) 5 · (w 2) 5 (6) 2 · (w −2) 2

The power of a product rule = 3 5 w 2·5 6 2 w −2·2 The power rule = 243 w 10 36 w −4 Simplify. = 27 w 10−(−4) 4

The quotient rule and reduce fraction = 27 w 14 4 Simplify.

Simplify each expression and write the answer with positive exponents only.

1. (2u v −2) −3
2. x 8 · x −12 ·x
3. (e 2 f −3 f −1) 2
4. (9 r −5 s 3)(3 r 6 s −4)
5. (4 9 t w −2) −3 (4 9 t w −2) 3
6. (2 h 2 k) 4 (7 h −1 k 2) 2

---

1. v 6 8 u 3
2. 1 x 3
3. e 4 f 4
4. 27r s
5. 1
6. 16 h 10 49

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron,

which is about 0.0000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

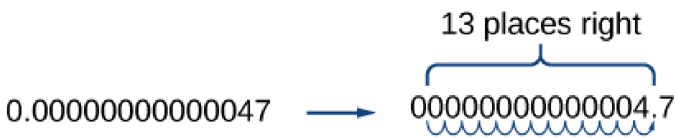
For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

$$2{,}780{,}418 \longrightarrow 2.\overbrace{780418}^{\text{6 places left}}$$

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

2.780418×10^{6}

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.



Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

Scientific Notation

A number is written in **scientific notation** if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

1. Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m
2. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
3. Number of stars in Andromeda Galaxy: 1,000,000,000,000
4. Diameter of electron: 0.00000000000094 m
5. Probability of being struck by lightning in any single year: 0.00000143

---

1. 24,000,000,000,000,000,000,000 m 24,000,000,000,000,000,000,000 m ←22 places 2.4×10^{22} m
2. 1,300,000,000,000,000,000,000 m 1,300,000,000,000,000,000,000 m ←21 places 1.3×10^{21} m
3. 1,000,000,000,000 1,000,000,000,000 ←12 places 1×10^{12}
4. 0.00000000000094 m 0.00000000000094 m →13 places 9.4×10^{-13} m
5. 0.00000143 0.00000143 →6 places 1.43×10^{-6}

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

Write each number in scientific notation.

1. U.S. national debt per taxpayer (April 2014): \$152,000
2. World population (April 2014): 7,158,000,000
3. World gross national income (April 2014): \$85,500,000,000,000
4. Time for light to travel 1 m: 0.00000000334 s
5. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

---

1. $\$1.52 \times 10^{5}$
2. 7.158×10^{9}
3. $\$8.55 \times 10^{13}$
4. 3.34×10^{-9}
5. 7.15×10^{-8}

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

1. 3.547×10^{14}
2. -2×10^{6}
3. 7.91×10^{-7}
4. -8.05×10^{-12}

---

1. 3.547×10^{14} $3.54700000000000 \rightarrow 14$ places $354{,}700{,}000{,}000{,}000$
2. -2×10^{6} $-2.000000 \rightarrow 6$ places $-2{,}000{,}000$
3.

7.91×10^{-7} 0000007.91 →7 places
0.000000791

4. -8.05×10^{-12} -000000000008.05
→12 places -0.00000000000805

Convert each number in scientific notation to standard notation.

1. 7.03×10^{5}
2. -8.16×10^{11}
3. -3.9×10^{-13}
4. 8×10^{-6}

---

1. 703,000
2. $-816,000,000,000$
3. $-0.000\,000\,000\,000\,39$
4. 0.000008

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents,

makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^{4} average drops. Therefore, there are approximately $3 \cdot (1.32 \times 10^{21}) \cdot (1.22 \times 10^{4}) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^{4}) \cdot (5 \times 10^{6}) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

Using Scientific Notation

Perform the operations and write the answer in scientific notation.

1. $(8.14 \times 10^{-7})(6.5 \times 10^{10})$
2. $(4 \times 10^{5}) \div (-1.52 \times 10^{9})$
3. $(2.7 \times 10^{5})(6.04 \times 10^{13})$
4. $(1.2 \times 10^{8}) \div (9.6 \times 10^{5})$
5. $(3.33 \times 10^{4})(-1.05 \times 10^{7})(5.62 \times 10^{5})$

---

1. $(8.14 \times 10^{-7})(6.5 \times 10^{10}) = (8.14 \times 6.5)(10^{-7} \times 10^{10})$ Commutative and associative properties of multiplication $= (52.91)(10^{3})$ Product rule of exponents $= 5.291 \times 10^{4}$ Scientific notation
2. $(4 \times 10^{5}) \div (-1.52 \times 10^{9}) = \left(\frac{4}{-1.52}\right)\left(\frac{10^{5}}{10^{9}}\right)$ Commutative and associative properties of multiplication $\approx (-2.63)(10^{-4})$ Quotient rule of exponents $= -2.63 \times 10^{-4}$ Scientific notation
3. $(2.7 \times 10^{5})(6.04 \times 10^{13}) = (2.7 \times 6.04)(10^{5} \times 10^{13})$ Commutative and associative properties of multiplication $= (16.308)(10^{18})$ Product rule of exponents $= 1.6308 \times 10^{19}$ Scientific notation
4. $(1.2 \times 10^{8}) \div (9.6 \times 10^{5}) = \left(\frac{1.2}{\right.$

$9.6)(10^8 10^5)$ Commutative and associative properties of multiplication $= (0.125)(10^3)$ Quotient rule of exponents $= 1.25 \times 10^2$ Scientific notation

5.

$$(3.33 \times 10^4)(-1.05 \times 10^7)(5.62 \times 10^5) = [3.33 \times (-1.05) \times 5.62](10^4 \times 10^7 \times 10^5) \approx (-19.65)(10^{16}) = -1.965 \times 10^{17}$$

Perform the operations and write the answer in scientific notation.

1. $(-7.5 \times 10^8)(1.13 \times 10^{-2})$
2. $(1.24 \times 10^{11}) \div (1.55 \times 10^{18})$
3. $(3.72 \times 10^9)(8 \times 10^3)$
4. $(9.933 \times 10^{23}) \div (-2.31 \times 10^{17})$
5. $(-6.04 \times 10^9)(7.3 \times 10^2)(-2.81 \times 10^2)$

1. -8.475×10^6
2. 8×10^{-8}
3. 2.976×10^{13}
4. -4.3×10^6

5. $\approx 1.24 \times 10^{15}$

Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

The population was $308{,}000{,}000 = 3.08 \times 10^8$.

The national debt was $\$17{,}547{,}000{,}000{,}000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.
$(1.75 \times 10^{13}) \div (3.08 \times 10^8) = \left(\frac{1.75}{3.08}\right) \cdot \left(\frac{10^{13}}{10^8}\right) \approx 0.57 \times 10^5 = 5.7 \times 10^4$

The debt per citizen at the time was about

$\$5.7 \times 10^4$, or \$57,000.

An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Number of cells: 3×10^{13}; length of a cell: 8×10^{-6} m; total length: 2.4×10^{8} m or $240,000,000$ m.

Access these online resources for additional instruction and practice with exponents and scientific notation.

- Exponential Notation
- Properties of Exponents
- Zero Exponent
- Simplify Exponent Expressions
- Quotient Rule for Exponents
- Scientific Notation

- Converting to Decimal Notation

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Key Concepts

- Products of exponential expressions with the

same base can be simplified by adding exponents. See [link].
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See [link].
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See [link].
- An expression with exponent zero is defined as 1. See [link].
- An expression with a negative exponent is defined as a reciprocal. See [link] and [link].
- The power of a product of factors is the same as the product of the powers of the same factors. See [link].
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See [link].
- The rules for exponential expressions can be combined to simplify more complicated expressions. See [link].
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See [link] and [link].
- Scientific notation may be used to simplify calculations with very large or very small numbers. See [link] and [link].

Section Exercises

Verbal

Is 2 3 the same as 3 2 ? Explain.

No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2 3 is the same as $2 \times 2 \times 2$, which is 8. 3 2 is the same as 3×3, which is 9.

When can you add two exponents?

What is the purpose of scientific notation?

It is a method of writing very small and very large numbers.

Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

9 2

81

$15 \; -2$

$3\,2 \;\times\; 3\,3$

243

$4\,4 \;\div 4$

$(\,2\,2\,) \;-2$

$1 \; 16$

$(5-8)\;0$

$11\;3 \;\div\; 11\;4$

$1\;11$

$6\,5 \;\times\; 6 \;-7$

(8 0) 2

1

5 −2 ÷ 5 2

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

4 2 × 4 3 ÷ 4 −4

---

4 9

6 12 6 9

(12 3 × 12) 10

---

12 40

10 6 ÷ (10 10) −2

7 −6 × 7 −3

$\frac{1}{7^9}$

$(3^3 \div 3^4)^5$

For the following exercises, express the decimal in scientific notation.

0.0000314

3.14×10^{-5}

148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

1.6×10^{10}

16,000,000,000

9.8×10^{-9}

Algebraic

For the following exercises, simplify the given

expression. Write answers with positive exponents.

a 3 a 2 a

---

a 4

m n 2 m −2

(b 3 c 4) 2

---

b 6 c 8

(x −3 y 2) −5

a b 2 ÷ d −3

---

a b 2 d 3

(w 0 x 5) −1

m 4 n 0

---

m 4

$y^{-4} (y^2)^2$

$p^{-4} q^2 p^2 q^{-3}$

$q^5 p^6$

$(l \times w)^2$

$(y^7)^3 \div x^{14}$

$y^{21} x^{14}$

$(a^2 3)^2$

$5^2 m \div 5^0 m$

25

$(16 x)^2 y^{-1}$

$2^3 (3a)^{-2}$

$72a^2$

$(ma^6)^2 \frac{1}{m^3a^2}$

$(b^{-3}c)^3$

---

c^3b^9

$(x^2y^{13} \div y^0)^2$

$(9z^3)^{-2}y$

$y81z^6$

Real-World Applications

To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

A dime is the thinnest coin in U.S. currency. A dime’s thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

0.00135 m

The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

1.0995×10^{12}

The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

0.00000000003397 in.

The value of the services sector of the U.S. economy in the first quarter of 2012 was

$10,633.6 billion. Rewrite this amount in scientific notation.

Technology

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

(12 3 m 33 4 −3) 2

12,230,590,464 m 66

17 3 ÷ 15 2 x 3

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

(3 2 a 3) −2 (a 4 2 2) 2

---

a 14 1296

(6 2 −24) 2 ÷ (x y) −5

m 2 n 3 a 2 c −3 · a −7 n −2 m 2 c 4

---

n a 9 c

(x 6 y 3 x 3 y −3 · y −7 x −3) 10

((a b 2 c) −3 b −3) 2

---

1 a 6 b 6 c 6

Avogadro’s constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro’s constant in standard notation.

Planck’s constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck’s constant in standard notation.

0.000000000000000000000000000000000662606957

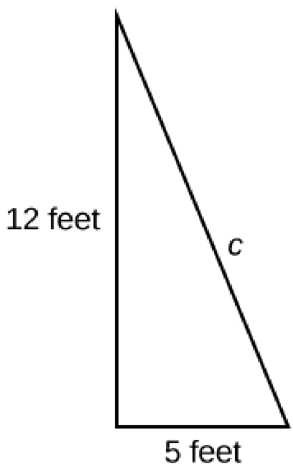
Glossary

scientific notation
: a shorthand notation for writing very large or very small numbers in the form $a \times 10^n$ where $1 \leq |a| < 10$ and n is an integer

Radicals and Rational Exponents
In this section students will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in [link], and use the Pythagorean Theorem.



$$a^2 + b^2 = c^2 \quad 5^2 + 12^2 = c^2 \quad 169 = c^2$$

Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.

Principal Square Root

The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a **radical expression,** with a symbol called a **radical** over the term called the **radicand**: a .

Does $25 = \pm 5$?

No. Although both $5 \; 2$ *and* $(-5) \; 2$ *are* 25, *the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is* $25 = 5$.

Evaluating Square Roots

Evaluate each expression.

1. 100
2. 16
3. $25 + 144$

4. $49 - 81$

---

1. $100 = 10$ because $10\ 2 = 100$
2. $16 = 4 = 2$ because $4\ 2 = 16$ and $2\ 2 = 4$
3. $25+144 = 169 = 13$ because $13\ 2 = 169$
4. $49 - 81 = 7-9 = -2$ because $7\ 2 = 49$ and $9\ 2 = 81$

For $25+144$, can we find the square roots before adding?

No. $25 + 144 = 5+12 = 17$. *This is not equivalent to* $25+144 = 13$. *The order of operations requires us to add the terms in the radicand before finding the square root.*

Evaluate each expression.

1. 225
2. 81
3. $25-9$
4. $36 + 121$

1. 15
2. 3
3. 4
4. 17

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots,* which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite 15 as $3 \cdot 5$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule for Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square

roots of a and b.
ab = a · b

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

1. 300
2. 162 a 5 b 4

---

1. 100·3
   Factor perfect square from radicand. 100 · 3
   Write radical expression as product of radical expre
   10 3 Simplify.
2.

81 a 4 b 4 ·2a
Factor perfect square from radicand. 81 a 4 b 4 · 2a
Write radical expression as product of radical expre
9 a 2 b 2 2a Simplify.

Simplify 50 x 2 y 3 z .

5| x || y | 2yz . Notice the absolute value signs around *x* and *y*? That's because their value must be positive!

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression.
12 · 3

12·3
Express the product as a single radical expression.
36 Simplify. 6

Simplify 50x · 2x assuming x > 0.

10| x |

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square

roots, using the *quotient rule for simplifying square roots.* It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite 5 2 as 5 2 .

The Quotient Rule for Simplifying Square Roots

The square root of the quotient a b is equal to the quotient of the square roots of a and b, where $b \neq 0$.

a b = a b

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

5 36

5 36
Write as quotient of two radical expressions. 5
6 Simplify denominator.

Simplify 2 x 2 9 y 4 .

x 2 3 y 2 . We do not need the absolute value signs for y 2 because that term will always be nonnegative.

Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

234 x 11 y 26 x 7 y

234 x 11 y 26 x 7 y
Combine numerator and denominator into one radical e
9 x 4 Simplify fraction. 3 x 2

Simplify square root.

Simplify 9 a 5 b 14 3 a 4 b 5 .

b 4 3ab

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of 2 and 3 2 is 4 2 . However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression 18 can be written with a 2 in the radicand, as 3 2 , so 2 + 18 = 2 +3 2 =4 2 .

Given a radical expression requiring addition or subtraction of square roots, solve.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Adding Square Roots

Add $5\ 12\ + 2\ 3$.

We can rewrite $5\ 12$ as $5\ 4 \cdot 3$. According the product rule, this becomes $5\ 4\ 3$. The square root of 4 is 2, so the expression becomes $5(2)\ 3$, which is $10\ 3$. Now we can the terms have the same radicand so we can add.

$10\ 3\ + 2\ 3\ = 12\ 3$

Add $5\ + 6\ 20$.

$13\ 5$

Subtracting Square Roots

Subtract 20 72 a 3 b 4 c $-$14 8 a 3 b 4 c .

Rewrite each term so they have equal radicands.
20 72 a 3 b 4 c = 20 9 4 2 a a 2 (b 2) 2 c = 20(3)(2)|a| b 2 2ac = 120|a| b 2 2ac
14 8 a 3 b 4 c = 14 2 4 a a 2 (b 2) 2 c = 14(2)|a| b 2 2ac = 28|a| b 2 2ac

Now the terms have the same radicand so we can subtract.
120|a| b 2 2ac $-$28|a| b 2 2ac =92|a| b 2 2ac

Subtract 3 80x $-$4 45x .

0

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator*.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b c$, multiply by $c c$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b c$, then the conjugate is $a-b c$
.

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Rationalizing a Denominator Containing a Single Term

Write 2 3 3 10 in simplest form.

The radical in the denominator is 10 . So multiply the fraction by 10 10 . Then simplify.

2 3 3 10 · 10 10 2 30 30 30 15

Write 12 3 2 in simplest form.

6 6

Given an expression with a radical term and a constant in the denominator, rationalize the

denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Rationalizing a Denominator Containing Two Terms

Write $4\ 1+\ 5$ in simplest form.

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\ 5$ is $1-\ 5$. Then multiply the fraction by $1-\ 5$ $1-\ 5$.

$4\ 1+\ 5 \cdot 1-\ 5\ 1-\ 5\ 4-4\ 5\ -4$

Use the distributive property. $5\ -1$ Simplify.

Write $7\ 2+\ 3$ in simplest form.

$14-7\ 3$

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3=8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3=8$, we say that 2 is the cube root of 8.

The *n*th root of a is a number that, when raised to the *n*th power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5=-243$. If a is a real number with at least one *n*th root, then the **principal *n*th root** of a is the number with the same sign as a that, when raised to the *n*th power, equals a.

The principal *n*th root of a is written as a n , where n is a positive integer greater than or equal to 2. In the radical expression, n is called the **index** of the radical.

Principal *n*th Root

If a is a real number with at least one *n*th root, then the **principal *n*th root** of a, written as a n , is the number with the same sign as a that, when raised to the *n*th power, equals a. The **index** of the radical is n.

Simplifying *n*th Roots

Simplify each of the following:

1. −32 5
2. 4 4 · 1,024 4
3. − 8 x 6 125 3
4. 8 3 4 − 48 4

---

1. −32 5 = −2 because (−2) 5 = −32
2. First, express the product as a single radical expression. 4,096 4 =8 because 8 4 =4,096
3. − 8 x 6 3 125 3

Write as quotient of two radical expressions.
−2 x 2 5 Simplify.

4. 8 3 4 −2 3 4

Simplify to get equal radicands. 6 3 4
Add.

Simplify.

1. −216 3
2. 3 80 4 5 4
3. 6 9,000 3 +7 576 3

1. −6
2. 6
3. 88 9 3

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be

negative.
a 1 n = a n

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an *n*th root. The numerator tells us the power and the denominator tells us the root.
a m n = (a n) m = a m n

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Rational Exponents
Rational exponents are another way to express principal *n*th roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is
a m n = (a n) m = a m n

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.

2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

The 2 tells us the power and the 3 tells us the root.

$343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = \sqrt[3]{343^2}$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$343^{\frac{2}{3}} = (\sqrt[3]{343})^2 = 7^2 = 49$

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

(9) 5 = 3 5 =243

Writing Radicals as Rational Exponents

Write 4 a 2 7 using a rational exponent.

The power is 2 and the root is 7, so the rational exponent will be 2 7 . We get 4 a 2 7 . Using properties of exponents, we get 4 a 2 7 =4 a −2 7 .

Write x (5y) 9 using a rational exponent.

x (5y) 9 2

Simplifying Rational Exponents

Simplify:

1. 5(2 x 3 4)(3 x 1 5)
2. (16 9) − 1 2

1.

   30 x 3 4 x 1 5 Multiply the coefficients.
   30 x 3 4 + 1 5
   Use properties of exponents. 30 x 19 20
   Simplify.

2.

   (9 16) 1 2
   Use definition of negative exponents. 9 16
   Rewrite as a radical. 9 16
   Use the quotient rule. 3 4 Simplify.

Simplify (8x) 1 3 (14 x 6 5).

---

28 x 23 15

Access these online resources for additional instruction and practice with radicals and rational exponents.

- Radicals
- Rational Exponents
- Simplify Radicals
- Rationalize Denominator

Key Concepts

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See [link].
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See [link] and [link].
- If a and b are nonnegative, the square root of the quotient a b is equal to the quotient of the square roots of a and b See [link] and [link].
- We can add and subtract radical expressions if they have the same radicand and the same index. See [link] and [link].
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See [link] and [link].
- The principal *n*th root of a is the number with

the same sign as a that when raised to the *n*th power equals a. These roots have the same properties as square roots. See [link].
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See [link] and [link].
- The properties of exponents apply to rational exponents. See [link].

Section Exercises

Verbal

What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.

When there is no index, it is assumed to be 2 or the square root. The expression would only be equal to the radicand if the index were 1.

Where would radicals come in the order of operations? Explain why.

Every number will have two square roots. What

is the principal square root?

The principal square root is the nonnegative root of the number.

Can a radical with a negative radicand have a real square root? Why or why not?

Numeric

For the following exercises, simplify each expression.

256

16

256

4(9 + 16)

10

289 − 121

196

14

1

98

---

7 2

27 64

81 5

---

9 5 5

800

169 + 144

25

8 50

18 162

2

192

14 6 -6 24

---

2 6

15 5 $+7$ 45

150

---

5 6

96 100

(42)(30)

---

6 35

12 3 −4 75

4 225

---

2 15

405 324

360 361

---

6 10 19

5 1+ 3

8 1− 17

− 1+ 17 2

16 4

$$\frac{128\ 3\ +3\ 2\ 3}{7\ 2\ 3}$$

$-32\ 243\ 5$

$$\frac{15\ 125\ 4\ 5\ 4}{15\ 5}$$

$3\ -432\ 3\ +\ 16\ 3$

Algebraic

For the following exercises, simplify each expression.

$$\frac{400\ x\ 4}{20\ x\ 2}$$

$4\ y\ 2$

$49p$

7 p

(144 p 2 q 6) 1 2

m 5 2 289

---

17 m 2 m

9 3 m 2 + 27

3 a b 2 −b a

2b a

4 2n 16 n 4

225 x 3 49x

15x 7

3 44z + 99z

50 y 8

---

5 y 4 2

490b c 2

32 14d

---

4 7d 7d

q 3 2 63p

8 1 − 3x

---

2 2 + 2 6x 1 − 3x

20 121 d 4

w 3 2 32 − w 3 2 50

− w 2w

108 x 4 + 27 x 4

12x 2 + 2 3

---

3 x − 3x 2

147 k 3

125 n 10

---

5 n 5 5

42q 36 q 3

81m 361 m 2

---

9 m 19m

72c −2 2c

144 324 d 2

---

2 3d

24 x 6 3 + 81 x 6 3

162 x 6 16 x 4 4

---

3 2 x 2 4 2

64y 3

128 z 3 3 − −16 z 3 3

6z 2 3

1,024 c 10 5

Real-World Applications

A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance

between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $90,000+160,000$. What is the length of the guy wire?

---

500 feet

A car accelerates at a rate of $6-4t$ m/s 2 where *t* is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

$8 - 16\ 4 - 2 - 2\ 1\ 2$

$-5\ 2\ -6\ 7$

$4\ 3\ 2 - 16\ 3\ 2\ 8\ 1\ 3$

$m\ n\ 3\ a\ 2\ c\ -3 \cdot a\ -7\ n\ -2\ m\ 2\ c\ 4$

---

mnc a 9 cmn

a a− c

x 64y +4 y 128y

---

2 2 x+ 2 4

(250 x 2 100 b 3)(7 b 125x)

64 3 + 256 4 64 + 256

---

3 3

Glossary

index
: the number above the radical sign indicating the *n*th root

principal *n*th root
: the number with the same sign as a that when raised to the *n*th power equals a

principal square root
: the nonnegative square root of a number a

that, when multiplied by itself, equals a

radical
: the symbol used to indicate a root

radical expression
: an expression containing a radical symbol

radicand
: the number under the radical symbol

Real Numbers: Algebra Essentials
In this section students will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire

crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

The real number line Sets of numbers

N: the set of natural numbers

W: the set of whole numbers

I: the set of integers

Q: the set of rational numbers

Q´: the set of irrational numbers

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1,2,3,...\}$ where the ellipsis (…) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0,1,2,3,...\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{...,-3,-2,-1,0,1,2,3,...\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$\ldots,-3,-2,-1,$ negative integers $0,$ zero $1,2,3,\cdots$ positive integers

The set of **rational numbers** is written as $\{ m\,n \mid m \text{ and } n \text{ are integers and } n \neq 0 \}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8}=1.875$, or
2. a repeating decimal: $\frac{4}{11}=0.36363636\ldots=0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Writing Integers as Rational Numbers

Write each of the following as a rational number.

1. 7
2. 0
3. –8

Write a fraction with the integer in the numerator and 1 in the denominator.

1. $7=\frac{7}{1}$
2. $0=\frac{0}{1}$
3. $-8=-\frac{8}{1}$

Write each of the following as a rational number.

1. 11
2. 3
3. –4

1. 11 1
2. 3 1
3. − 4 1

Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

1. − 5 7
2. 15 5
3. 13 25

Write each fraction as a decimal by dividing the numerator by the denominator.

1. − 5 7 = −0. 714285 ——— , a repeating

decimal
2. $\frac{15}{5} = 3$ (or 3.0), a terminating decimal
3. $\frac{13}{25} = 0.52$, a terminating decimal

Write each of the following rational numbers as either a terminating or repeating decimal.

1. $\frac{68}{17}$
2. $\frac{8}{13}$
3. $-\frac{17}{20}$

1. 4 (or 4.0), terminating;
2. $0.\overline{615384}$, repeating;
3. –0.85, terminating

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$, but was something else. Or a

garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$\{ h | h \text{ is not a rational number} \}$

Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

1. 25
2. 33 9
3. 11
4. 17 34
5. 0.3033033303333…

---

1. 25 : This can be simplified as 25 = 5.

Therefore, 25 is rational.

2. 33 9 : Because it is a fraction, 33 9 is a rational number. Next, simplify and divide.
   33 9 = 33 11 9 3 = 11 3 =3. 6 ¯

   So, 33 9 is rational and a repeating decimal.
3. 11 : This cannot be simplified any further. Therefore, 11 is an irrational number.
4. 17 34 : Because it is a fraction, 17 34 is a rational number. Simplify and divide.
   17 34 = 17 1 34 2 = 1 2 =0.5

   So, 17 34 is rational and a terminating decimal.
5. 0.3033033303333… is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Determine whether each of the following

numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

1. 7 77
2. 81
3. 4.27027002700027…
4. 91 13
5. 39

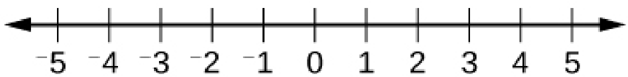
1. rational and repeating;
2. rational and terminating;
3. irrational;
4. rational and repeating;
5. irrational

Real Numbers

Given any number *n*, we know that *n* is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive

nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in [link].



Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

1. − 10 3
2. 5
3. − 289
4. -6π
5. 0.615384615384…

1. − 10 3 is negative and rational. It lies to the left of 0 on the number line.
2. 5 is positive and irrational. It lies to the right of 0.
3. − 289 = − 17 2 = −17 is negative and rational. It lies to the left of 0.
4. −6π is negative and irrational. It lies to the left of 0.
5. 0.615384615384… is a repeating decimal so it is rational and positive. It lies to the right of 0.

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

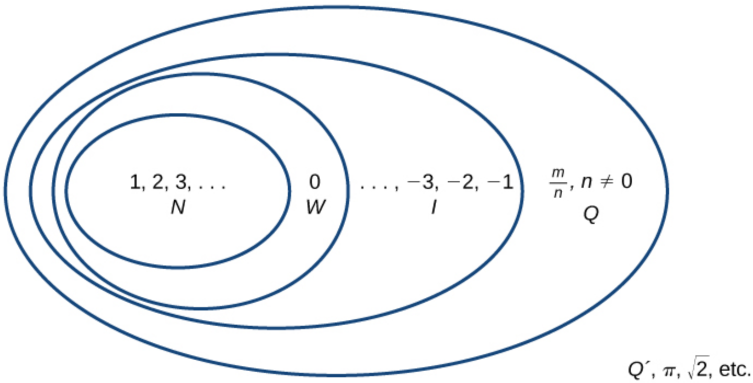
1. 73
2. −11.411411411…
3. 47 19
4. − 5 2
5. 6.210735

1. positive, irrational; right
2. negative, rational; left

3. positive, rational; right
4. negative, irrational; left
5. positive, rational; right

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as [link].



Sets of Numbers
The set of **natural numbers** includes the numbers used for counting: $\{1,2,3,...\}$.
The set of **whole numbers** is the set of natural

numbers plus zero: $\{0,1,2,3,...\}$.
The set of **integers** adds the negative natural numbers to the set of whole numbers: $\{...,-3,-2,-1,0,1,2,3,...\}$.
The set of **rational numbers** includes fractions written as $\{\frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0\}$.
The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \mid h \text{ is not a rational number}\}$.

Differentiating the Sets of Numbers

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q′*).

1. 36
2. 8 3
3. 73
4. -6
5. 3.2121121112…

	N	*W*	*I*	*Q*	*Q′*
a. 36 =6	X	X	X	X	
b. 8 3 =2. 6 ‾				X	
c. 73					X
d. 6			X	X	
e. 3.2121121112...					X

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q′*).

1. − 35 7
2. 0
3. 169
4. 24
5. 4.763763763…

	N	W	I	Q	Q'
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. 169	X	X	X	X	
d. 24					X
e. 4.763763763...				X	

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4\cdot4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.

$$a^n = \underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{n \text{ factors}}$$

In this notation, a^n is read as the *n*th power of a, where a is called the **base** and n is called the **exponent.** A term in exponential notation may be part of a mathematical expression, which is a

combination of numbers and operations. For example, $24+6\cdot\ 2\ 3\ -\ 4\ 2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let’s take a look at the expression provided.

$24+6\cdot\ 2\ 3\ -\ 4\ 2$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify $4\ 2$ as 16.

$24+6\cdot\ 2\ 3\ -\ 4\ 2\ 24+6\cdot\ 2\ 3\ -16$

Next, perform multiplication or division, left to right.
$24+6\cdot\ 2\ 3\ -16\ 24+4-16$

Lastly, perform addition or subtraction, left to right.
$24+4-16 \qquad 28-16 \qquad\qquad 12$

Therefore, $24+6\cdot\ 2\ 3\ -\ 4\ 2\ =12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

Order of Operations
Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:
P(arentheses)
E(xponents)
M(ultiplication) and **D**(ivision)
A(ddition) and **S**(ubtraction)

Given a mathematical expression, simplify it

using the order of operations.

Simplify any expressions within grouping symbols. Simplify any expressions containing exponents or radicals. Perform any multiplication and division in order, from left to right. Perform any addition and subtraction in order, from left to right.

Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

1. $(3\cdot2) 2 -4(6+2)$
2. $5 2 -4 7 - 11-2$
3. $6-| 5-8 |+3(4-1)$
4. $14-3\cdot2 2\cdot5- 3 2$
5. $7(5\cdot3)-2[(6-3)- 4 2]+1$

1.

$(3\cdot2)2-4(6+2) = (6)2-4(8)$
Simplify parentheses $= 36-4(8)$
Simplify exponent $= 36-32$
Simplify multiplication $= 4$
Simplify subtraction

2.

$52\text{-}47-11-2 = 52-47-9$
Simplify grouping symbols (radical) $=$

$52-47-3$ Simplify radical $= 25-47-3$
Simplify exponent $= 217-3$
Simplify subtraction in numerator $= 3-3$
Simplify division $= 0$
Simplify subtraction

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

3.

$6-|5-8|+3(4-1) = 6-|-3|+3(3)$
Simplify inside grouping symbols $=$
$6-3+3(3)$ Simplify absolute value $=$
$6-3+9$ Simplify multiplication $= 3+9$
Simplify subtraction $= 12$
Simplify addition

4.

$14-3\cdot22\cdot5-32 = 14-3\cdot22\cdot5-9$
Simplify exponent $= 14-610-9$
Simplify products $= 81$
Simplify differences $= 8$
Simplify quotient

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

5.

$7(5\cdot3)-2[\ (6-3)-42\ ]+1 = 7(15)-2[$

(3) − 42] + 1 Simplify inside parentheses = 7(15) − 2(3 − 16) + 1 Simplify exponent = 7(15) − 2(−13) + 1 Subtract = 105 + 26 + 1 Multiply = 132 Add

Use the order of operations to evaluate each of the following expressions.

1. 5 2 − 4 2 +7 (5−4) 2
2. 1+ 7·5−8·4 9−6
3. | 1.8−4.3|+0.4 15+10
4. 1 2 [5· 3 2 − 7 2]+ 1 3 · 9 2
5. [(3−8) 2 −4]−(3−8)

---

1. 10
2. 2
3. 4.5
4. 25
5. 26

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.
$a+b=b+a$

We can better see this relationship when using real numbers.
$(-2)+7=5$ and $7+(-2)=5$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.
$a \cdot b=b \cdot a$

Again, consider an example with real numbers.
$(-11)\cdot(-4)=44$ and $(-4)\cdot(-11)=44$

It is important to note that neither subtraction nor

division is commutative. For example, $17-5$ is not the same as $5-17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.
$a(bc)=(ab)c$

Consider this example.
$(3 \cdot 4) \cdot 5 = 60$ and $3 \cdot (4 \cdot 5) = 60$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.
$a+(b+c)=(a+b)+c$

This property can be especially helpful when dealing with negative integers. Consider this example.
$[15+(-9)]+23=29$ and $15+[(-9)+23]=29$

Are subtraction and division associative? Review these examples.
$8-(3-15) \overset{?}{=} (8-3)-15$ $64 \div (8 \div 4) \overset{?}{=} (64 \div 8) \div 4$ $8-(-12) = 5-15$ $64 \div 2 \overset{?}{=} 8 \div 4$ $20 \neq -10$ $32 \neq 2$

As we can see, neither subtraction nor division is associative.

Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.
$a\cdot(b+c)=a\cdot b+a\cdot c$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4\cdot[12+(-7)] &= \overbrace{4\cdot 12} + \overbrace{4\cdot(-7)} \\ &= 48+(-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by –7, and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.
$6+(3\cdot 5) \overset{?}{=} (6+3)\cdot(6+5)$ $6+(15) \overset{?}{=} (9)\cdot(11)$ $21 \neq 99$

A special case of the distributive property occurs when a sum of terms is subtracted.

$a-b=a+(-b)$

For example, consider the difference $12-(5+3)$. We can rewrite the difference of the two terms 12 and $(5+3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5+3)$, we add the opposite.

$12+(-1)\cdot(5+3)$

Now, distribute -1 and simplify the result.

$12-(5+3) = 12+(-1)\cdot(5+3) =$
$12+[(-1)\cdot5+(-1)\cdot3] = 12+(-8) = 4$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$12-(5+3) = 12+(-5-3) = 12+(-8) = 4$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$a+0=a$

The **identity property of multiplication** states that there is a unique number, called the multiplicative

identity (1) that, when multiplied by a number, results in the original number.

$a \cdot 1 = a$

For example, we have $(-6) + 0 = -6$ and $23 \cdot 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The **inverse property of addition** states that, for every real number *a*, there is a unique number, called the additive inverse (or opposite), denoted $-a$, that, when added to the original number, results in the additive identity, 0.

$a + (-a) = 0$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number *a*, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$a \cdot \frac{1}{a} = 1$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted

$\frac{1}{a}$, is $-\frac{3}{2}$ because

$a\cdot\frac{1}{a}=\left(-\frac{2}{3}\right)\cdot\left(-\frac{3}{2}\right)=1$

Properties of Real Numbers

The following properties hold for real numbers *a*, *b*, and *c*.

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative Property	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Distributive Property	$a\cdot(b+c)=a\cdot b+a\cdot c$	
Identity Property		
Inverse Property		

Using Properties of Real Numbers

Use the properties of real numbers to rewrite

and simplify each expression. State which properties apply.

1. $3\cdot6+3\cdot4$
2. $(5+8)+(-8)$
3. $6-(15+9)$
4. $47\cdot(23\cdot74)$
5. $100\cdot[0.75+(-2.38)]$

---

1. $3\cdot6+3\cdot4 = 3\cdot(6+4)$ Distributive property $= 3\cdot10$ Simplify $= 30$ Simplify
2. $(5+8)+(-8) = 5+[8+(-8)]$ Associative property of addition $= 5+0$ Inverse property of addition $= 5$ Identity property of addition
3. $6-(15+9) = 6+[(-15)+(-9)]$ Distributive property $= 6+(-24)$ Simplify $= -18$ Simplify
4. $47\cdot(23\cdot74) = 47\cdot(74\cdot23)$ Commutative property of multiplication $= (47\cdot74)\cdot23$ Associative property of multiplication $= 1\cdot23$ Inverse property of multiplication $= 23$ Identity property of multiplication
5. $100\cdot[0.75+(-2.38)] =$

$100 \cdot 0.75 + 100 \cdot (-2.38)$
Distributive property $= 75 + (-238)$
Simplify $= -163$ Simplify

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

1. $(- \ 23\ 5\ ) \cdot [\ 11 \cdot (- \ 5\ 23\ )\ ]$
2. $5 \cdot (\ 6.2 + 0.4\ )$
3. $18 - (\ 7 - 15\ )$
4. $17\ 18\ + [\ 4\ 9\ + (- \ 17\ 18\ )\ ]$
5. $6 \cdot (\ -3\ ) + 6 \cdot 3$

1. 11, commutative property of multiplication, associative property of multiplication, inverse property of multiplication, identity property of multiplication;
2. 33, distributive property;
3. 26, distributive property;
4. $4\ 9$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition;

5. 0, distributive property, inverse property of addition, identity property of addition

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5$, $\frac{4}{3}\pi r^3$, or $2m^3n^2$. In the expression $x+5$, 5 is called a **constant** because it does not vary and *x* is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$(-3)^5 = (-3)\cdot(-3)\cdot(-3)\cdot(-3)\cdot(-3)$ $x^5 = x\cdot x\cdot x\cdot x\cdot x$ $(2\cdot 7)^3 = (2\cdot 7)\cdot(2\cdot 7)\cdot(2\cdot 7)$ $(yz)^3 = (yz)\cdot(yz)\cdot(yz)$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of

constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

1. $x + 5$
2. 4 3 π r 3
3. 2 m 3 n 2

	Constants	Variables
a. $x + 5$	5	x
b. 4 3 π r 3	4 3 ,π	r
c. 2 m 3 n 2	2	m,n

List the constants and variables for each algebraic expression.

1. 2πr(r+h)
2. $2(L + W)$
3. 4 y 3 +y

	Constants	Variables
a. 2πr(r+h)	2,π	r,h
b. 2(L + W)	2	L, W
c. 4 y 3 +y	4	y

Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x-7$ for each value for *x*.

1. $x=0$
2. $x=1$
3. $x=\ 1\ 2$
4. $x=-4$

---

1. Substitute 0 for x.
   $2x-7\ =\ 2(0)-7\ =\ 0-7\ =\ -7$
2. Substitute 1 for x.
   $2x-7\ =\ 2(1)-7\ =\ 2-7\ =\ -5$
3. Substitute $\ 1\ 2\ $ for x.
   $2x-7\ =\ 2(12)-7\ =\ 1-7\ =\ -6$
4. Substitute -4 for x.
   $2x-7\ =\ 2(-4)-7\ =\ -8-7\ =\ -15$

Evaluate the expression $11-3y$ for each value for *y*.

1. $y=2$
2. $y=0$
3. $y=\ 2\ 3$

4. $y = -5$

1. 5;
2. 11;
3. 9;
4. 26

Evaluating Algebraic Expressions

Evaluate each expression for the given values.

1. $x+5$ for $x=-5$
2. $t\ 2t-1$ for $t=10$
3. $4\ 3\ \pi\ r\ 3$ for $r=5$
4. $a+ab+b$ for $a=11, b=-8$
5. $2\ m\ 3\ n\ 2$ for $m=2, n=3$

1. Substitute -5 for x.
   $x+5 = (-5)+5 = 0$
2. Substitute 10 for t.
   $t2t-1 = (10)2(10)-1 = 1020-1 = 1019$
3. Substitute 5 for r.
   $43\pi r3 = 43\pi(5)3 = 43\pi(125) = 5003\ \pi$
4. Substitute 11 for a and –8 for b.
   $a+ab+b = (11)+(11)(-8)+(-8) =$

$11-88-8 = -85$

5. Substitute 2 for m and 3 for n.
   $2m3n2 = 2(2)3(3)2 = 2(8)(9) = 144 = 12$

Evaluate each expression for the given values.

1. $y+3\ y-3$ for $y=5$
2. $7-2t$ for $t=-2$
3. $1\ 3\ \pi\ r\ 2$ for $r=11$
4. $(\ p\ 2\ q\ )\ 3$ for $p=-2, q=3$
5. $4(\ m-n\ )-5(\ n-m\ )$ for $m=\ 2\ 3\ , n=\ 1\ 3$

1. 4;
2. 11;
3. $121\ 3\ \pi$;
4. 1728;
5. 3

Formulas

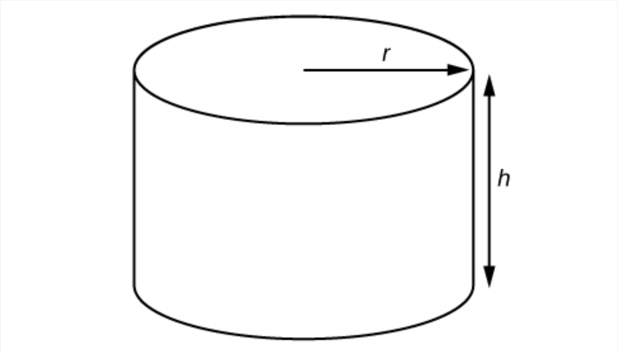
An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x+1=7$ has the unique solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2(3)+1=7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A=\pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S=2\pi r(r+h)$. See [link]. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

Right circular cylinder

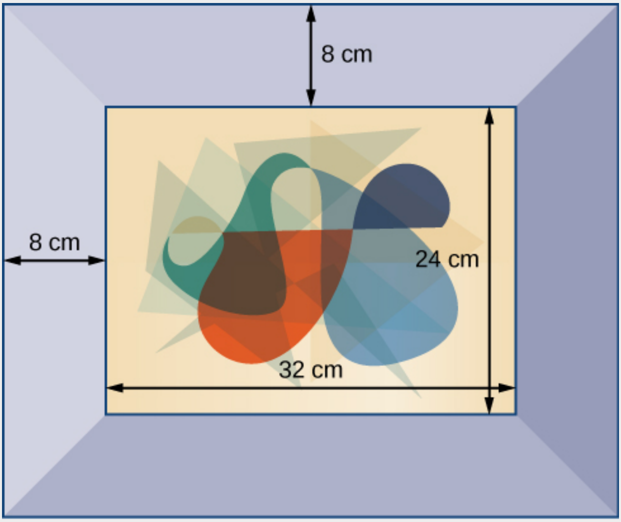


Evaluate the expression $2\pi r(r+h)$ for $r=6$ and $h=9$.

$S = 2\pi r(r+h) = 2\pi(6)[(6)+(9)] = 2\pi(6)(15)$
$= 180\pi$

The surface area is 180π square inches.

A photograph with length *L* and width *W* is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm2) is found to be $A=(L+16)(W+16)-L\cdot W$. See [link]. Find the area of a matte for a photograph with length 32 cm and width 24 cm.



1,152 cm^2

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Simplifying Algebraic Expressions

Simplify each algebraic expression.

1. $3x-2y+x-3y-7$
2. $2r-5(\ 3-r\ )+4$
3. $(\ 4t-\ 5\ 4\ s\ )-(\ 2\ 3\ t+2s\ )$
4. $2mn-5m+3mn+n$

1.

   $3x-2y+x-3y-7\ =\ 3x+x-2y-3y-7$
   Commutative property of addition $=\ 4x$
   $-5y-7$ Simplify

2.

   $2r-5(3-r)+4\ =\ 2r-15+5r+4$
   Distributive property $=\ 2r+5r-15+4$
   Commutative property of addition $=\ 7r$
   -11 Simplify

3.

   $(\ 4t-\ 5\ 4\ s\ )-(\ 2\ 3\ t+2s\ )\ =\ 4t-54s$
   $-23t-2s$ Distributive property $=\ 4t$
   $-23t-54s-2s$
   Commutative property of addition $=\ 103t$
   $-\ 134s$ Simplify

4.

   $2mn-5m+3mn+n\ =\ 2mn+3mn-5m$
   $+n$ Commutative property of addition $=$
   $5mn-5m+n$ Simplify

Simplify each algebraic expression.

1. $2\ 3\ y-2(\ 4\ 3\ y+z\ )$
2. $5\ t\ -2-\ 3\ t\ +1$
3. $4p(\ q-1\ )+q(\ 1-p\ )$
4. $9r-(\ s+2r\ )+(\ 6-s\ )$

---

1. $-2y-2z$ or $-2(\ y+z\ )$;
2. $2\ t\ -1$;
3. $3pq-4p+q$;
4. $7r-2s+6$

Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P=L+W+L+W$. Simplify this expression.

$P\ =\ L+W+L+W$ $P\ =\ L+L+W+W$
Commutative property of addition $P\ =\ 2L$
$+2W$ Simplify $P\ =\ 2(L+W\ )$
Distributive property

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

$A = P(1 + rt)$

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression
- Evaluate an Expression1
- Evaluate an Expression2

Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals. See [link] and [link].
- Determine whether a number is rational or irrational by writing it as a decimal. See [link].
- The rational numbers and irrational numbers

make up the set of real numbers. See [link]. A number can be classified as natural, whole, integer, rational, or irrational. See [link].

- The order of operations is used to evaluate expressions. See [link].
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See [link].
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See [link]. They take on a numerical value when evaluated by replacing variables with constants. See [link], [link], and [link]
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See [link] and [link].

Verbal

Is 2 an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational.

What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. This is because the same operation is performed (either addition or subtraction), so the terms can be re-ordered.

Numeric

For the following exercises, simplify the given expression.

$10+2\times(5-3)$

$6 \div 2 - (81 \div \ 3\ 2)$

-6

$18 + \ (6 - 8)\ 3$

$-2 \times \ [16 \div \ (8 - 4)\ 2]\ 2$

-2

$4 - 6 + 2 \times 7$

$3(5 - 8)$

-9

$4 + 6 - 10 \div 2$

$12 \div (36 \div 9) + 6$

9

$(4+5)2 \div 3$

$3-12 \times 2+19$

-2

$2+8 \times 7 \div 4$

$5+(6+4)-11$

4

$9-18 \div 3\ 2$

$14 \times 3 \div 7-6$

0

$9-(3+11) \times 2$

$6+2 \times 2-1$

9

$64 \div (8 + 4 \times 2)$

$9 + 4(2\ 2)$

25

$(12 \div 3 \times 3)\ 2$

$25 \div\ 5\ 2\ - 7$

-6

$(15 - 7) \times (3 - 7)$

$2 \times 4 - 9(\ -1)$

17

$4\ 2\ - 25 \times\ 1\ 5$

$12(3-1)\div 6$

4

Algebraic

For the following exercises, solve for the variable.

$8(x+3)=64$

$4y+8=2y$

-4

$(11a+3)-18a=-4$

$4z-2z(1+4)=36$

-6

$4y(7-2)^2=-200$

$-(2x)^2+1=-3$

± 1

$8(2+4)-15b=b$

$2(11c-4)=36$

2

$4(3-1)x=4$

$\frac{1}{4}(8w-4^2)=0$

2

For the following exercises, simplify the expression.

$4x+x(13-7)$

$2y-(4)^2y-11$

$-14y-11$

$\frac{a}{2^3}(64)-12a\div 6$

$8b - 4b(\ 3\ ) + 1$

$-4b + 1$

$5l \div 3l \times (\ 9 - 6\ )$

$7z - 3 + z \times\ \ 6\ 2$

$43z - 3$

$4 \times 3 + 18x \div 9 - 12$

$9(\ y + 8\ ) - 27$

$9y + 45$

$(\ 9\ 6\ t - 4\ )2$

$6 + 12b - 3 \times 6b$

$-6b + 6$

$18y-2(1+7y)$

$(4\ 9)2 \times 27x$

$16x\ 3$

$8(3-m)+1(-8)$

$9x+4x(2+3)-4(2x+3x)$

$9x$

$5\ 2\ -4(3x)$

Real-World Applications

For the following exercises, consider this scenario: Fred earns \$40 mowing lawns. He spends \$10 on mp3s, puts half of what is left in a savings account, and gets another \$5 for washing his neighbor’s car.

Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

$1\ 2\ (40 - 10) + 5$

How much money does Fred keep?

For the following exercises, solve the given problem.

According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

---

irrational number

Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold

and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

Write the equation that describes the situation.

$g+400-2(600)=1200$

Solve for *g*.

For the following exercise, solve the given problem.

Ramon runs the marketing department at his company. His department gets a budget every year, and every year, he must spend the entire budget without going over. If he spends less than the budget, then his department gets a smaller budget the following year. At the beginning of this year, Ramon got $2.5 million for the annual marketing budget. He must spend the budget such that $2{,}500{,}000-x=0$. What property of addition tells us what the value of *x* must be?

---

inverse property of addition

Technology

For the following exercises, use a graphing calculator to solve for *x*. Round the answers to the nearest hundredth.

$0.5 (12.3) 2 - 48x = 3 5$

$(0.25 - 0.75) 2 x - 7.2 = 9.9$

68.4

Extensions

If a whole number is not a natural number, what must the number be?

Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

---

true

Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

Determine whether the simplified expression is rational or irrational: $-18-4(5)(-1)$.

---

irrational

Determine whether the simplified expression is rational or irrational: $-16+4(5)+5$.

The division of two whole numbers will always result in what type of number?

---

rational

What property of real numbers would simplify the following expression: $4+7(x-1)$?

Glossary

algebraic expression
: constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition
: the sum of three numbers may be grouped differently without affecting the result; in

symbols, $a+(b+c)=(a+b)+c$

associative property of multiplication
: the product of three numbers may be grouped differently without affecting the result; in symbols, $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

base
: in exponential notation, the expression that is being multiplied

commutative property of addition
: two numbers may be added in either order without affecting the result; in symbols, $a+b=b+a$

commutative property of multiplication
: two numbers may be multiplied in any order without affecting the result; in symbols, $a\cdot b=b\cdot a$

constant
: a quantity that does not change value

distributive property
: the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a\cdot(b+c)=a\cdot b+a\cdot c$

equation
: a mathematical statement indicating that two expressions are equal

exponent
: in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation
: a shorthand method of writing products of the same factor

formula
: an equation expressing a relationship between constant and variable quantities

identity property of addition
: there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a + 0 = a$

identity property of multiplication
: there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1 = a$

integers
: the set consisting of the natural numbers, their opposites, and 0: $\{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

inverse property of addition
: for every real number a, there is a unique

number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0; in symbols, $a+(-a)=0$

inverse property of multiplication
: for every non-zero real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a\cdot\frac{1}{a}=1$

irrational numbers
: the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

natural numbers
: the set of counting numbers: $\{1,2,3,\ldots\}$

order of operations
: a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

rational numbers
: the set of all numbers of the form $\frac{m}{n}$, where m and n are integers and $n\neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line

a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers

the sets of rational numbers and irrational numbers taken together

variable

a quantity that may change value

whole numbers

the set consisting of 0 plus the natural numbers: $\{ 0,1,2,3,\ldots \}$

Systems of Measurement

By the end of this section, you will be able to:

- Make unit conversions in the US system
- Use mixed units of measurement in the US system
- Make unit conversions in the metric system
- Use mixed units of measurement in the metric system
- Convert between the US and the metric systems of measurement
- Convert between Fahrenheit and Celsius temperatures

A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **The Properties of Real Numbers**.

Make Unit Conversions in the U.S. System

There are two systems of measurement commonly used around the world. Most countries use the metric system. The U.S. uses a different system of

measurement, usually called the **U.S. system**. We will look at the U.S. system first.

The U.S. system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart, and gallons. Both the U.S. system and the metric system measure time in seconds, minutes, and hours.

The equivalencies of measurements are shown in [link]. The table also shows, in parentheses, the common abbreviations for each measurement.

U.S. System of Measurement

Length 1 foot (ft.) = 12 inches (in.) 1 yard (yd.) = 3 feet (ft.) 1 mile [illegible]	Volume 3 teaspoons (t) = 1 tablespoon (T) [illegible] tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl. oz.) 1 pint (pt.) = 2 cups (C) 1 quart (qt.) = 2 pints (pt.) 1 gallon (gal) = 4 quarts (qt.)
Weight 1 pound (lb.) = 16 ounces (oz.) 1 ton = 2000 pounds (lb.)	Time 1 minute (min) = 60 seconds (sec) 1 hour (hr) = 60 minutes (min) 1 day = 24

hours(hr)1 week(wk) = 7 days1 year(yr) = 365 days

In many real-life applications, we need to convert between units of measurement, such as feet and yards, minutes and seconds, quarts and gallons, etc. We will use the identity property of multiplication to do these conversions. We'll restate the identity property of multiplication here for easy reference.

Identity Property of Multiplication

For any real numbera:a·1 = a1·a = a1is themultiplicative identity

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to change inches to feet. We know that 1 foot is equal to 12 inches, so we will write 1 as the fraction 1foot12inches. When we multiply by this fraction we do not change the value, but just change the units.

But 12inches1foot also equals 1. How do we decide whether to multiply by 1foot12inches or 12inches1foot? We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like factors and "divide out"

common units like we do common factors. If we want to convert 66 inches to feet, which multiplication will eliminate the inches?

$$66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \quad \text{or} \quad 66 \text{ inches} \cdot \frac{12 \text{ inches}}{1 \text{ foot}}$$

$$\text{The first form works since } 66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}.$$

The inches divide out and leave only feet. The second form does not have any units that will divide out and so will not help us.

How to Make Unit Conversions

MaryAnne is 66 inches tall. Convert her height into feet.

Solution

Step 1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.	Multiply 66 inches by 1, writing 1 as a fraction relating inches and feet. We need inches in the denominator so that the inches will divide out!	$66 \text{ inches} \cdot 1$ $66 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$

Step 2. Multiply.	Think of 66 inches as $\frac{66 \text{ inches}}{1}$.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$

Step 3. Simplify the fraction.	Notice: inches divide out.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$ $\frac{66 \text{ feet}}{12}$
Step 4. Simplify.	Divide 66 by 12.	5.5 feet

Lexie is 30 inches tall. Convert her height to feet.

2.5 feet

Rene bought a hose that is 18 yards long. Convert the length to feet.

54 feet

Make Unit Conversions.

Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed. Multiply. Simplify the fraction. Simplify.

When we use the identity property of multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

Ndula, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.

Solution

We will convert 3.2 tons into pounds. We will use the identity property of multiplication, writing 1 as the fraction 2000pounds1ton.

	3.2 tons
Multiply the measurement to be converted, by 1.	3.2 tons·1
Write 1 as a fraction relating tons and pounds.	3.2 tons·2,000 pounds1 ton
Simplify.	$\frac{3.2\ \cancel{\text{tons}} \cdot 2{,}000\ \text{pounds}}{1\ \cancel{\text{ton}}}$
Multiply.	6,400 pounds
	Ndula weighs almost 6,400 pounds.

Arnold’s SUV weighs about 4.3 tons. Convert the weight to pounds.

8,600 pounds

The Carnival *Destiny* cruise ship weighs 51,000

tons. Convert the weight to pounds.

102,000,000 pounds

Sometimes, to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

Juliet is going with her family to their summer home. She will be away from her boyfriend for 9 weeks. Convert the time to minutes.

Solution

To convert weeks into minutes we will convert weeks into days, days into hours, and then hours into minutes. To do this we will multiply by conversion factors of 1.

	9 weeks	

Write 1 as $\frac{7\ \text{days}}{1\ \text{week}}$, $\frac{24\ \text{hours}}{1\ \text{day}}$, and $\frac{60\ \text{minutes}}{1\ \text{hour}}$.	$\frac{9\ \text{wk}}{1}\cdot\frac{7\ \text{days}}{1\ \text{wk}}\cdot\frac{24\ \text{hr}}{1\ \text{day}}\cdot\frac{60\ \text{min}}{1\ \text{hr}}$
Divide out the common units.	$\frac{9\ \cancel{\text{wk}}}{1}\cdot\frac{7\ \cancel{\text{days}}}{1\ \cancel{\text{wk}}}\cdot\frac{24\ \cancel{\text{hr}}}{1\ \cancel{\text{day}}}\cdot\frac{60\ \text{min}}{1\ \cancel{\text{hr}}}$
Multiply.	$\frac{9\cdot7\cdot24\cdot60\ \text{min}}{1\cdot1\cdot1\cdot1}$
Multiply.	90,720 min

Juliet and her boyfriend will be apart for 90,720 minutes (although it may seem like an eternity!).

The distance between the earth and the moon is about 250,000 miles. Convert this length to yards.

440,000,000 yards

The astronauts of Expedition 28 on the International Space Station spend 15 weeks in

space. Convert the time to minutes.
151,200 minutes

How many ounces are in 1 gallon?

Solution

We will convert gallons to ounces by multiplying by several conversion factors. Refer to [link].

Multiply the measurement to be converted by 1.	1 gallon 1 gallon1·4 quarts1 gallon·2 pints1 quart·2 cups1 pint·8 ounces1 cup
Use conversion factors to get to the right unit. Si	$\frac{\cancel{1\text{ gallon}}}{1}\cdot\frac{\cancel{4\text{ quarts}}}{\cancel{1\text{ gallon}}}\cdot\frac{\cancel{2\text{ pints}}}{\cancel{1\text{ quart}}}\cdot\frac{\cancel{2\text{ cups}}}{\cancel{1\text{ pint}}}\cdot\frac{8\text{ ounces}}{\cancel{1\text{ cup}}}$
Multiply.	1·4·2·2·8 ounces1·1·1·1·1

Simplify.	128 ounces There are 128 ounces in a gallon.

How many cups are in 1 gallon?
16 cups

How many teaspoons are in 1 cup?
48 teaspoons

Use Mixed Units of Measurement in the U.S. System

We often use mixed units of measurement in everyday situations. Suppose Joe is 5 feet 10 inches tall, stays at work for 7 hours and 45 minutes, and then eats a 1 pound 2 ounce steak for dinner—all these measurements have mixed units.

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units!

Seymour bought three steaks for a barbecue. Their weights were 14 ounces, 1 pound 2 ounces and 1 pound 6 ounces. How many total pounds of steak did he buy?

Solution

We will add the weights of the steaks to find the total weight of the steaks.

Add the ounces. Then add the pounds.

$$\begin{array}{rr} & 14 \text{ ounces} \\ 1 \text{ pound} & 2 \text{ ounces} \\ +\ 1 \text{ pound} & 6 \text{ ounces} \\ \hline 2 \text{ pounds} & 22 \text{ ounces} \end{array}$$

Convert 22 ounces to pounds and ounces.	2 pounds + 1 pound, 6 ounces
Add the pounds.	3 pounds, 6 ounces
	Seymour bought 3 pounds 6 ounces of steak.

Laura gave birth to triplets weighing 3 pounds 3 ounces, 3 pounds 3 ounces, and 2 pounds 9 ounces. What was the total birth weight of the three babies?

8 lbs.15 oz

Stan cut two pieces of crown molding for his family room that were 8 feet 7 inches and 12

feet 11 inches. What was the total length of the molding?

21 ft. 6 in.

Anthony bought four planks of wood that were each 6 feet 4 inches long. What is the total length of the wood he purchased?

Solution

We will multiply the length of one plank to find the total length.

Multiply the inches and then the feet.	6 feet 4 inches $\times$ 4 24 feet 16 inches
Convert the 16 inches to feet. Add the feet.	

[redacted]

Anthony bought 25 feet and 4 inches of wood.

Henri wants to triple his spaghetti sauce recipe that uses 1 pound 8 ounces of ground turkey. How many pounds of ground turkey will he need?
4 lbs. 8 oz.

Joellen wants to double a solution of 5 gallons 3 quarts. How many gallons of solution will she have in all?
11 gal. 2 qt.

Make Unit Conversions in the Metric System

In the **metric system**, units are related by powers of 10. The roots words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1,000 meters; the prefix *kilo* means *thousand*. One centimeter is 1100 of a meter, just like one cent is 1100 of one dollar.

The equivalencies of measurements in the metric system are shown in [link]. The common abbreviations for each measurement are given in parentheses.

Metric System of Measurement		
Length	Mass	Capacity
1 kilometer (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kiloliter (kL) = 1,000 L
1 hectometer (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectoliter (hL) = 100 L

1 dekameter (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekaliter (daL) = 10 L
1 meter (m) = 1 m	1 gram (g) = 1 g	1 liter (L) = 1 L
1 decimeter (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 deciliter (dL) = 0.1 L
1 centimeter (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centiliter (cL) = 0.01 L
1 millimeter (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters	1 gram = 100 centigrams	1 liter = 100 centiliters
1 meter = 1,000 millimeters	1 gram = 1,000 milligrams	1 liter = 1,000 milliliters

To make conversions in the metric system, we will use the same technique we did in the US system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

Have you ever run a 5K or 10K race? The length of those races are measured in kilometers. The metric system is commonly used in the United States when talking about the length of a race.

Nick ran a 10K race. How many meters did he run?

Solution

We will convert kilometers to meters using the identity property of multiplication.

	10 kilometers
Multiply the measurement to be converted by 1.	[redacted]
Write 1 as a fraction relating kilometers and me[redacted]	
Simplify.	[redacted]
Multiply.	10,000 meters
	Nick ran 10,000 meters.

Sandy completed her first 5K race! How many meters did she run?

5,000 meters

Herman bought a rug 2.5 meters in length. How many centimeters is the length?

250 centimeters

Eleanor's newborn baby weighed 3,200 grams. How many kilograms did the baby weigh?

Solution

We will convert grams into kilograms.

[redacted]

Multiply the measurement to be co[redacted]

Write 1 as a function relating kilograms and gra[redacted]

Simplify.

[redacted]

Multiply.	3,200 kilograms1,000
Divide.	3.2 kilograms
	The baby weighed 3.2 kilograms.

Kari’s newborn baby weighed 2,800 grams. How many kilograms did the baby weigh?

2.8 kilograms

Anderson received a package that was marked 4,500 grams. How many kilograms did this package weigh?
4.5 kilograms

As you become familiar with the metric system you may see a pattern. Since the system is based on multiples of ten, the calculations involve multiplying by multiples of ten. We have learned how to simplify these calculations by just moving the decimal.

To multiply by 10, 100, or 1,000, we move the decimal to the right one, two, or three places, respectively. To multiply by 0.1, 0.01, or 0.001, we move the decimal to the left one, two, or three places, respectively.

We can apply this pattern when we make measurement conversions in the metric system. In [link], we changed 3,200 grams to kilograms by multiplying by 11000 (or 0.001). This is the same as moving the decimal three places to the left.

$3{,}200 \cdot \frac{1}{1{,}000}$	$3{,}200.$
3.2	3.2

Convert ⓐ 350 L to kiloliters ⓑ 4.1 L to milliliters.

Solution

ⓐ We will convert liters to kiloliters. In [link], we see that $1 \text{ kiloliter} = 1{,}000 \text{ liters}$.

	350 L
Multiply by 1, writing 1 as a fraction relating liters to kiloliters.	$350 \text{ L} \cdot \frac{1 \text{ kL}}{1{,}000 \text{ L}}$
Simplify.	$350 \text{L} \cdot \frac{1 \text{ kL}}{1{,}000 \text{L}}$
	0.35 kL
Move the decimal 3 units to the left. (350.)	

ⓑ We will convert liters to milliliters. From [link] we see that $1 \text{ liter} = 1{,}000 \text{ milliliters}$.

[redacted]	
Multiply by 1, writing 1 as a fraction rel[redacted] milliliters.	
Simplify.	

[redacted]

Move the decimal 3 units to the right.

[redacted]

[redacted]

Convert: ⓐ 725 L to kiloliters ⓑ 6.3 L to milliliters

ⓐ 0.725 kiloliters ⓑ 6,300 milliliters

Convert: ⓐ 350 hL to liters ⓑ 4.1 L to centiliters

ⓐ 35,000 liters ⓑ 410 centiliters

Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires the same care we used in the US system. But it may be easier because of the relation of the units to the powers of 10. Make sure to add or subtract like units.

Ryland is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Ryland than his younger brother?

Solution

We can convert both measurements to either centimeters or meters. Since meters is the larger unit, we will subtract the lengths in meters. We convert 85 centimeters to meters by moving the decimal 2 places to the left.

Write the 85 centimeters as meters.		1.60m -0.85m_____ 0.75m

Ryland is 0.75 m taller than his brother.

Mariella is 1.58 meters tall. Her daughter is 75 centimeters tall. How much taller is Mariella than her daughter? Write the answer in centimeters.

83 centimeters

The fence around Hank's yard is 2 meters high. Hank is 96 centimeters tall. How much shorter than the fence is Hank? Write the answer in meters.

1.04 meters

Dena’s recipe for lentil soup calls for 150 milliliters of olive oil. Dena wants to triple the recipe. How many liters of olive oil will she need?

Solution

We will find the amount of olive oil in millileters then convert to liters.

	Triple 150 mL
Translate to algebra.	$3 \cdot 150 \text{mL}$
Multiply.	450 mL
Convert to liters.	$450 \cdot \frac{0.001 \text{L}}{1 \text{mL}}$
Simplify.	0.45 L
	Dena needs 0.45 liters of olive oil.

A recipe for Alfredo sauce calls for 250 milliliters of milk. Renata is making pasta with Alfredo sauce for a big party and needs to

multiply the recipe amounts by 8. How many liters of milk will she need?

2 liters

To make one pan of baklava, Dorothea needs 400 grams of filo pastry. If Dorothea plans to make 6 pans of baklava, how many kilograms of filo pastry will she need?

2.4 kilograms

This ruler shows inches and centimeters. This measuring cup shows ounces and milliliters. This scale shows pounds and kilograms.

Convert Between the U.S. and the Metric Systems of Measurement

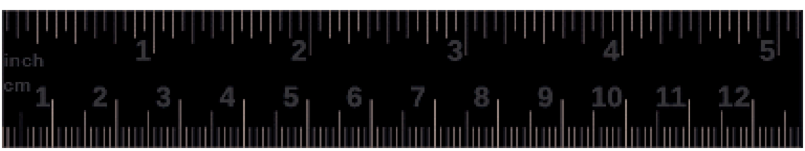
Many measurements in the United States are made in metric units. Our soda may come in 2-liter bottles, our calcium may come in 500-mg capsules,

and we may run a 5K race. To work easily in both systems, we need to be able to convert between the two systems.

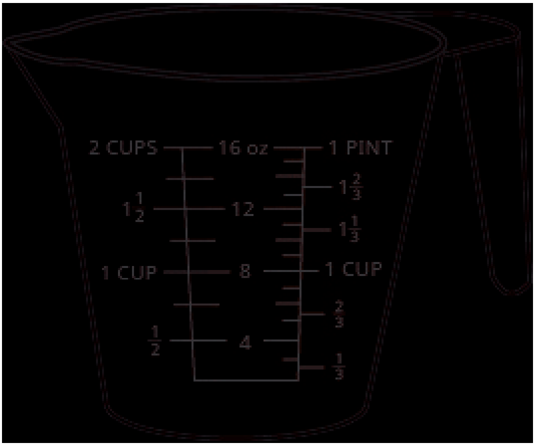
[link] shows some of the most common conversions.

Conversion Factors Between U.S. and Metric Systems		
Length	Mass	Capacity
1 in. = 2.54 cm1 ft. = 0.305 m1 yd. = 0.914 m1 mi. = 1.61 km1 m = 3.28 ft.	1 lb. = 0.45 kg1 oz. = 28 g1 kg = 2.2 lb.	1 qt. = 0.95 L1 fl. oz. = 30 mL1 L = 1.06 qt.

[link] shows how inches and centimeters are related on a ruler.



[link] shows the ounce and milliliter markings on a measuring cup.



[link] shows how pounds and kilograms marked on a bathroom scale.

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

Lee's water bottle holds 500 mL of water. How

many ounces are in the bottle? Round to the nearest tenth of an ounce.

Solution

Multiply by a unit conversion factor relating mL and ounces.	500 mL 500milliliters·1ounce30milliliters
Simplify.	50 ounce30
Divide.	16.7 ounces.
	The water bottle has 16.7 ounces.

How many quarts of soda are in a 2-L bottle?

2.12 quarts

How many liters are in 4 quarts of milk?

3.8 liters

Soleil was on a road trip and saw a sign that said the next rest stop was in 100 kilometers. How many miles until the next rest stop?

Solution

	100 kilometers
Multiply by a unit conversion factor relating km and mi.	100kilometers·1mile1.61kilomet
Simplify.	100 miles1.61
Divide.	62 miles.
	Soleil will travel 62 miles.

The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet.
19,335.6 feet

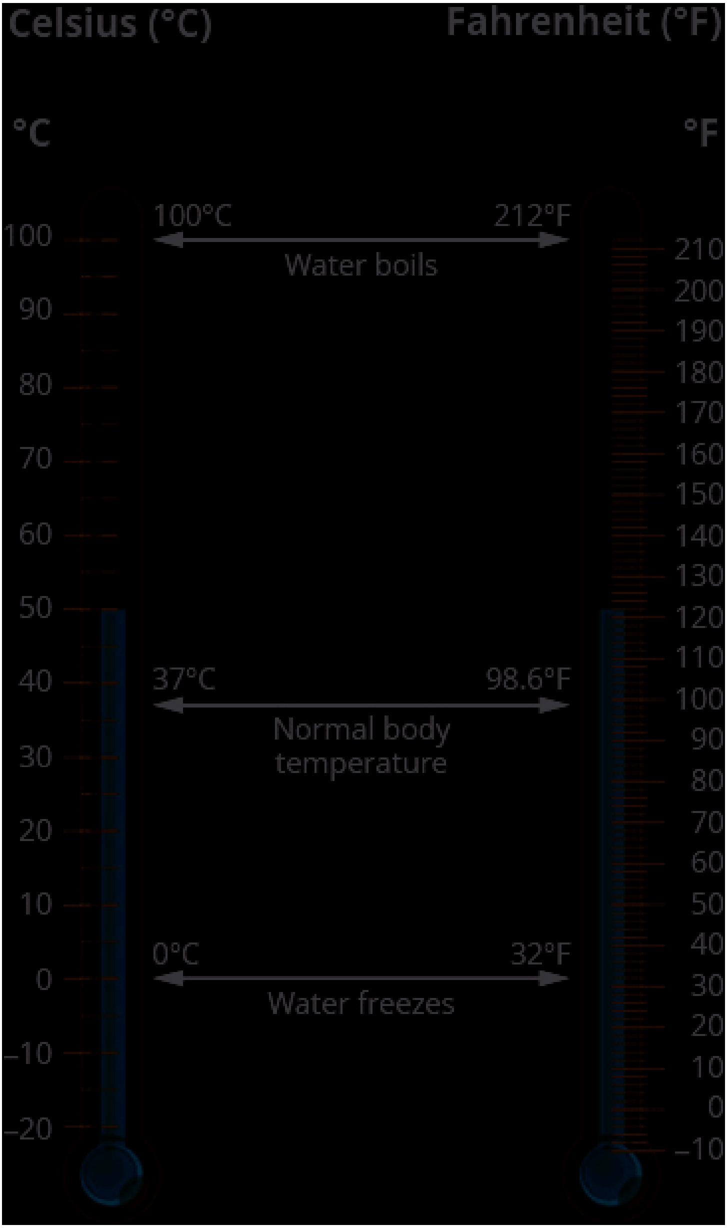
The flight distance from New York City to London is 5,586 kilometers. Convert the distance to miles.
3469.57 miles

The diagram shows normal body temperature, along with the freezing and boiling temperatures of water in degrees Fahrenheit and degrees Celsius.

Convert between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for 22°C, what does that mean?

The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F. The metric system uses degrees Celsius, written °C. [link] shows the relationship between the two systems.



Temperature Conversion

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula
$C=59(F-32)$.
To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula
$F=95C+32$.

Convert 50° Fahrenheit into degrees Celsius.

Solution

We will substitute 50°F into the formula to find C.

[redacted]

[redacted] or F.

Simplify in parentheses.

[redacted]

Multiply.

[redacted]

So we found that 50°F is equivalent to 10°C.

Convert the Fahrenheit temperature to degrees Celsius: 59° Fahrenheit.

15°C

Convert the Fahrenheit temperature to degrees Celsius: 41° Fahrenheit.

5°C

While visiting Paris, Woody saw the temperature was 20° Celsius. Convert the temperature into degrees Fahrenheit.

Solution

We will substitute 20°C into the formula to find F.

[redacted]

[redacted]

Multiply.

[redacted]

Add.

[redacted]

So we found that 20°C is equivalent to 68°F.

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Helsinki, Finland, was 15° Celsius.

59°F

Convert the Celsius temperature to degrees Fahrenheit: the temperature in Sydney, Australia, was 10° Celsius.

50°F

Key Concepts

- **Metric System of Measurement**
  - **Length**
    1 kilometer (km) = 1,000 m 1 hectometer (hm) = 100 m 1 dekameter (dam) = 10 m 1

meter (m) = 1 m 1 decimeter (dm) = 0.1 m 1 centimeter (cm) = 0.01 m 1 millimeter (mm) = 0.001 m 1 meter = 100 centimeters 1 meter = 1,000 millimeters

  - **Mass**
    1 kilogram (kg) = 1,000 g 1 hectogram (hg) = 100 g 1 dekagram (dag) = 10 g 1 gram (g) = 1 g 1 decigram (dg) = 0.1 g 1 centigram (cg) = 0.01 g 1 milligram (mg) = 0.001 g 1 gram = 100 centigrams 1 gram = 1,000 milligrams
  - **Capacity**
    1 kiloliter (kL) = 1,000 L 1 hectoliter (hL) = 100 L 1 dekaliter (daL) = 10 L 1 liter (L) = 1 L 1 deciliter (dL) = 0.1 L 1 centiliter (cL) = 0.01 L 1 milliliter (mL) = 0.001 L 1 liter = 100 centiliters 1 liter = 1,000 milliliters

- **Temperature Conversion**

  - To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula $C = 59(F - 32)$
  - To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula $F = 95C + 32$

Section Exercises

Practice Makes Perfect

Make Unit Conversions in the U.S. System

In the following exercises, convert the units.

A park bench is 6 feet long. Convert the length to inches.

---

72 inches

A floor tile is 2 feet wide. Convert the width to inches.

A ribbon is 18 inches long. Convert the length to feet.

1.5 feet

Carson is 45 inches tall. Convert his height to feet.

A football field is 160 feet wide. Convert the width to yards.

5313 yards

On a baseball diamond, the distance from home plate to first base is 30 yards. Convert the distance to feet.

Ulises lives 1.5 miles from school. Convert the distance to feet.

7,920 feet

Denver, Colorado, is 5,183 feet above sea level. Convert the height to miles.

A killer whale weighs 4.6 tons. Convert the weight to pounds.

9,200 pounds

Blue whales can weigh as much as 150 tons. Convert the weight to pounds.

An empty bus weighs 35,000 pounds. Convert the weight to tons.

1712 tons

At take-off, an airplane weighs 220,000 pounds. Convert the weight to tons.

Rocco waited 112 hours for his appointment. Convert the time to seconds.

5,400 s

Misty’s surgery lasted 214 hours. Convert the time to seconds.

How many teaspoons are in a pint?

---

96 teaspoons

How many tablespoons are in a gallon?

JJ’s cat, Posy, weighs 14 pounds. Convert her weight to ounces.

---

224 ounces

April’s dog, Beans, weighs 8 pounds. Convert his weight to ounces.

Crista will serve 20 cups of juice at her son’s party. Convert the volume to gallons.

---

114 gallons

Lance needs 50 cups of water for the runners in a race. Convert the volume to gallons.

Jon is 6 feet 4 inches tall. Convert his height to inches.

---

76 in.

Faye is 4 feet 10 inches tall. Convert her height to inches.

The voyage of the *Mayflower* took 2 months and 5 days. Convert the time to days.

---

65 days

Lynn’s cruise lasted 6 days and 18 hours. Convert the time to hours.

Baby Preston weighed 7 pounds 3 ounces at birth. Convert his weight to ounces.

---

115 ounces

Baby Audrey weighted 6 pounds 15 ounces at birth. Convert her weight to ounces.

Use Mixed Units of Measurement in the U.S. System

In the following exercises, solve.

Eli caught three fish. The weights of the fish were 2 pounds 4 ounces, 1 pound 11 ounces, and 4 pounds 14 ounces. What was the total weight of the three fish?

---

8 lbs. 13 oz.

Judy bought 1 pound 6 ounces of almonds, 2 pounds 3 ounces of walnuts, and 8 ounces of cashews. How many pounds of nuts did Judy

buy?

One day Anya kept track of the number of minutes she spent driving. She recorded 45, 10, 8, 65, 20, and 35. How many hours did Anya spend driving?

3.05 hours

Last year Eric went on 6 business trips. The number of days of each was 5, 2, 8, 12, 6, and 3. How many weeks did Eric spend on business trips last year?

Renee attached a 6 feet 6 inch extension cord to her computer’s 3 feet 8 inch power cord. What was the total length of the cords?

---

10 ft. 2 in.

Fawzi’s SUV is 6 feet 4 inches tall. If he puts a 2 feet 10 inch box on top of his SUV, what is the total height of the SUV and the box?

Leilani wants to make 8 placemats. For each placemat she needs 18 inches of fabric. How

many yards of fabric will she need for the 8 placemats?

---

4 yards

Mireille needs to cut 24 inches of ribbon for each of the 12 girls in her dance class. How many yards of ribbon will she need altogether?

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

Ghalib ran 5 kilometers. Convert the length to meters.

5,000 meters

Kitaka hiked 8 kilometers. Convert the length to meters.

Estrella is 1.55 meters tall. Convert her height to centimeters.

---

155 centimeters

The width of the wading pool is 2.45 meters. Convert the width to centimeters.

Mount Whitney is 3,072 meters tall. Convert the height to kilometers.

3.072 kilometers

The depth of the Mariana Trench is 10,911 meters. Convert the depth to kilometers.

June’s multivitamin contains 1,500 milligrams of calcium. Convert this to grams.

1.5 grams

A typical ruby-throated hummingbird weights 3 grams. Convert this to milligrams.

One stick of butter contains 91.6 grams of fat. Convert this to milligrams.

91,600 milligrams

One serving of gourmet ice cream has 25 grams of fat. Convert this to milligrams.

The maximum mass of an airmail letter is 2 kilograms. Convert this to grams.

2,000 grams

Dimitri's daughter weighed 3.8 kilograms at birth. Convert this to grams.

A bottle of wine contained 750 milliliters. Convert this to liters.

0.75 liters

A bottle of medicine contained 300 milliliters. Convert this to liters.

Use Mixed Units of Measurement in the Metric System

In the following exercises, solve.

Matthias is 1.8 meters tall. His son is 89 centimeters tall. How much taller is Matthias

than his son?

---

91 centimeters

Stavros is 1.6 meters tall. His sister is 95 centimeters tall. How much taller is Stavros than his sister?

A typical dove weighs 345 grams. A typical duck weighs 1.2 kilograms. What is the difference, in grams, of the weights of a duck and a dove?

---

855 grams

Concetta had a 2-kilogram bag of flour. She used 180 grams of flour to make biscotti. How many kilograms of flour are left in the bag?

Harry mailed 5 packages that weighed 420 grams each. What was the total weight of the packages in kilograms?

2.1 kilograms

One glass of orange juice provides 560 milligrams of potassium. Linda drinks one glass of orange juice every morning. How many grams of potassium does Linda get from her orange juice in 30 days?

Jonas drinks 200 milliliters of water 8 times a day. How many liters of water does Jonas drink in a day?

1.6 liters

One serving of whole grain sandwich bread provides 6 grams of protein. How many milligrams of protein are provided by 7 servings of whole grain sandwich bread?

Convert Between the U.S. and the Metric Systems of Measurement

In the following exercises, make the unit conversions. Round to the nearest tenth.

Bill is 75 inches tall. Convert his height to centimeters.

190.5 centimeters

Frankie is 42 inches tall. Convert his height to centimeters.

Marcus passed a football 24 yards. Convert the pass length to meters

---

21.9 meters

Connie bought 9 yards of fabric to make drapes. Convert the fabric length to meters.

Each American throws out an average of 1,650 pounds of garbage per year. Convert this weight to kilograms.

---

750 kilograms

An average American will throw away 90,000 pounds of trash over his or her lifetime. Convert this weight to kilograms.

A 5K run is 5 kilometers long. Convert this length to miles.

3.1 miles

Kathryn is 1.6 meters tall. Convert her height to feet.

Dawn’s suitcase weighed 20 kilograms. Convert the weight to pounds.

---

44 pounds

Jackson’s backpack weighed 15 kilograms. Convert the weight to pounds.

Ozzie put 14 gallons of gas in his truck. Convert the volume to liters.

53.2 liters

Bernard bought 8 gallons of paint. Convert the volume to liters.

Convert between Fahrenheit and Celsius Temperatures

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

86° Fahrenheit

30°C

77° Fahrenheit

104° Fahrenheit

40°C

14° Fahrenheit

72° Fahrenheit

22.2°C

4° Fahrenheit

0° Fahrenheit

−17.8°C

120° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

5° Celsius

$41^{\circ}F$

25° Celsius

-10° Celsius

$14^{\circ}F$

-15° Celsius

22° Celsius

$71.6^{\circ}F$

8° Celsius

43° Celsius

109.4°F

16° Celsius

Everyday Math

Nutrition Julian drinks one can of soda every day. Each can of soda contains 40 grams of sugar. How many kilograms of sugar does Julian get from soda in 1 year?

14.6 kilograms

Reflectors The reflectors in each lane-marking stripe on a highway are spaced 16 yards apart. How many reflectors are needed for a one mile long lane-marking stripe?

Writing Exercises

Some people think that 65°to75° Fahrenheit is the ideal temperature range.

ⓐ What is your ideal temperature range? Why do you think so?
ⓑ Convert your ideal temperatures from Fahrenheit to Celsius.

Answers may vary.

ⓐ Did you grow up using the U.S. or the metric system of measurement?
ⓑ Describe two examples in your life when you had to convert between the two systems of measurement.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
define U.S. units of measurement and convert from one unit to another.			
use U.S units of measurement.			
define metric units of measurement and convert from one unit to another.			
use metric units of measurement.			
convert between the U.S. and the metric system of measurement.			
convert between Fahrenheit and Celsius temperatures.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

Chapter Review Exercises

Introduction to Whole Numbers

Use Place Value with Whole Number

In the following exercises find the place value of each digit.

26,915

ⓐ 1 ⓑ 2 ⓒ 9 ⓓ 5 ⓔ 6

ⓐ tens ⓑ ten thousands ⓒ hundreds ⓓ ones ⓔ thousands

359,417

ⓐ 9 ⓑ 3 ⓒ 4 ⓓ 7 ⓔ 1

58,129,304

ⓐ 5 ⓑ 0 ⓒ 1 ⓓ 8 ⓔ 2

ⓐ ten millions ⓑ tens ⓒ hundred thousands ⓓ millions ⓔ ten thousands

9,430,286,157

ⓐ 6 ⓑ 4 ⓒ 9 ⓓ 0 ⓔ 5

In the following exercises, name each number.

6,104

---

six thousand, one hundred four

493,068

3,975,284

---

three million, nine hundred seventy-five thousand, two hundred eighty-four

85,620,435

In the following exercises, write each number as a whole number using digits.

three hundred fifteen

315

sixty-five thousand, nine hundred twelve

ninety million, four hundred twenty-five thousand, sixteen

90,425,016

one billion, forty-three million, nine hundred twenty-two thousand, three hundred eleven

In the following exercises, round to the indicated place value.

Round to the nearest ten.

ⓐ 407 ⓑ 8,564

ⓐ 410 ⓑ 8,560

Round to the nearest hundred.

ⓐ 25,846 ⓑ 25,864

In the following exercises, round each number to the nearest ⓐ hundred ⓑ thousand ⓒ ten thousand.

864,951

ⓐ 865,000 ⓑ 865,000 ⓒ 860,000

3,972,849

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, by 3, by 5, by 6, and by 10.

168

---

by2,3,6

264

375

by3,5

750

1430

---

by2,5,10

1080

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

420

$2\cdot2\cdot3\cdot5\cdot7$

115

225

3·3·5·5
2475
1560
2·2·2·3·5·13
56
72
2·2·2·3·3
168
252
2·2·3·3·7
391

In the following exercises, find the least common

multiple of the following numbers using the multiples method.

6,15

30

60, 75

In the following exercises, find the least common multiple of the following numbers using the prime factors method.

24, 30

120

70, 84

Use the Language of Algebra

Use Variables and Algebraic Symbols

In the following exercises, translate the following from algebra to English.

$25-7$

25 minus 7, the difference of twenty-five and seven

$5 \cdot 6$

$45 \div 5$

45 divided by 5, the quotient of forty-five and five

$x+8$

$42 \geq 27$

forty-two is greater than or equal to twenty-seven

$3n=24$

$3 \leq 20 \div 4$

3 is less than or equal to 20 divided by 4, three is less than or equal to the quotient of twenty and four

$a \neq 7 \cdot 4$

In the following exercises, determine if each is an expression or an equation.

$6 \cdot 3 + 5$

---

expression

$y - 8 = 32$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

35

243

108

In the following exercises, simplify

$6+10/2+2$

13

$9+12/3+4$

$20\div(4+6)\cdot 5$

10

$33\div(3+8)\cdot 2$

$(4+5)2$

41

$(4+5)2$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

$9x+7$ when $x=3$

34

$5x-4$ when $x=6$

$x4$ when $x=3$

81

$3x$ when $x=3$

$x2+5x-8$ when $x=6$

58

$2x+4y-5$ when
$x=7, y=8$

Simplify Expressions by Combining Like Terms

In the following exercises, identify the coefficient of each term.

12n

12

9x2

In the following exercises, identify the like terms.

3n,n2,12,12p2,3,3n2

12and3,n2and3n2

5,18r2,9s,9r,5r2,5s

In the following exercises, identify the terms in each expression.

11x2 + 3x + 6

11x2,3x,6

22y3 + y + 15

In the following exercises, simplify the following

expressions by combining like terms.

$17a + 9a$

$26a$

$18z + 9z$

$9x + 3x + 8$

$12x + 8$

$8a + 5a + 9$

$7p + 6 + 5p - 4$

$12p + 2$

$8x + 7 + 4x - 5$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the following

phrases into algebraic expressions.

the sum of 8 and 12

$8+12$

the sum of 9 and 1

the difference of x and 4

$x-4$

the difference of x and 3

the product of 6 and y

$6y$

the product of 9 and y

Adele bought a skirt and a blouse. The skirt cost \$15 more than the blouse. Let b represent the cost of the blouse. Write an expression for the cost of the skirt.

$b+15$

Marcella has 6 fewer boy cousins than girl cousins. Let g represent the number of girl cousins. Write an expression for the number of boy cousins.

Add and Subtract Integers

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

ⓐ $6\_\_2$
ⓑ $-7\_\_4$
ⓒ $-9\_\_-1$
ⓓ $9\_\_-3$

ⓐ $>$ ⓑ $<$ ⓒ $<$ ⓓ $>$

ⓐ $-5\_\_1$
ⓑ $-4\_\_-9$
ⓒ $6\_\_10$
ⓓ $3\_\_-8$

In the following exercises,, find the opposite of each number.

ⓐ -8 ⓑ 1

ⓐ 8 ⓑ -1

ⓐ -2 ⓑ 6

In the following exercises, simplify.

$-(-19)$

19

$-(-53)$

In the following exercises, simplify.

$-m$ when
ⓐ $m=3$
ⓑ $m=-3$

ⓐ -3 ⓑ 3

$-p$ when
ⓐ $p=6$
ⓑ $p=-6$

Simplify Expressions with Absolute Value

In the following exercises,, simplify.

ⓐ $|7|$ ⓑ $|-25|$ ⓒ $|0|$

ⓐ 7 ⓑ 25 ⓒ 0

ⓐ $|5|$ ⓑ $|0|$ ⓒ $|-19|$

In the following exercises, fill in $<$, $>$, or $=$ for each of the following pairs of numbers.

ⓐ -8__$|-8|$
ⓑ $-|-2|$__-2

ⓐ $<$ ⓑ $=$

ⓐ $|-3|$__$-|-3|$
ⓑ 4__$-|-4|$

In the following exercises, simplify.

$|8-4|$

4

$|9-6|$

$8(14-2|-2|)$

80

$6(13-4|-2|)$

In the following exercises, evaluate.

ⓐ $|x|$ when $x=-28$ ⓑ $|x|$ when $x=-15$

ⓐ 28 ⓑ 15

ⓐ $|y|$ when $y=-37$
ⓑ $|-z|$ when $z=-24$

Add Integers

In the following exercises, simplify each expression.

$-200+65$

-135

$-150+45$

$2+(-8)+6$

0

$4+(-9)+7$

$140+(-75)+67$

132

$-32+24+(-6)+10$

Subtract Integers

In the following exercises, simplify.

$9 - 3$

6

$-5 - (-1)$

ⓐ $15 - 6$ ⓑ $15 + (-6)$

ⓐ 9 ⓑ 9

ⓐ $12 - 9$ ⓑ $12 + (-9)$

ⓐ $8 - (-9)$ ⓑ $8 + 9$

ⓐ 17 ⓑ 17

ⓐ $4 - (-4)$ ⓑ $4 + 4$

In the following exercises, simplify each expression.

$10 - (-19)$

29

$11-(-18)$

$31-79$

-48

$39-81$

$-31-11$

-42

$-32-18$

$-15-(-28)+5$

18

$71+(-10)-8$

$-16-(-4+1)-7$

-20

$-15-(-6+4)-3$

Multiply Integers

In the following exercises, multiply.

$-5(7)$

-35

$-8(6)$

$-18(-2)$

36

$-10(-6)$

Divide Integers

In the following exercises, divide.

$-28 \div 7$

-4

$56 \div (-7)$

$-120 \div (-20)$

6

$-200 \div 25$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

$-8(-2) - 3(-9)$

43

$-7(-4) - 5(-3)$

$(-5)3$

-125

$(-4)3$

$-4 \cdot 2 \cdot 11$

-88

$-5 \cdot 3 \cdot 10$

$-10(-4) \div (-8)$

-5

$-8(-6) \div (-4)$

$31 - 4(3 - 9)$

55

$24-3(2-10)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

$x+8$ when ⓐ $x=-26$ⓑ $x=-95$

ⓐ -18 ⓑ -87

$y+9$ when ⓐ $y=-29$ ⓑ $y=-84$

When $b=-11$, evaluate:ⓐ $b+6$ ⓑ $-b+6$

ⓐ -5 ⓑ 17

When $c=-9$, evaluate:
ⓐ $c+(-4)>$ ⓑ $-c+(-4)$

$p2-5p+2$ when
$p=-1$

8

$q2-2q+9$ when $q=-2$

$6x-5y+15$ when $x=3$ and $y=-1$

38

$3p-2q+9$ when $p=8$ and $q=-2$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

the sum of -4 and -17, increased by 32

$(-4+(-17))+32;11$

ⓐ the difference of 15 and -7 ⓑ subtract 15 from -7

the quotient of -45 and -9

$-45-9;5$

the product of -12 and the difference of candd

Use Integers in Applications

In the following exercises, solve.

Temperature The high temperature one day in Miami Beach, Florida, was $76°$. That same day, the high temperature in Buffalo, New York was $-8°$. What was the difference between the temperature in Miami Beach and the temperature in Buffalo?

---

84 degrees

Checking Account Adrianne has a balance of $-\$22$ in her checking account. She deposits \$301 to the account. What is the new balance?

Visualize Fractions

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.

14

28,312,416 answers may vary

13

56

1012,1518,2024 answers may vary

27

Simplify Fractions

In the following exercises, simplify.

721

13

824

1520

34

1218

−168192

−78

−140224

11x11y

---

xy

15a15b

Multiply Fractions

In the following exercises, multiply.

25·13

215

12·38

$712(-821)$

-29

$512(-815)$

$-28p(-14)$

$7p$

$-51q(-13)$

$145(-15)$

-42

$-1(-38)$

Divide Fractions

In the following exercises, divide.

$12 \div 14$

2

$12 \div 18$

$-45 \div 47$

-75

$-34 \div 35$

$58 \div a10$

$254a$

$56 \div c15$

$7p12 \div 21p8$

29

$5q12 \div 15q8$

$25 \div (-10)$

-125

$-18 \div -(92)$

In the following exercises, simplify.

2389

34

45815

-9103

-310

258

$r5s3$

$3r5s$

$-x6-89$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

$4+118$

158

$9+37$

$307-12$

-6

$154-9$

$22-1419-13$

43

$15+918+12$

$5 \cdot 8 - 10$

-4

$3 \cdot 4 - 24$

$15 \cdot 5 - 522 \cdot 10$

52

$12 \cdot 9 - 323 \cdot 18$

$2 + 4(3) - 3 - 22$

-2

$7 + 3(5) - 2 - 32$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

the quotient of *c* and the sum of *d* and 9.

---

cd + 9

the quotient of the difference of *h* and *k*, and − 5.

Add and Subtract Fractions

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

49 + 19

59

29 + 59

y3 + 23

---

y + 23

7p + 9p

$-18+(-38)$

-12

$-18+(-58)$

In the following exercises, subtract.

$45-15$

35

$45-35$

$y17-917$

$y-917$

$x19-819$

$-8d-3d$

$-11d$

$-7c-7c$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

$13+15$

815

$14+15$

$15-(-110)$

310

$12-(-16)$

$23+34$

1712

$34+25$

$1112 - 38$

1324

$58 - 712$

$-916 - (-45)$

1980

$-720 - (-58)$

$1 + 56$

116

$1 - 59$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

$(15)22 + 32$

1275

$(13)25+22$

$23+1234-23$

14

$34+1256-23$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

$x+12$ when
ⓐ $x=-18$
ⓑ $x=-12$

ⓐ 38 ⓑ 0

$x+23$ when
ⓐ $x=-16$
ⓑ $x=-53$

$4p2q$ when $p=-12$ and $q=59$

59

$5m2n$ when $m=-25$ and $n=13$

$u+vw$ when
$u=-4, v=-8, w=2$

-6

$m+np$ when
$m=-6, n=-2, p=4$

Decimals

Name and Write Decimals

In the following exercises, write as a decimal.

Eight and three hundredths

8.03

Nine and seven hundredths

One thousandth

0.001

Nine thousandths

In the following exercises, name each decimal.

7.8

---

seven and eight tenths

5.01

0.005

---

five thousandths

0.381

Round Decimals

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

5.7932

ⓐ 5.79 ⓑ 5.8 ⓒ 6

3.6284

12.4768

ⓐ 12.48 ⓑ 12.5 ⓒ 12

25.8449

Add and Subtract Decimals

In the following exercises, add or subtract.

$18.37 + 9.36$

27.73

$256.37 - 85.49$

$15.35 - 20.88$

-5.53

$37.5 + 12.23$

$-4.2 + (-9.3)$

-13.5

$-8.6 + (-8.6)$

$100 - 64.2$

35.8

$100 - 65.83$

$2.51 + 40$

42.51

$9.38 + 60$

Multiply and Divide Decimals

In the following exercises, multiply.

$(0.3)(0.4)$

0.12

$(0.6)(0.7)$

$(8.52)(3.14)$

26.7528

$(5.32)(4.86)$

$(0.09)(24.78)$

2.2302

$(0.04)(36.89)$

In the following exercises, divide.

$0.15 \div 5$

0.03

$0.27 \div 3$

$\$8.49 \div 12$

$\$0.71$

$\$16.99 \div 9$

$12 \div 0.08$

150

$5 \div 0.04$

Convert Decimals, Fractions, and Percents

In the following exercises, write each decimal as a fraction.

0.08

225

0.17

0.425

1740

0.184

1.75

74

0.035

In the following exercises, convert each fraction to a decimal.

25

0.4	
45	
−38	
−0.375	
−58	
59	
0.5–	
29	
12 + 6.5	
7	
14 + 10.75	

In the following exercises, convert each percent to a

decimal.

5%

0.05

9%

40%

0.4

50%

115%

1.15

125%

In the following exercises, convert each decimal to a percent.

0.18

18%

0.15

0.009

0.9%

0.008

1.5

150%

2.2

The Real Numbers

Simplify Expressions with Square Roots

In the following exercises, simplify.

64

8

144

-25

-5

-81

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write as the ratio of two integers.

ⓐ 9 ⓑ 8.47

ⓐ 91 ⓑ 847100

ⓐ -15 ⓑ 3.591

In the following exercises, list the ⓐ rational numbers, ⓑ irrational numbers.

0.84,0.79132…,1.3–

ⓐ 0.84,1.3– ⓑ 0.79132…,

2.38–,0.572,4.93814…

In the following exercises, identify whether each number is rational or irrational.

ⓐ 121 ⓑ 48

ⓐ rational ⓑ irrational

ⓐ 56 ⓑ 16

In the following exercises, identify whether each number is a real number or not a real number.

ⓐ -9 ⓑ -169

ⓐ not a real number ⓑ real number

ⓐ -64 ⓑ -81

In the following exercises, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers for each set of numbers.

$-4, 0, 56, 16, 18, 5.2537...$

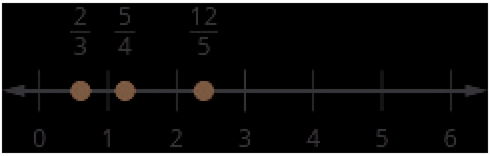
ⓐ 0,16 ⓑ $-4, 0, 16$ ⓒ $-4, 0, 56, 16$ ⓓ $18, 5.2537...$ ⓔ $-4, 0, 56, 16, 18, 5.2537...$

$-4, 0.36—, 133, 6.9152..., 48, 1012$

Locate Fractions on the Number Line

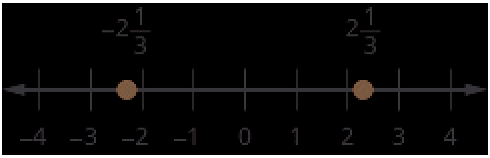
In the following exercises, locate the numbers on a number line.

$23, 54, 125$



$13, 74, 135$

$213, -213$



$135, -135$

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

-1__-18

$<$

-314__-4

-79__-49

$<$

-2__-198

Locate Decimals on the Number Line

In the following exercises, locate on the number line.

0.3

-0.2

-2.5

2.7

In the following exercises, order each of the following pairs of numbers, using $<$ or $>$.

$0.9\_\_0.6$

$>$

$0.7\_\_0.8$

$-0.6\_\_-0.59$

$<$

$-0.27\_\_-0.3$

Properties of Real Numbers

Use the Commutative and Associative Properties

In the following exercises, use the Associative Property to simplify.

$-12(4m)$

$-48m$

$30(56q)$

$(a+16)+31$

$a+47$

$(c+0.2)+0.7$

In the following exercises, simplify.

$6y+37+(-6y)$

37

$14+1115+(-14)$

$1411+359+(-1411)$

359

$-18\cdot15\cdot29$

$(712+45)+15$

1712

$(3.98d+0.75d)+1.25d$

$11x+8y+16x+15y$

$27x+23y$

$52m+(-20n)+(-18m)+(-5n)$

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number.

ⓐ 13
ⓑ 5.1
ⓒ -14
ⓓ -85

ⓐ -13 ⓑ -5.1 ⓒ 14 ⓓ 85

ⓐ -78
ⓑ -0.03
ⓒ 17

ⓓ 125

In the following exercises, find the multiplicative inverse of each number.

ⓐ 10 ⓑ -49 ⓒ 0.6

ⓐ 110 ⓑ -94 ⓒ 53

ⓐ -92 ⓑ -7 ⓒ 2.1

Use the Properties of Zero

In the following exercises, simplify.

$83 \cdot 0$

0

09

50

---

undefined

$0 \div 23$

In the following exercises, simplify.

$43 + 39 + (-43)$

39

$(n + 6.75) + 0.25$

$513 \cdot 57 \cdot 135$

57

$16 \cdot 17 \cdot 12$

$23 \cdot 28 \cdot 37$

8

$9(6x - 11) + 15$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the Distributive Property.

$7(x+9)$

$7x+63$

$9(u-4)$

$-3(6m-1)$

$-18m+3$

$-8(-7a-12)$

$13(15n-6)$

$5n-2$

$(y+10)\cdot p$

$(a-4)-(6a+9)$

$-5a-13$

$4(x+3)-8(x-7)$

Systems of Measurement

1.1 Define U.S. Units of Measurement and Convert from One Unit to Another

In the following exercises, convert the units. Round to the nearest tenth.

A floral arbor is 7 feet tall. Convert the height to inches.

---

84 inches

A picture frame is 42 inches wide. Convert the width to feet.

Kelly is 5 feet 4 inches tall. Convert her height to inches.

---

64 inches

A playground is 45 feet wide. Convert the width to yards.

The height of Mount Shasta is 14,179 feet. Convert the height to miles.

2.7 miles

Shamu weights 4.5 tons. Convert the weight to pounds.

The play lasted 134 hours. Convert the time to minutes.

---

105 minutes

How many tablespoons are in a quart?

Naomi’s baby weighed 5 pounds 14 ounces at birth. Convert the weight to ounces.

---

94 ounces

Trinh needs 30 cups of paint for her class art

project. Convert the volume to gallons.

Use Mixed Units of Measurement in the U.S. System.

In the following exercises, solve.

John caught 4 lobsters. The weights of the lobsters were 1 pound 9 ounces, 1 pound 12 ounces, 4 pounds 2 ounces, and 2 pounds 15 ounces. What was the total weight of the lobsters?

---

10 lbs. 6 oz.

Every day last week Pedro recorded the number of minutes he spent reading. The number of minutes were 50, 25, 83, 45, 32, 60, 135. How many hours did Pedro spend reading?

Fouad is 6 feet 2 inches tall. If he stands on a rung of a ladder 8 feet 10 inches high, how high off the ground is the top of Fouad's head?

---

15 feet

Dalila wants to make throw pillow covers. Each cover takes 30 inches of fabric. How many yards of fabric does she need for 4 covers?

Make Unit Conversions in the Metric System

In the following exercises, convert the units.

Donna is 1.7 meters tall. Convert her height to centimeters.

---

170 centimeters

Mount Everest is 8,850 meters tall. Convert the height to kilometers.

One cup of yogurt contains 488 milligrams of calcium. Convert this to grams.

0.488 grams

One cup of yogurt contains 13 grams of protein. Convert this to milligrams.

Sergio weighed 2.9 kilograms at birth. Convert this to grams.

2,900 grams

A bottle of water contained 650 milliliters. Convert this to liters.

Use Mixed Units of Measurement in the Metric System

In the following exerices, solve.

Minh is 2 meters tall. His daughter is 88 centimeters tall. How much taller is Minh than his daughter?

1.12 meter

Selma had a 1 liter bottle of water. If she drank 145 milliliters, how much water was left in the bottle?

One serving of cranberry juice contains 30 grams of sugar. How many kilograms of sugar are in 30 servings of cranberry juice?

0.9 kilograms

One ounce of tofu provided 2 grams of protein. How many milligrams of protein are provided by 5 ounces of tofu?

Convert between the U.S. and the Metric Systems of Measurement

In the following exercises, make the unit conversions. Round to the nearest tenth.

Majid is 69 inches tall. Convert his height to centimeters.

175.3 centimeters

A college basketball court is 84 feet long. Convert this length to meters.

Caroline walked 2.5 kilometers. Convert this length to miles.

1.6 miles

Lucas weighs 78 kilograms. Convert his weight to pounds.

Steve's car holds 55 liters of gas. Convert this to gallons.

14.6 gallons

A box of books weighs 25 pounds. Convert the weight to kilograms.

Convert between Fahrenheit and Celsius Temperatures

In the following exercises, convert the Fahrenheit temperatures to degrees Celsius. Round to the nearest tenth.

95° Fahrenheit

35° C

23° Fahrenheit

20° Fahrenheit

–6.7° C

64° Fahrenheit

In the following exercises, convert the Celsius temperatures to degrees Fahrenheit. Round to the nearest tenth.

30° Celsius

86° F

–5° Celsius

–12° Celsius

10.4° F

24° Celsius

Chapter Practice Test

Write as a whole number using digits: two hundred five thousand, six hundred seventeen.

205,617

Find the prime factorization of 504.

Find the Least Common Multiple of 18 and 24.

72

Combine like terms: $5n + 8 + 2n - 1$.

In the following exercises, evaluate.

$-|x|$ when $x = -2$

-2

$11 - a$ when $a = -3$

Translate to an algebraic expression and simplify: twenty less than negative 7.

$-7 - 20; -27$

Monique has a balance of $-\$18$ in her checking account. She deposits \$152 to the account. What is the new balance?

Round 677.1348 to the nearest hundredth.

677.13

Convert 45 to a decimal.

Convert 1.85 to a percent.

185%

Locate $23, -1.5, and 94$ on a number line.

In the following exercises, simplify each expression.

$4+10(3+9)-52$

99

$-85+42$

$-19-25$

-44

$(-2)4$

$-5(-9)\div 15$

3

$38\cdot 1112$

$45\div 920$

169

$12+3\cdot 515-6$

$m7+107$

$m+107$

$712-38$

$-5.8+(-4.7)$

-10.5

$100-64.25$

$(0.07)(31.95)$

2.2365

$9 \div 0.05$

$-14(57p)$

$-10p$

$(u+8)-9$

$6x+(-4y)+9x+8y$

$15x + 4y$

023

750

---

undefined

$-2(13q - 5)$

A movie lasted 123 hours. How many minutes did it last? *(1 hour = 60 minutes)*

---

100 minutes

Mike’s SUV is 5 feet 11 inches tall. He wants to put a rooftop cargo bag on the the SUV. The cargo bag is 1 foot 6 inches tall. What will the total height be of the SUV with the cargo bag on the roof? *(1 foot = 12 inches)*

Jennifer ran 2.8 miles. Convert this length to kilometers. *(1 mile = 1.61 kilometers)*

4.508 km

Use a General Strategy to Solve Linear Equations

By the end of this section, you will be able to:

- Solve linear equations using a general strategy
- Classify equations
- Solve equations with fraction or decimal coefficients

Before you get started, take this readiness quiz.

Simplify: $32(12x+20)$.
If you missed this problem, review [link].

$18x+30$

Simplify: $5-2(n+1)$.
If you missed this problem, review [link].

$3-2n$

Find the LCD of 56 and 14.
If you missed this problem, review [link].

12

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Solution of an Equation
A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.

Determine Whether a Number is a Solution to an Equation.

Substitute the number for the variable in the equation. Simplify the expressions on both sides of the equation. Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Determine whether the values are solutions to the equation: $5y+3=10y-4$.

ⓐ $y=\frac{3}{5}$ ⓑ $y=\frac{7}{5}$

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution

for the variable.

ⓐ

[redacted]

[redacted]

Multiply.

[redacted]

Simplify.

[redacted]

Since $y = 35$ does not result in a true equation, $y = 35$ is not a solution to the equation $5y + 3 = 10y - 4$.

ⓑ

[redacted]

[redacted]

Multiply.

[redacted]

Simplify.

[redacted]

Since $y = 75$ results in a true equation, $y = 75$ is a solution to the equation $5y + 3 = 10y - 4$.

Determine whether the values are solutions to the equation: $9y + 2 = 6y + 3$.

ⓐ $y = 43$ ⓑ $y = 13$

ⓐ no ⓑ yes

Determine whether the values are solutions to the equation: $4x-2=2x+1$.

ⓐ $x=32$ ⓑ $x=-12$

ⓐ yes ⓑ no

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation**.

Linear Equation

A **linear equation** is an equation in one variable that can be written, where *a* and *b* are real numbers and $a \neq 0$, as:

$$ax+b=0$$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

How to Solve a Linear Equation Using a General Strategy

Solve: $7(n-3)-8=-15.$

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do—all n's are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$
Step 4. Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5. Check the solution.	Let $n=2$ Subtract.	**Check:** $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15\checkmark$

Solve: $2(m-4)+3=-1$.
$m=2$

Solve: $5(a-3)+5=-10$.
$a=0$

These steps are summarized in the General Strategy for Solving Linear Equations below.

Solve linear equations using a general strategy.

Simplify each side of the equation as much as possible.
Use the Distributive Property to remove any parentheses.
Combine like terms. Collect all the variable terms on one side of the equation.
Use the Addition or Subtraction Property of Equality. Collect all the constant terms on the other

side of the equation.
Use the Addition or Subtraction Property of Equality. Make the coefficient of the variable term equal to 1.
Use the Multiplication or Division Property of Equality.
State the solution to the equation. Check the solution.
Substitute the solution into the original equation to make sure the result is a true statement.

Solve: $23(3m - 6) = 5 - m$.

[redacted]

Distribute.

[redacted]

Add m to both sides to get the va[redacted]

on the left.
Simplify.

[redacted]

Add 4 to both sides to get constants only on the right.

[redacted]

Simplify.

[redacted]

Divide both sides by three.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let m = 3.

[redacted]

[redacted]

Solve: $13(6u+3)=7-u$.

$u=2$

Solve: $23(9x-12)=8+2x$.

$x=4$

We can solve equations by getting all the variable terms to either side of the equal sign. By collecting the variable terms on the side where the coefficient of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

Solve: $4(x-1)-2=5(2x+3)+6$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Subtract 4x from each side to get the variables only on the right since $10 > 4$.

[redacted]

Simplify.

[redacted]

Subtract

21 [redacted]
each side to get the constants on left. Simplify.

[redacted]

Divide both sid [redacted]
6. Simplify.

[redacted]

Check:

[redacted]

Let $x = -92$.

[redacted]

[redacted]

Solve: $6(p-3)-7=5(4p+3)-12.$

$p=-2$

Solve: $8(q+1)-5=3(2q-4)-1.$

$q=-8$

Solve: $10[3-8(2s-5)]=15(40-5s).$

[redacted]

Simplify from the innermost parentheses first.

[redacted]

Combine like terms in the brackets.

[redacted]

Distribute.

[redacted]

Add 160s to both sides to get the variables to the right.

[redacted]

Simplify.

[redacted]

Subtract 600 from both sides to get the constants to the left.

[redacted]

Simplify.

[redacted]

Divide both sides by 85.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $s = -2$.

[redacted]

Solve: $6[4-2(7y-1)] = 8(13-8y)$.

$y = -175$

Solve: $12[1 - 5(4z - 1)] = 3(24 + 11z)$.

$z = 0$

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x + 8 = -13$ is true when we replace the variable, x, with the value -3, but not true when we replace x with any other value. An equation like this is called a **conditional equation**. All the equations we have solved so far are conditional equations.

Conditional Equation
An equation that is true for one or more values of the variable and false for all other values of the

variable is a **conditional equation**.

Now let's consider the equation $7y + 14 = 7(y + 2)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for *y*.

Solve:

	[redacted]
Distribute.	[redacted]
Subtract 7y to each side to get the y's to one side.	[redacted]
Simplify—the *y*'s are eliminated.	[redacted]
	But $14 = 14$ is true.

This means that the equation $7y + 14 = 7(y + 2)$ is true for any value of *y*. We say the solution to the

equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

> **Identity**
> An equation that is true for any value of the variable is called an **identity**.
> The solution of an identity is all real numbers.

What happens when we solve the equation $-8z = -8z + 9$?

Solve:

	[redacted]
Add 8z to both sides to leave the constant alone on [redacted]	
Simplify—the z's are eliminated.	[redacted]

But $0 \neq 9$.

Solving the equation $-8z = -8z + 9$ led to the false statement $0 = 9$. The equation $-8z = -8z + 9$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

Contradiction

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

The next few examples will ask us to classify an equation as conditional, an identity, or as a contradiction.

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Subtract 12n from each side to get the n’s to one side.

[redacted]

Simplify.

[redacted]

This is a true statement.

The equation is an identity.
The solution is all real numbers.

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4+9(3x-7)=-42x-13+23(3x-2)$.

identity; all real numbers

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$.

identity; all real numbers

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8+3(a-4)=0$.

$8+3(a-4)=0$

Distribute.

[redacted]

Combine like terms.

[redacted]

Add 4 to both sides.

[redacted]

Simplify.

[redacted]

Divide.

[redacted]

Simplify.

[redacted]

The equation is true when $a = 43$.

This is a conditional equation. The solution is $a = 43$.

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$.

conditional equation; $q = -911$

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$.

conditional equation; $k=20114$

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $5m+3(9+3m)=2(7m-11)$.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Subtract 14m from both sides.

[redacted]

Simplify.

[redacted]

But $27 \neq -22$.

The equation is a contradiction.
It has no solution.

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $12c+5(5+3c)=3(9c-4)$.

contradiction; no solution

Classify the equation as a conditional

equation, an identity, or a contradiction and then state the solution: $4(7d+18)=13(3d-2)-11d$.

contradiction; no solution

We summarize the methods for classifying equations in the table.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Contradiction	False for all values of the variable	No solution

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

How to Solve Equations with Fraction or Decimal Coefficients

Solve: $112x+56=34$.

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$ LCD = 12
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.	Multiply both sides of the equation by the LCD, 12.	$12\left(\frac{1}{12}x + \frac{5}{6}\right) = 12\left(\frac{3}{4}\right)$
	Use the Distributive Property.	$12 \cdot \frac{1}{12}x + 12 \cdot \frac{5}{6} = 12 \cdot \frac{3}{4}$
	Simplify—and notice, no more fractions!	$x + 10 = 9$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10.	$x + 10 - 10 = 9 - 10$
	Simplify.	$x = -1$
		Check: $\frac{1}{12}x + \frac{5}{6} = \frac{3}{4}$ $\frac{1}{12}(-1) + \frac{5}{6} \stackrel{?}{=} \frac{3}{4}$ $-\frac{1}{12} + \frac{5}{6} \stackrel{?}{=} \frac{3}{4}$ $-\frac{1}{12} + \frac{10}{12} \stackrel{?}{=} \frac{9}{12}$ $\frac{9}{12} = \frac{9}{12}$ ✓

Solve: $14x + 12 = 58$.

$x = 12$

Solve: $18x+12=14$.

$x=-2$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the General Strategy for Solving Linear Equations.

Solve Equations with Fraction or Decimal Coefficients.

Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation. Multiply both sides of the equation by that LCD. This clears the fractions and decimals. Solve using the General Strategy for Solving Linear Equations.

Solve: $5=12y+23y-34y$.

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the [redacted]

The LCD is 12.

Multiply both sides of the eq[redacted]

Distribute.

[redacted]

Simplify—notice, no more fra[redacted]

Combine like terms.

[redacted]

Divide by five.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $y = 12.$

[redacted]

Solve: $7 = 12x + 34x - 23x.$

$x = 12$

Solve: $-1 = 12u + 14u - 23u.$

$u = -12$

In the next example, we'll distribute before we clear the fractions.

Solve: $12(y-5)=14(y-1)$.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

Multiply by the LCD, four.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to the right.

[redacted]

Simplify.

[redacted]

An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.

[redacted]

Multiply by the LCD, 4.

[redacted]

Multiply four times the fractions.

[redacted]

Distribute.

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to the right.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $y = 9$.

[redacted]

Finish the check on your own.

Solve: $15(n+3)=14(n+2)$.

$n=2$

Solve: $12(m-3)=14(m-7)$.

$m=-1$

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

Solve: $4q+32+6=3q+54$

[redacted]

Multiply both sides by the LC[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

Collect the variables to the left.

[redacted]

Simplify.

[redacted]

Collect the constants to the right.

[redacted]

Simplify.

[redacted]

Divide both

sid[redacted]

Simplify.

[redacted]

Check:

[redacted]

Let $q = -5$.

[redacted]

Finish the check on your own.

Solve: $3r + 56 + 1 = 4r + 33$.

$r = 1$

Solve: $2s + 32 + 1 = 3s + 24$.

$s = -8$

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7 = 710$ and $0.29 = 29100$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

Solve: $0.25x + 0.05(x + 3) = 2.85$.

Look at the decimals and think of the equivalent fractions:
$0.25 = 25100, 0.05 = 5100, 2.85 = 285100.$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

[redacted]

Distribute first.

[redacted]

Combine like terms.

[redacted]

To clear decimals, multiply by 100.

[redacted]

Distribute.

[redacted]

Subtract 15 from both sides.

[redacted]

Simplify.

[redacted]

Divide by 30.

[redacted]

Simplify.

[redacted]

Check it yourself by substituting x = 9 into the original equation.

Solve: $0.25n + 0.05(n + 5) = 2.95$.
$n = 9$

Solve: $0.10d + 0.05(d - 5) = 2.15$.
$d = 16$

Key Concepts

- **How to determine whether a number is a solution to an equation**

  Substitute the number in for the variable in the equation. Simplify the expressions on both sides of the equation. Determine whether the resulting equation is true.
  If it is true, the number is a solution.
  If it is not true, the number is not a solution.

- **How to Solve Linear Equations Using a General Strategy**

  Simplify each side of the equation as much as possible.
  Use the Distributive Property to remove any parentheses.
  Combine like terms. Collect all the variable terms on one side of the equation.
  Use the Addition or Subtraction Property of Equality. Collect all the constant terms on the other side of the equation.
  Use the Addition or Subtraction Property of Equality. Make the coefficient of the variable term equal to 1.
  Use the Multiplication or Division Property of Equality.
  State the solution to the equation. Check the solution.
  Substitute the solution into the original equation to make sure the result is a true statement.

- **How to Solve Equations with Fraction or Decimal Coefficients**

  Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation. Multiply both sides of the equation by that LCD. This clears the fractions and decimals. Solve using the General Strategy

for Solving Linear Equations.

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

$6y + 10 = 12y$

ⓐ $y = 53$ ⓑ $y = -12$

ⓐ yes ⓑ no

$4x + 9 = 8x$

ⓐ $x = -78$ ⓑ $x = 94$

$8u - 1 = 6u$

ⓐ $u = -12$ ⓑ $u = 12$

ⓐ no ⓑ yes

$9v - 2 = 3v$

ⓐ $v = -13$ ⓑ $v = 13$

In the following exercises, solve each linear equation.

$15(y - 9) = -60$

$y = 5$

$-16(3n + 4) = 32$

$-(w - 12) = 30$

$w = -18$

$-(t - 19) = 28$

$51 + 5(4 - q) = 56$

$q = 3$

$-6 + 6(5 - k) = 15$

$3(10-2x)+54=0$

$x=14$

$-2(11-7x)+54=4$

$23(9c-3)=22$

$c=4$

$35(10x-5)=27$

$15(15c+10)=c+7$

$c=52$

$14(20d+12)=d+7$

$3(4n-1)-2=8n+3$

$n=2$

$9(2m-3)-8=4m+7$

$12+2(5-3y)=-9(y-1)-2$

$y=-5$

$-15+4(2-5y)=-7(y-4)+4$

$5+6(3s-5)=-3+2(8s-1)$

$s=10$

$-12+8(x-5)=-4+3(5x-2)$

$4(p-4)-(p+7)=5(p-3)$

$p=-4$

$3(a-2)-(a+6)=4(a-1)$

$4[5-8(4c-3)]=12(1-13c)-8$

$c=-4$

$5[9-2(6d-1)]=11(4-10d)-139$

$3[-9+8(4h-3)]=2(5-12h)-19$

$h=34$

$3[-14+2(15k-6)]=8(3-5k)-24$

$5[2(m+4)+8(m-7)]=2[3(5+m)-(21-3m)]$

$m=6$

$10[5(n+1)+4(n-1)]=11[7(5+n)-(25-3n)]$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

$23z+19=3(5z-9)+8z+46$

identity; all real numbers

$15y+32=2(10y-7)-5y+46$

$18(5j-1)+29=47$

conditional equation; $j=25$

$24(3d-4)+100=52$

$22(3m-4)=8(2m+9)$

conditional equation; $m=165$

$30(2n-1)=5(10n+8)$

$7v+42=11(3v+8)-2(13v-1)$

contradiction; no solution

$18u-51=9(4u+5)-6(3u-10)$

$45(3y-2)=9(15y-6)$

---

contradiction; no solution

$60(2x-1)=15(8x+5)$

$9(14d+9)+4d=13(10d+6)+3$

---

identity; all real numbers

$11(8c+5)-8c=2(40c+25)+5$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

$14x-12=-34$

$x=-1$

$34x-12=14$

$56y - 23 = -32$

$y = -1$

$56y - 13 = -76$

$12a + 38 = 34$

$a = 34$

$58b + 12 = -34$

$2 = 13x - 12x + 23x$

$x = 4$

$2 = 35x - 13x + 25x$

$13w + 54 = w - 14$

$w = 94$

$12a - 14 = 16a + 112$

$13b + 15 = 25b - 35$

$b = 12$

$13x + 25 = 15x - 25$

$14(p - 7) = 13(p + 5)$

$p = -41$

$15(q + 3) = 12(q - 3)$

$12(x + 4) = 34$

$x = -52$

$13(x + 5) = 56$

$4n + 84 = n3$

$n = -3$

$3p + 63 = p2$

$3x + 42 + 1 = 5x + 108$

$x = -2$

$10y - 23 + 3 = 10y + 19$

$7u - 14 - 1 = 4u + 85$

$u = 3$

$3v - 62 + 5 = 11v - 45$

In the following exercises, solve each equation with decimal coefficients.

$0.4x + 0.6 = 0.5x - 1.2$

$x = 18$

$0.7x + 0.4 = 0.6x + 2.4$

$0.9x - 1.25 = 0.75x + 1.75$

$x = 20$

$1.2x - 0.91 = 0.8x + 2.29$

$0.05n + 0.10(n + 8) = 2.15$

$n = 9$

$0.05n + 0.10(n + 7) = 3.55$

$0.10d + 0.25(d + 5) = 4.05$

$d = 8$

$0.10d + 0.25(d + 7) = 5.25$

Everyday Math

Fencing Micah has 74 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L+2(L-2.5)=74$.

$L=19.75$ feet

Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was eight less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-8)=22.82$ for s, to find the number of 49-cent stamps Paula bought.

Writing Exercises

Using your own words, list the steps in the general strategy for solving linear equations.

Answers will vary.

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

What is the first step you take when solving the equation $3-7(y-4)=38$? Why is this your first step?

Answers will vary.

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Answers will vary.

For the equation $0.35x+2.1=3.85$, how do you clear the decimal?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
solve linear equations using a general strategy.			
classify equations.			
solve equations with fraction or decimal coefficients.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Whom can you ask for help?Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

conditional equation
: An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
: An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity
: An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation
: A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation
: A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

Use a Problem Solving Strategy
By the end of this section, you will be able to:

- Use a problem solving strategy for word problems
- Solve number word problems
- Solve percent applications
- Solve simple interest applications

Before you get started, take this readiness quiz.

1. Translate “six less than twice *x*” into an algebraic expression.
   If you missed this problem, review [link].
2. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
3. Convert 0.6 to a percent.
   If you missed this problem, review [link].

Have you ever had any negative experiences in the past with word problems? When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. Realize that your negative experiences with word problems are in your past. To move forward you need to calm your fears and change your negative feelings.

Start with a fresh slate and begin to think positive thoughts. Repeating some of the following statements may be helpful to turn your thoughts positive. Thinking positive thoughts is a first step towards success.

I think I can! I think I can!

While word problems were hard in the past, I think I can try them now.

I am better prepared now—I think I will begin to understand word problems.

I am able to solve equations because I practiced many problems and I got help when I needed it—I can try that
with word problems.

It may take time, but I can begin to solve word problems.

You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. What was the snowfall last season at the ski resort?

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	What was the snowfall last season?
Step 3. Name what we are looking for and choose a variable to represent it.	Let $s =$ the snowfall last season.
Step 4. Translate. Restate the problem in on	

The normal snowfall was twelve more than twice the amount last year

the important information.

Translate into an equation.

[redacted]

Step 5. Solve the equation.

[redacted]

Subtract 12 from each side.

[redacted]

Simplify.

[redacted]

Divide each side by two.

[redacted]

Simplify.

[redacted]

Step 6. Check: First, is our answer reasonable? Yes, having 25 inches of snow seems OK.
The problem says the normal snowfall is twelve inches more than twice the number of last season.
Twice 25 is 50 and 12 more than that is 62.

Step 7. Answer the question.	The snowfall last season was 25 inches.

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. How many notebooks did he buy?

He bought two notebooks.

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

He did seven crosswords puzzles.

We summarize an effective strategy for problem solving.

Use a Problem Solving Strategy for word problems.

Read the problem. Make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation. **Solve** the equation using proper algebra techniques. **Check** the answer in the problem to make sure it makes sense. **Answer** the question with a complete sentence.

Solve Number Word Problems

We will now apply the problem solving strategy to "number word problems." Number word problems give some clues about one or more numbers and we use these clues to write an equation. Number word problems provide good practice for using the

Problem Solving Strategy.

The sum of seven times a number and eight is thirty-six. Find the number.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number
Step 3. Name what you are looking for and choose a variable to represent it.	Let $n =$ the number.
Step 4. Translate: Restate the problem as on[redacted] Translate into an equation.	
[redacted]	
Step 5. Solve the equation. Su[redacted]	

each side and simplify.
Divide each side by seven and simplify.

[redacted]

[redacted]

Step 6. Check.
Is the sum of seven times four plus eight equal to 36?
$7 \cdot 4 + 8 = ?3628 + 8 = ?$
$3636 = 36 ✓$

Step 7. Answer the question.

The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

The sum of four times a number and two is fourteen. Find the number.

3

The sum of three times a number and seven is twenty-five. Find the number.
6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two numbers
Step 3. Name what you are looking for by choosing a variable to represent the first number. "One number is nine less than the other."	Let $n = 1\text{st number}$. $n - 9 = 2\text{nd number}$
Step 4. Translate. Write as one sentence. Translate into an equation.	The sum of two numbers is negative fifteen.

$$\underbrace{1^{st}\text{ number} + 2^{nd}\text{ number}}_{n + n - 9} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\text{negative fifteen}}_{-15}$$

Step 5. Solve the equation. Combine like terms. Add nine to each side and simplify. Simplify.	

$$n + n - 9 = -15$$

$$2n - 9 = -15$$

$$2n = -6$$

[redacted]

[redacted]

[redacted]

[redacted]

Step 6. Check.
Is -12 nine less than -3?
$-3-9 \stackrel{?}{=} -12$ $-12 = -12$✓
Is their sum -15?
$-3+(-12) \stackrel{?}{=} -15$ $-15 = -15$✓

Step 7. Answer the question. The numbers are -3 and -12.

The sum of two numbers is negative twenty-three. One number is seven less than the other.

Find the numbers.

$-15, -8$

The sum of two numbers is negative eighteen. One number is forty more than the other. Find the numbers.

$-29, 11$

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of **consecutive integers** are:

$$1,2,3,4 \quad -10,-9,-8,-7 \quad 150,151,152,153$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as *n*, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{array}{ll} n & \text{1st integer} \\ n+1 & \text{2nd consecutive integer} \\ n+2 & \text{3rd consecutive integer etc.} \end{array}$$

We will use this notation to represent consecutive integers in the next example.

Find three consecutive integers whose sum is -54.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive integers
Step 3. Name each of the three numbers	Let $n = 1$st integer. $n+1 = 2$nd consecutive integer $n+2 = 3$rd consecutive integer
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three integers is -54.
[redacted]	
Step 5. Solve the	

eq[redacted]
Combine like terms.
Subtract three from each side.
Di[redacted]
three.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

Step 6. Check.
$-19+(-18)+(-17)=$
$-54\ -54=-54$✓

Step 7. Answer the question.	The three consecutive integers are $-17, -18$, and -19.

Find three consecutive integers whose sum is -96.

$-33, -32, -31$

Find three consecutive integers whose sum is -36.

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of consecutive even integers are:

$$24,\ 26,\ 28$$
$$-12, -10, -8$$

Notice each integer is two more than the number preceding it. If we call the first one *n*, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

$$n \quad \text{1st even integer} \quad n+2 \quad \text{2nd consecutive even integer} \quad n+4 \quad \text{3rd consecutive even integer} \quad \text{etc.}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

$$63,\ 65,\ 67$$
$$n, n+2, n+4$$

$$n \quad \text{1st odd integer} \quad n+2 \quad \text{2nd consecutive odd integer} \quad n+4 \quad \text{3rd consecutive odd integer} \quad \text{etc.}$$

Does it seem strange to have to add two (an even number) to get the next odd number? Do we get an

odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same—to get to the next odd or the next even integer, add two.

Find three consecutive even integers whose sum is 120.
Step 1. Readthe problem.Step 2. Identifywhat you are looking for.three consecutive even integersStep 3. Name.Let$n = 1$steven integer.$n + 2 = 2$ndconsecutive even integer$n + 4 = 3$rdconsecutive even integerStep 4. Translate.Restate as one sentence.The sum of the three even integers is120.Translate into an equation.$n + n + 2 + n + 4 = 120$Step 5. Solvethe equation.$n + n + 2 + n + 4 = 120$Combine like terms.$3n + 6 = 120$Subtract 6 from each side.$3n = 114$Divide each side by 3.$n = 38$1stinteger$n + 2$2ndinteger$38 + 240$$n + 4$3rdinteger$38 + 442$Step 6. Check.$38 + 40 + 42 \overset{?}{=} 120$$120 = 120$✓Step 7. Answerthe question.The three consecutive

integers are 38, 40, and 42.

Find three consecutive even integers whose sum is 102.

32, 34, 36

Find three consecutive even integers whose sum is -24.

$-10, -8, -6$

When a number problem is in a real life context, we still use the same strategies that we used for the previous examples.

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Step 1. Readthe problem. Step 2. Identifywhat you are looking for. How much does the husband earn? Step 3. Name. Choose a variable to representLeth $=$ the amount the husband earns.the amount the husband earns.The wife earns $16,000 less than twice that.Step 4. Translate.Restate the problem in one sentencewith all the important information.Translate into an equation.2h $-16{,}000 =$ the amount the wife earnsTogether the husband and wife earn $110,000. $h + 2h - 16{,}000 = 110{,}000$ Step 5. Solvethe equation. Combine like terms. Add 16,000 to both sides and simplify. Divide each side by three. $h + 2h - 16{,}000 = 110{,}000$ $3h - 16{,}000 = 110{,}000$ $3h = 126{,}000$ $h = 42{,}000$ $42,000 amount husband earns $2h - 16{,}000$ amount wife earns $2(42{,}000) - 16{,}000$ $84{,}000 - 16{,}000$ $68{,}000$ Step 6. Check:If the wife earns $68,000 and the husbandearns $42,000, is that $110,000? Yes! Step 7. Answerthe question.The husband earns $42,000 a year.

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,400. This was $1,600 less than six times the cost in 1975. What was the average cost of a car in 1975?

The average cost was $5,000.

US Census data shows that the median price of new home in the U.S. in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

The median price was $19,300.

Solve Percent Applications

There are several methods to solve percent

equations. In algebra, it is easiest if we just translate English sentences into algebraic equations and then solve the equations. Be sure to change the given percent to a decimal before you use it in the equation.

Translate and solve:

ⓐ What number is 45% of 84? ⓑ 8.5% of what amount is \$4.76? ⓒ 168 is what percent of 112?

ⓐ

[redacted]

Translate into alg[redacted] n = the number.

Multiply.

[redacted]

37.8 is 45% of 84.

ⓑ

[redacted]

Translate.
Let n = the amount.

[redacted]

Multiply.

[redacted]

Divide both sides by 0.085 and simplify.

[redacted]

8.5% of $56 is $4.76

©

We are asked to fin [redacted] percent, so we must have our result in percent form.

Translate into alg [redacted] $p =$ the percent.

Multiply.

[redacted]

Divide both sides by 11 [redacted] simplify.

Convert to percent.

[redacted]

168 is 150% of 112.

Translate and solve: ⓐ What number is 45% of 80? ⓑ 7.5% of what amount is $1.95? ⓒ 110 is what percent of 88?

ⓐ 36 ⓑ $26 ⓒ 125%

Translate and solve: ⓐ What number is 55% of 60? ⓑ 8.5% of what amount is $3.06? ⓐ 126 is what percent of 72?

ⓐ 33 ⓑ $36 ⓐ 175%

Now that we have a problem solving strategy to

refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we will solve involve everyday situations, you can rely on your own experience.

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

What are you asked to find?	What total amount of protein is recommended?
Choose a variable to represent it.	Let $a =$ total amount of protein.
Write a sentence that gives the inf	12 g is 24% of the total amount
Translate into an	

eq[redacted]
Solve.

[redacted]

Check: Does this make sense?
Yes, 24% is about 14 of the total and
12 is about 14 of 50.

Write a complete sentence to answer the question.	The amount of protein that is recommended is 50 g.

One serving of wheat square cereal has 7 grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

25 grams

One serving of rice cereal has 190 mg of

sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

2,375 mg

Remember to put the answer in the form requested. In the next example we are looking for the percent.

Veronica is planning to make muffins from a mix. The package says each muffin will be 240 calories and 60 calories will be from fat. What percent of the total calories is from fat?

What are you asked to find?	What percent of the total calories is fat?
Choose a variable to represent it.	Let p = percent of fat.
Write a sentence that	

giv… information to find it.	What percent of 240 is 60?
Translate the sentence into an equation.	$p \cdot 240 = 60$
Multiply.	$240p = 60$
Divide both sides by 240.	$p = 0.25$
Put in percent form.	$p = 25\%$
Check: does this make sense? Yes, 25% is one-fourth; 60 is one-fourth of 240. So, 25% makes sense.	
Write a complete sentence to answer the question.	Of the total calories in each muffin, 25% is fat.

Mitzi received some gourmet brownies as a gift. The wrapper said each 28% brownie was 480 calories, and had 240 calories of fat. What

percent of the total calories in each brownie comes from fat? Round the answer to the nearest whole percent.

50%

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat? Round the answer to the nearest whole percent.

40%

It is often important in many fields—business, sciences, pop culture—to talk about how much an amount has increased or decreased over a certain period of time. This increase or decrease is generally expressed as a percent and called the **percent change**.

To find the percent change, first we find the amount

of change, by finding the difference of the new amount and the original amount. Then we find what percent the amount of change is of the original amount.

Find percent change.

Find the amount of change.
$\text{change} = \text{new amount} - \text{original amount}$ Find what percent the amount of change is of the original amount.
change is what percent of the original amount?

Recently, the California governor proposed raising community college fees from $36 a unit to $46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)

Find the amount of change.	$46 - 36 = 10$

Find the percent.	Change is what percent of the original amount?
Let p = the percent.	
[redacted]	
Translate to an equation.	
[redacted]	
Simplify.	
[redacted]	
Divide both sides by 36.	
[redacted]	
Change to percent form; round to the nearest tenth.	
[redacted]	
Write a complete sentence to answer the question.	The new fees are approximately a 27.8% increase over the old fees.

Remember to round the division to the nearest thousandth in order to round the percent to the nearest tenth.

Find the percent change. (Round to the nearest tenth of a percent.) In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.
8.8%

Find the percent change. (Round to the nearest tenth of a percent.) In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was 2.25.
50%

Applications of discount and mark-up are very common in retail settings.

When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

Discount
amount of discount = discount rate·original pricesale price = original amount–discount price
The sale price should always be less than the original price.

Mark-up
amount of mark-up = mark-up rate·original pricelist price = original cost + mark-up
The list price should always be more than the original cost.

Liam's art gallery bought a painting at an original cost of \$750. Liam marked the price up 40%. Find ⓐ the amount of mark-up and

ⓑ the list price of the painting.

ⓐ

Identify what you are asked to find, and choose a variable to represent it.	What is the amount of mark-up? Let m = the amount of mark up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the $750 original cost
Translate into an equation.	$m = 0.40 \times 750$
Solve the equation.	$m = 300$
Write a complete sentence.	The mark-up on the painting was $300.

ⓑ

Identify what you are asked to find, and choose a variable to represent it.	What is the list price? Let p = the list price.
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	[redacted]
Solve the equation.	[redacted]
Check.	Is the list price more than the original cost? Is $1,050 more than $750? Yes.
Write a complete sentence.	The list price of the painting was $1,050.

Find ⓐ the amount of mark-up and ⓑ the list price: Jim's music store bought a guitar at original cost $1,200. Jim marked the price up 50%.

ⓐ $600 ⓑ $1,800

Find ⓐ the amount of mark-up and ⓑ the list price: The Auto Resale Store bought Pablo's Toyota for $8,500. They marked the price up 35%.

ⓐ $2,975 ⓑ $11,475

Solve Simple Interest Applications

Interest is a part of our daily lives. From the interest earned on our savings to the interest we pay on a car loan or credit card debt, we all have some experience with interest in our lives.

The amount of money you initially deposit into a bank is called the **principal**, P, and the bank pays you **interest**, I. When you take out a loan, you pay interest on the amount you borrow, also called the principal.

In either case, the interest is computed as a certain

percent of the principal, called the **rate of interest**, r. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable t, (for time) represents the number of years the money is saved or borrowed.

Interest is calculated as simple interest or compound interest. Here we will use simple interest.

Simple Interest

If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is

$I = \text{interest} \quad I = Prt \quad \text{where} \quad P = \text{principal} \quad r = \text{rate} \quad t = \text{time}$

Interest earned or paid according to this formula is called **simple interest**.

The formula we use to calculate interest is $I = Prt$. To use the formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Areli invested a principal of $950 in her bank account that earned simple interest at an interest rate of 3%. How much interest did she earn in five years?

$I=?P=\$950r=3\%t=5\text{years}$

Identify what you are asked to find, and choose aWhat is the simple interest?variable to represent it.Let$I=$interest.Write the formula.$I=Prt$Substitute in the given information.$I=(950)(0.03)(5)$Simplify.$I=142.5$Check.Is $142.50 a reasonable amount of interest on $950? Yes.Write a complete sentence.The interest is $142.50.

Nathaly deposited $12,500 in her bank account where it will earn 4% simple interest. How much interest will Nathaly earn in five years?

He will earn $2,500.

Susana invested a principal of $36,000 in her bank account that earned simple interest at an interest rate of 6.5%. How much interest did she earn in three years?

She earned $7,020.

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we do not know the rate.

Hang borrowed $7,500 from her parents to pay her tuition. In five years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of simple interest?

$I = \$1500 P = \$7500 r = ? t = 5 years$

Identify what you are asked to find, and chooseWhat is the rate of simple interest?a variable to represent it.Write the

formula.Substitute in the given information.Multiply.Divide.Change to percent form.Let $r =$ rate of interest.$I = Prt$ $1,500 = (7,500)r(5)$ $1,500 = 37,500r$ $0.04 =$ $(7,500)(0.04)(5)$ $1,500 = 1,500$ ✓ Write a complete sentence.The rate of interest was 4%.

Jim lent his sister \$5,000 to help her buy a house. In three years, she paid him the \$5,000, plus \$900 interest. What was the rate of simple interest?

The rate of simple interest was 6%.

Loren lent his brother \$3,000 to help him buy a car. In four years, his brother paid him back the \$3,000 plus \$660 in interest. What was the rate of simple interest?

The rate of simple interest was 5.5%.

In the next example, we are asked to find the principal—the amount borrowed.

Sean's new car loan statement said he would pay \$4,866,25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?

$I=4{,}866.25$ $P=?$ $r=8.5\%$ $t=5\text{years}$

Identify what you are asked to find,What is the amount borrowed (the principal)?and choose a variable to represent it.Write the formula.Substitute in the given information.Multiply.Divide.Let$P=\text{principal}$ borrowed.$I=Prt$ $4{,}866.25=P(0.085)(5)$ $4{,}866.25=0.425P$ $11{,}450=P$Check.$I=Prt$ $4{,}866.25\overset{?}{=}(11{,}450)(0.085)(5)$ $4{,}866.25=4{,}866.25$✓Write a complete sentence.The principal was \$11,450.

Eduardo noticed that his new car loan papers stated that with a 7.5% simple interest rate, he

would pay $6,596.25 in interest over five years. How much did he borrow to pay for his car?

He paid $17,590.

In five years, Gloria's bank account earned $2,400 interest at 5% simple interest. How much had she deposited in the account?

She deposited $9,600.

Access this online resource for additional instruction and practice with using a problem solving strategy.

- Begining Arithmetic Problems

Key Concepts

- **How To Use a Problem Solving Strategy for Word Problems**

  Read the problem. Make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation. **Solve** the equation using proper algebra techniques.

  Check the answer in the problem to make sure it makes sense. **Answer** the question with a complete sentence.

- **How To Find Percent Change**

  Find the amount of change
  $\text{change} = \text{new amount} - \text{original amount}$ Find what percent the amount of change is of the original amount.
  change is what percent of the original amount?

- **Discount**

amount of discount = discount rate·original pricesale price = original amount − discount

- **Mark-up**
  amount of mark-up = mark-up rate·original costlist price = original cost + mark up
- **Simple Interest**
  If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is:
  I = interestI = PrtwhereP = principalr = ratet = time

Practice Makes Perfect

Use a Problem Solving Strategy for Word Problems

List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Answers will vary.

List five negative thoughts that you have said to

yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

---

six boys

There are 18 Cub Scouts in Troop 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

---

58 hardback books

Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Solve Number Word Problems

In the following exercises, solve each number word problem.

The difference of a number and 12 is three. Find the number.

15

The difference of a number and eight is four. Find the number.

The sum of three times a number and eight is 23. Find the number.

5

The sum of twice a number and six is 14. Find the number.

The difference of twice a number and seven is 17. Find the number.

12

The difference of four times a number and seven is 21. Find the number.

Three times the sum of a number and nine is 12. Find the number.

-5

Six times the sum of a number and eight is 30. Find the number.

One number is six more than the other. Their sum is 42. Find the numbers.

18, 24

One number is five more than the other. Their sum is 33. Find the numbers.

The sum of two numbers is 20. One number is four less than the other. Find the numbers.

8, 12

The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

32, 46

One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

4, 10

The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

The sum of two consecutive integers is 77. Find the integers.

38, 39

The sum of two consecutive integers is 89. Find the integers.

The sum of three consecutive integers is 78. Find the integers.

25, 26, 27

The sum of three consecutive integers is 60. Find the integers.

Find three consecutive integers whose sum is -36.

$-11, -12, -13$

Find three consecutive integers whose sum is -3.

Find three consecutive even integers whose sum is 258.

84, 86, 88

Find three consecutive even integers whose sum is 222.

Find three consecutive odd integers whose sum is -213.

$-69, -71, -73$

Find three consecutive odd integers whose sum is -267.

Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul

pay for rent?

$750

Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?

$13,500

Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Percent Applications

In the following exercises, translate and solve.

ⓐ What number is 45% of 120? ⓑ 81 is 75% of what number? ⓐ What percent of 260 is 78?

ⓐ 54 ⓑ 108 ⓐ 30%

ⓐ What number is 65% of 100? ⓑ 93 is 75% of what number? ⓐ What percent of 215 is 86?

ⓐ 250% of 65 is what number? ⓑ 8.2% of what amount is $2.87? ⓐ 30 is what percent of 20?

ⓐ 162.5 ⓑ $35 ⓐ 150%

ⓐ 150% of 90 is what number? ⓑ 6.4% of what amount is $2.88? ⓐ 50 is what percent of 40?

In the following exercises, solve.

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

$11.88

When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

24.2 g

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

2407 mg

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

45%

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

25%

Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her

monthly pay goes to rent?

In the following exercises, solve.

Tamanika received a raise in her hourly pay, from \$15.50 to \$17.36. Find the percent change.

12%

Ayodele received a raise in her hourly pay, from \$24.50 to \$25.48. Find the percent change.

Annual student fees at the University of California rose from about \$4,000 in 2000 to about \$12,000 in 2010. Find the percent change.

200%

The price of a share of one stock rose from \$12.50 to \$50. Find the percent change.

A grocery store reduced the price of a loaf of bread from \$2.80 to \$2.73. Find the percent

change.

-2.5%

The price of a share of one stock fell from $8.75 to $8.54. Find the percent change.

Hernando's salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent change.

-11%

In ten years, the population of Detroit fell from 950,000 to about 712,500. Find the percent change.

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

ⓐ $26.97 ⓑ $17.98

Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate (Round to the nearest tenth of a percent if needed.)

Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

ⓐ $576 ⓑ 30%

Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%. What was the list price of the bracelet?

ⓐ $7.20 ⓑ $23.20

Regina bought a handmade quilt at original cost $120 to sell in her quilt store. She marked the price up 55%. What was the list price of the quilt?

Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up. What price did he charge his customers for the tomatoes?

ⓐ $0.20 ⓑ $0.80

Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up. What price did she charge her customers for the roses?

Solve Simple Interest Applications

In the following exercises, solve.

Casey deposited $1,450 in a bank account that earned simple interest at an interest rate of 4%. How much interest was earned in two years?

$116

Terrence deposited $5,720 in a bank account that earned simple interest at an interest rate of 6%. How much interest was earned in four years?

Robin deposited $31,000 in a bank account that earned simple interest at an interest rate of 5.2%. How much interest was earned in three years?

$4836

Carleen deposited $16,400 in a bank account that earned simple interest at an interest rate of 3.9% How much interest was earned in eight years?

Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of simple interest?

3%

Kenneth lent his niece $1,200 to buy a computer. Two years later, she paid him back

the $1,200, plus $96 interest. What was the rate of simple interest?

Lebron lent his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of simple interest?

3.75%

Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of simple interest?

In 10 years, a bank account that paid 5.25% simple interest earned $18,375 interest. What was the principal of the account?

$35,000

In 25 years, a bond that paid 4.75% simple interest earned $2,375 interest. What was the principal of the bond?

Joshua's computer loan statement said he would pay $1,244.34 in simple interest for a three-year loan at 12.4%. How much did Joshua borrow to buy the computer?

$3345

Margaret's car loan statement said she would pay $7,683.20 in simple interest for a five-year loan at 9.8%. How much did Margaret borrow to buy the car?

Everyday Math

Tipping At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

17.5%

Tipping Four friends went out to lunch and the bill came to $53.75 They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among

themselves. What percent tip did they leave?

Writing Exercises

What has been your past experience solving word problems? Where do you see yourself moving forward?

Answers will vary.

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Answers will vary.

Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of

their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can…	Confidently	With some help	No-I don't get it!
use a problem-solving strategy for word problems.			
solve number problems.			
solve percent applications.			
solve simple interest applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Solve a Formula for a Specific Variable

By the end of this section, you will be able to:

- Solve a formula for a specific variable
- Use formulas to solve geometry applications

Before you get started, take this readiness quiz.

Evaluate $2(x+3)$ when $x=5$.
If you missed this problem, review [link].

16

The length of a rectangle is three less than the width. Let *w* represent the width. Write an expression for the length of the rectangle.
If you missed this problem, review [link].

$w2-3w$

Evaluate 12bh when b = 14 and h = 9.
If you missed this problem, review [link].

63

Solve a Formula for a Specific Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in so many fields, it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula V = 13πr2h is used to find the volume of a right circular cone when given the radius of the base and height. In the

next example, we will solve this formula for the height.

Solve the formula V=13πr2h for *h*.

Write the formula. [redacted]

Remove the fraction on the right. [redacted]

Simplify. [redacted]

Divide both sides by πr[redacted]

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by

using the formula h=3Vπr2.

Use the formula A=12bh to solve for *b*.

b=2Ah

Use the formula A=12bh to solve for *h*.

h=2Ab

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

Solve the formula C = 59(F − 32) for *F*.

Write the formula.	$C=\frac{5}{9}(F-32)$
Remove the fraction on the right.	$\frac{9}{5}C=\frac{9}{5}\cdot\frac{5}{9}(F-32)$
Simplify.	$\frac{9}{5}C=(F-32)$
Add 32 to both sides.	$\frac{9}{5}C+32=F$

We can now use the formula F = 95C + 32 to find the Fahrenheit temperature when we know the Celsius temperature.

Solve the formula F = 95C + 32 for *C*.

$C=59(F-32)$

Solve the formula $A=12h(b+B)$ for *b*.

$b=2A-Bhh$

The next example uses the formula for the surface area of a right cylinder.

Solve the formula $S=2\pi r2+2\pi rh$ for *h*.

Write the formula.

[redacted]

Isolate the *h* term by

su[redacted]
each side.
Simplify.

[redacted]

Solve for *h* by dividing both sides by 2πr.

[redacted]

Simplify.

[redacted]

Solve the formula $A = P + Prt$ for *t*.

$t = A - PPr$

Solve the formula $A = P + Prt$ for *r*.

$r = A - PPt$

Sometimes we might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y.

Solve the formula $8x + 7y = 15$ for y.

We will isolate y on one side of the eq[redacted]

Subtract 6x from both sides to isolate the ter[redacted]

Simplify.

[redacted]

Divide both sides by 7 to make the coefficient of [redacted]

Simplify.

[redacted]

Solve the formula $4x + 7y = 9$ for *y*.

$y = 9 - 4x7$

Solve the formula $5x + 8y = 1$ for *y*.

$y = 1 - 5x8$

Use Formulas to Solve Geometry Applications

In this objective we will use some common geometry formulas. We will adapt our problem

solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

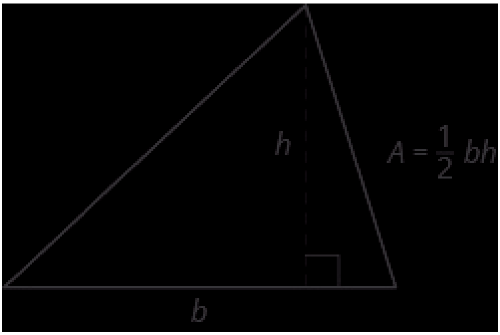
Solve geometry applications.

Read the problem and make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what we are looking for by choosing a variable to represent it. Draw the figure and label it with the given information. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve** the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

When we solve geometry applications, we often have to use some of the properties of the figures. We

will review those properties as needed.

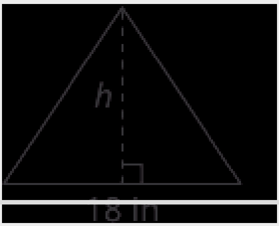
The next example involves the area of a triangle. The area of a triangle is one-half the base times the height. We can write this as A=12bh, where b = length of the base and h = height.



The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Step 1. Read the problem.	
Step 2. Identify what	height of a triangle

you are looking for.	
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area $=$ 126 sq. in.



Step 4. Translate.	
Write the appropriate formula.	$A = 12bh$
Substitute in the given information.	$126 = 12 \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$
Divide both sides by 9.	$14 = h$
Step 6. Check.	

$A = 12bh126 \stackrel{?}{=} 12 \cdot 18 \cdot 14126 = 126✓$

Step 7. Answer the question.	The height of the triangle is 14 inches.

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?
The window's height is 12 meters.

A triangular tent door has area 15 square feet. The height is five feet. What is the length of the base?
The length of the base is 6 feet.

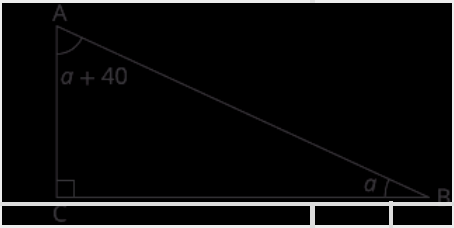
In the next example, we will work with a right triangle. To solve for the measure of each angle, we need to use two triangle properties. In any triangle, the sum of the measures of the angles is 180°. We can write this as a formula: $m\angle A + m\angle B + m\angle C = 180$. Also, since the triangle is a right triangle, we remember that a right triangle has one 90° angle.

Here, we will have to define one angle in terms of

another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let $a = 1\text{st angle}$. $a + 40 = 2\text{nd angle}$ $90 = 3\text{rd angle}$ (the right angle)
Draw the figure and label it with the given information.	



Step 4. Translate. Write the appropriate formula.

$m\angle A + m\angle B + m\angle C = 180$

Substitute into the formula.

$a + (a + 40) + 90 = 180$

Step 5. Solve the equation.

$$2a + 130 = 180$$
$$2a = 50$$
$$a = 25 \text{ first angle}$$

$$a + 40 \text{ second angle}$$
$$25 + 40$$
$$65$$

$$90 \quad \text{third angle}$$

Step 6. Check.

$25 + 65 + 90 \stackrel{?}{=} 180$
$180180 = 180 \checkmark$

Step 7. Answer the question.

The three angles measure 25°,65°, and 90°.

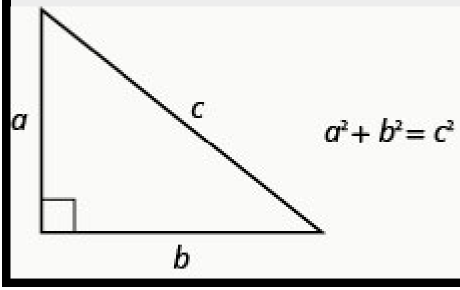
The measure of one angle of a right triangle is 50 more than the measure of the smallest angle. Find the measures of all three angles.
The measures of the angles are 20°, 70°, and 90°.

The measure of one angle of a right triangle is 30 more than the measure of the smallest angle. Find the measures of all three angles.
The measures of the angles are 30°, 60°, and 90°.

The next example uses another important geometry formula. The **Pythagorean Theorem** tells how the lengths of the three sides of a right triangle relate to each other. Writing the formula in every exercise and saying it aloud as you write it may help you memorize the Pythagorean Theorem.

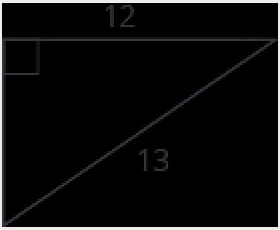
The Pythagorean Theorem

In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.

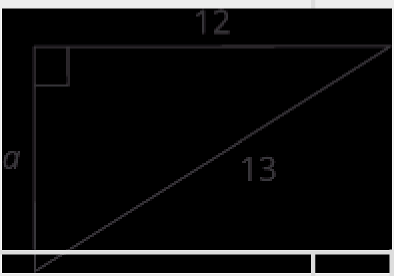


We will use the Pythagorean Theorem in the next example.

Use the Pythagorean Theorem to find the length of the other leg in



Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name. Choose a variable to represent it. Label side *a*.	Let $a =$ the leg of the triangle.



Step 4. Translate. Write the appropriate formula. Substitute.	$a^2 + b^2 = c^2$ $a^2 + 12^2 = 13^2$
Step 5. Solve the	$a^2 + 144 = 169$ $a^2 = 25$ $a = \sqrt{25}$ $a = 5$

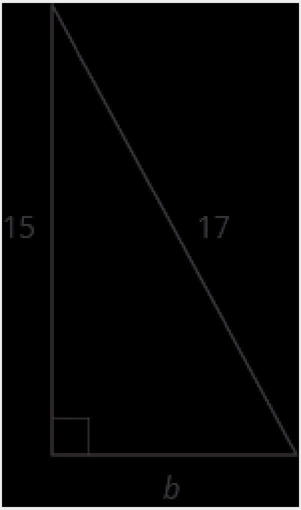
equation.
Isolate the variable term.
Use the definition of square root.
Simplify.

Step 6. Check.

$$5^2 + 12^2 \stackrel{?}{=} 13^2$$
$$25 + 144 \stackrel{?}{=} 169$$
$$169 = 169 \checkmark$$

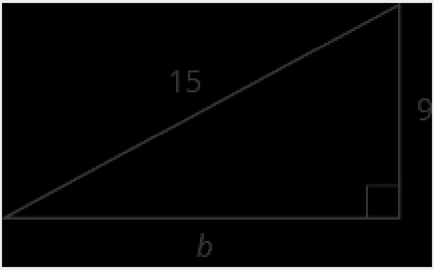
Step 7. Answer the question.	The length of the leg is 5.

Use the Pythagorean Theorem to find the length of the leg in the figure.



The length of the leg is 8.

Use the Pythagorean Theorem to find the length of the leg in the figure.



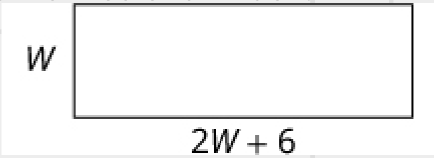
The length of the leg is 12.

The next example is about the perimeter of a rectangle. Since the perimeter is just the distance around the rectangle, we find the sum of the lengths of its four sides—the sum of two lengths and two widths. We can write is as $P = 2L + 2W$ where L is the length and W is the width. To solve the example, we will need to define the length in terms of the width.

The length of a rectangle is six centimeters more than twice the width. The perimeter is 96 centimeters. Find the length and width.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and the width
Step 3. Name. Choose a variable to represent	Let $w =$ width. $2w + 6 =$ length

the width.
The length is six more than twice the width.



$P = 96$ cm

Step 4. Translate.
Write the appropriate formula.

[redacted]

Substitute in the given information.

$$96 = 2(2W + 6) + 2W$$

Step 5. Solve the equation.

$$96 = 4W + 12 + 2W$$
$$96 = 6W + 12$$
$$84 = 6W$$
$$14 = W \text{ (width)}$$

$$2W + 6 \text{ (length)}$$
$$2(14) + 6$$
$$34 \quad \text{The length is 34 cm.}$$

Step 6. Check.

$P = 2L + 2W \quad 96 = ?$
$2 \cdot 34 + 2 \cdot 14 \quad 96 = 96$ ✓

Step 7. Answer the question.	The length is 34 cm and the width is 14 cm.

The length of a rectangle is seven more than twice the width. The perimeter is 110 inches. Find the length and width.

The length is 39 inches and the width is 16 inches.

The width of a rectangle is eight yards less than twice the length. The perimeter is 86 yards. Find the length and width.

The length is 17 yards and the width is 26 yards.

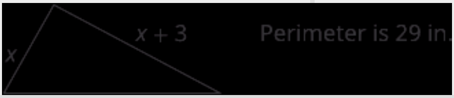
The next example is about the perimeter of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where *a*, *b*, and *c* are the lengths of the sides.

One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the lengths of the three sides of a triangle

Step 3. Name. Choose a variable to represent the length of the first side.

Let $x =$ length of 1st side. $x + 3 =$ length of 2nd side $2x + 2 =$ length of 3rd side



[redacted]

Step 4. Translate.
Write the appropriate formula.

[redacted]

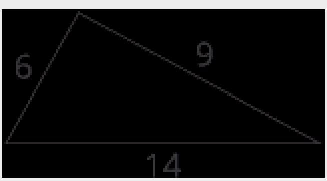
Substitute in the given information.

[redacted]

Step 5. Solve the equation.

[redacted]

Step 6. Check.



$$29 \stackrel{?}{=} 6+9+14$$
$$29 = 29 \checkmark$$

Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

The lengths of the sides of the triangle are 5, 11 and 12 inches.

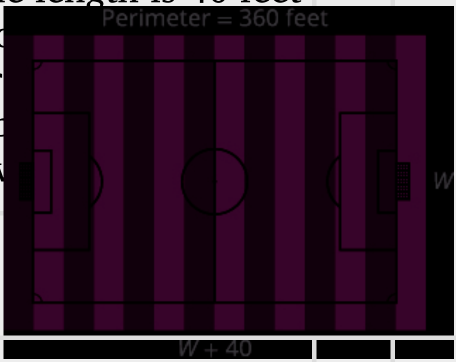
One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

The lengths of the sides of the triangle are 4, 7 and 9 feet.

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.

Step 1. Read the problem.	
Step 2. Identify what	the length and width of

we are looking for.	the soccer field
Step 3. Name. Choose a variable to represent it.	Let w = width. $w + 40 =$ length
The length is 40 feet mo Dr lab giv	



Step 4. Translate. Write the appropriate formula and substitute.

$$P = 2L + 2W$$

$$360 = 2(w + 40) + 2w$$

Step 5. Solve the equation.

$$360 = 2w + 80 + 2w$$
$$360 = 4w + 80$$
$$280 = 4w$$
$70 = w$ the width of the field

$w + 40$ the length of the field
$$70 + 40$$
$$110$$

Step 6. Check.

$P = 2L + 2W$ $360 \stackrel{?}{=} 2(110) + 2(70)$ $360 = 360$✓

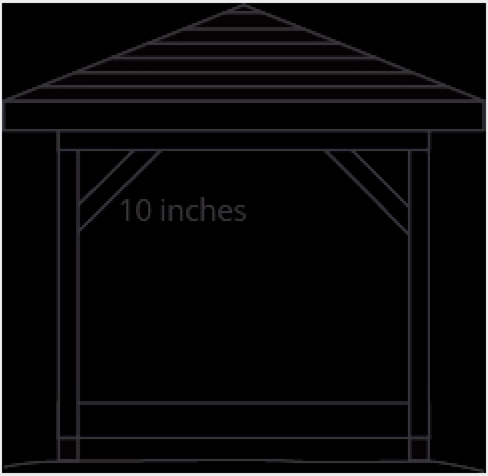
Step 7. Answer the question.	The length of the soccer field is 110 feet and the width is 70 feet.

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.
The length of the swimming pool is 70 feet and the width is 30 feet.

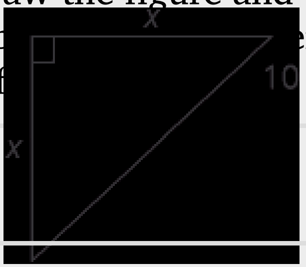
The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.
The length of the garden is 90 yards and the width is 60 yards.

Applications of these geometric properties can be found in many everyday situations as shown in the next example.

Kelvin is building a gazebo and wants to brace each corner by placing a 10” piece of wood diagonally as shown.

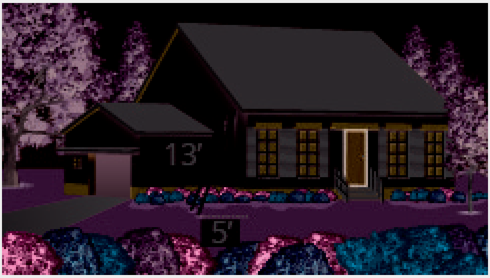


How far from the corner should he fasten the wood if wants the distances from the corner to be equal? Approximate to the nearest tenth of an inch.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let $x =$ the distance from the corner.
Draw the figure and lab en inf  	
Step 4. Translate.	
Write the appropriate formula and substitute.	$a2 + b2 = c2$ $x2 + x2 = 102$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x2 = 100x2 = 50x = 50x \approx 7.1$
Step 6. Check.	
$a2 + b2 = c2(7.1)2 + (7.1)2 \approx 102$Yes.	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1”

from the corner.

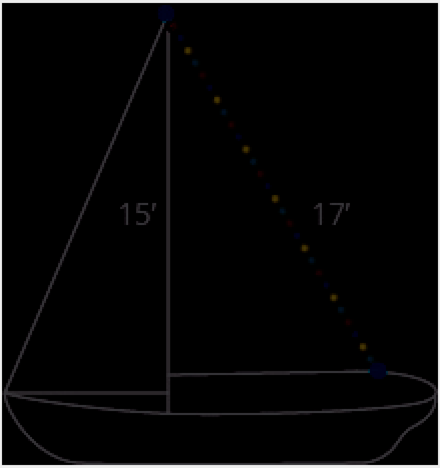
John puts the base of a 13-foot ladder five feet from the wall of his house as shown in the figure. How far up the wall does the ladder reach?



The ladder reaches 12 feet.

Randy wants to attach a 17-foot string of lights to the top of the 15 foot mast of his sailboat, as shown in the figure. How far from the base of the mast should he attach the end of the

light string?



He should attach the lights 8 feet from the base of the mast.

Access this online resource for additional instruction and practice with solving for a variable in literal equations.

- Solving Literal Equations

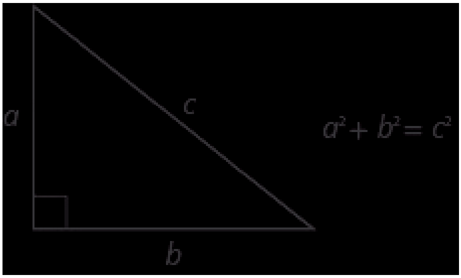
Key Concepts

- **How To Solve Geometry Applications**

  Read the problem and make sure all the words and ideas are understood. **Identify** what you are looking for. **Name** what you are looking for by choosing a variable to represent it. Draw the figure and label it with the given information. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve** the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

- **The Pythagorean Theorem**

  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



Practice Makes Perfect

Solve a Formula for a Specific Variable

In the following exercises, solve the given formula for the specified variable.

Solve the formula $C = \pi d$ for *d*.

$d = C\pi$

Solve the formula $C = \pi d$ for π.

Solve the formula $V = LWH$ for *L*.

$L = VWH$

Solve the formula $V = LWH$ for *H*.

Solve the formula $A = 12bh$ for *b*.

$b = 2Ah$

Solve the formula $A = 12bh$ for *h*.

Solve the formula
$A = 12d1d2$ for d1.

$d1 = 2Ad2$

Solve the formula
$A = 12d1d2$ for d2.

Solve the formula
$A = 12h(b1 + b2)$ for b1.

$b1 = 2Ah - b2$

Solve the formula
$A = 12h(b1 + b2)$ for b2.

Solve the formula
$h=54t+12at2$ for a.

$a=2h-108tt2$

Solve the formula
$h=48t+12at2$ for a.

Solve $180=a+b+c$ for a.

$a=180-b-c$

Solve $180=a+b+c$ for c.

Solve the formula
$A=12pl+B$ for p.

$p=2A-2Bl$

Solve the formula
$A=12pl+B$ for l.

Solve the formula
$P=2L+2W$ for L.

$L = P - 2W2$

Solve the formula
$P = 2L + 2W$ for *W*.

In the following exercises, solve for the formula for *y*.

Solve the formula
$8x + y = 15$ for *y*.

$y = 15 - 8x$

Solve the formula
$9x + y = 13$ for *y*.

Solve the formula
$-4x + y = -6$ for *y*.

$y = -6 + 4x$

Solve the formula
$-5x + y = -1$ for *y*.

Solve the formula
$x - y = -4$ for y.

$y = 4 + x$

Solve the formula
$x - y = -3$ for y.

Solve the formula
$4x + 3y = 7$ for y.

$y = 7 - 4x3$

Solve the formula
$3x + 2y = 11$ for y.

Solve the formula
$2x + 3y = 12$ for y.

$y = 12 - 2x3$

Solve the formula
$5x + 2y = 10$ for y.

Solve the formula
$3x - 2y = 18$ for y.

$y = 18 - 3x - 2$

Solve the formula
$4x - 3y = 12$ for y.

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

A triangular flag has area 0.75 square feet and height 1.5 foot. What is its base?

---

1 foot

A triangular window has area 24 square feet and height six feet. What is its base?

What is the base of a triangle with area 207 square inches and height 18 inches?

---

23 inches

What is the height of a triangle with area 893 square inches and base 38 inches?

The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

$45^{\circ}, 45^{\circ}, 90^{\circ}$

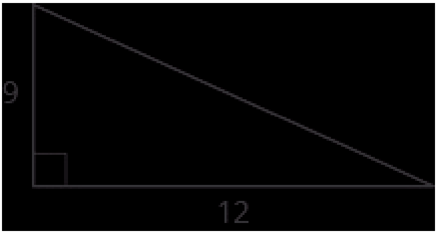
The measure of the smallest angle of a right triangle is 20° less than the measure of the next larger angle. Find the measures of all three angles.

The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

$30^{\circ}, 60^{\circ}, 90^{\circ}$

The angles in a triangle are such that one angle is 20 more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.



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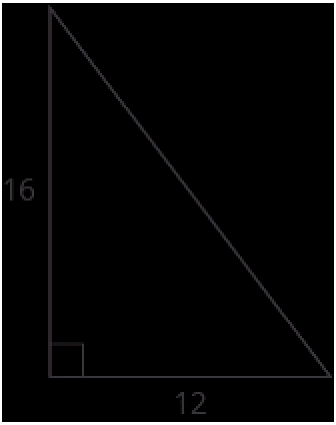

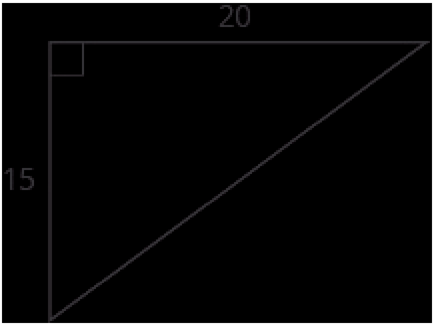

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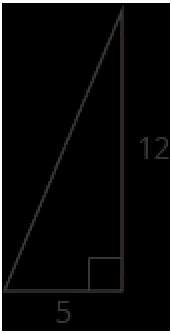
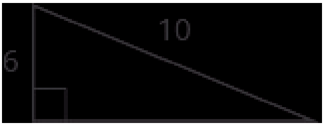


In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth if necessary.



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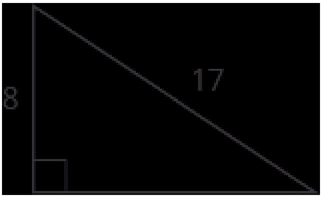

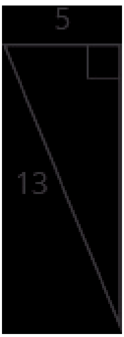

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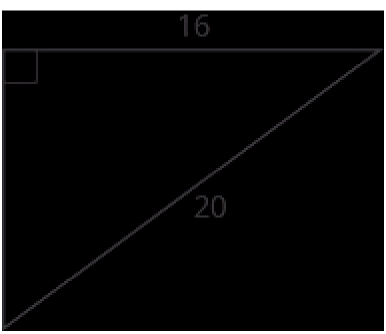

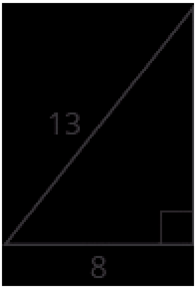

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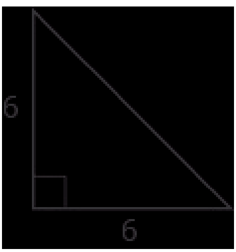

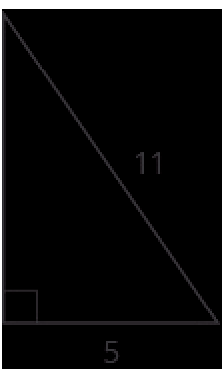

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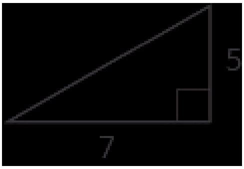


In the following exercises, solve using a geometry formula.

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

18 meters, 11 meters

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

13.5 m, 12.8 m

The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

---

25 ft, 50 ft

The length of the rectangle is three times the width. The perimeter of a rectangle is 72 feet. Find the length and width of the rectangle.

The length of the rectangle is three meters less than twice the width. The perimeter of a rectangle is 36 meters. Find the dimensions of the rectangle.

---

7 m, 11 m

The length of a rectangle is five inches more than twice the width. The perimeter is 34

inches. Find the length and width.

The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

---

12 ft, 13 ft, 14 ft

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

One side of a triangle is twice the smallest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

---

3 ft, 6 ft, 8 ft

One side of a triangle is three times the smallest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the

lengths of all three sides.

The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

---

120 yd, 160 yd

The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

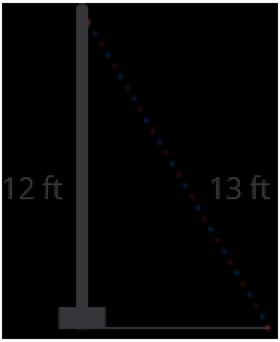
---

40 ft, 85 ft

A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

In the following exercises, solve. Approximate answers to the nearest tenth, if necessary.

A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display as shown. How far from the base of the pole should the end of the string of lights be anchored?



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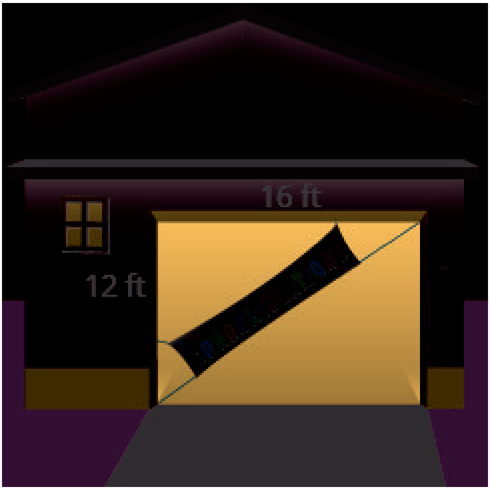
5 feet

am wants to put a banner across her garage door diagonally, as shown, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. Approximately how long should the banner be to fit the garage door?



Chi is planning to put a diagonal path of paving stones through her flower garden as shown. The flower garden is a square with side 10 feet. What will the length of the path be?

14.1 feet

Brian borrowed a 20-foot extension ladder to use when he paints his house. If he sets the base of the ladder six feet from the house as shown, how far up will the top of the ladder reach?

Everyday Math

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for *F* in the formula C = 59(F − 32) to find the Fahrenheit temperature.

104°F

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50° Fahrenheit. Solve for *C* in the formula F = 95C + 32 to find the Celsius temperature.

Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

---

240 ft

Jose just removed the children's play set from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep the dog out. He has a 50-foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side?

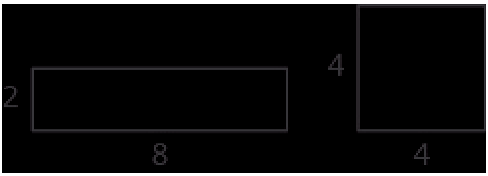
Writing Exercises

If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

Answers will vary.

If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

Look at the two figures below.



ⓐ Which figure looks like it has the larger area? Which looks like it has the larger perimeter?
ⓑ Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?
ⓒ Were the results of part (b) the same as your answers in part (a)? Is that surprising to you?

ⓐ Answers will vary. ⓑ The areas are the same. The 2×8 rectangle has a larger perimeter than the 4×4 square.
ⓒ Answers will vary.

Write a geometry word problem that relates to your life experience, then solve it and explain

all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve a formula for a specific variable.			
use formulas to solve geometry applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Mixture and Uniform Motion Applications

By the end of this section, you will be able to:

- Solve coin word problems
- Solve ticket and stamp word problems
- Solve mixture word problems
- Solve uniform motion applications

Before you get started, take this readiness quiz.

Simplify: $0.25x + 0.10(x + 4)$.
If you missed this problem, review [link].

$x = 6$

The number of adult tickets is three more than twice the number of children tickets. Let c represent the number of children tickets. Write an expression for the number of adult tickets.
If you missed this problem, review [link].

$2c + 3$

Convert 4.2% to a decimal.
If you missed this problem, review [link].

0.0042

Solve Coin Word Problems

Using algebra to find the number of nickels and pennies in a piggy bank may seem silly. You may wonder why we just don't open the bank and count them. But this type of problem introduces us to some techniques that will be useful as we move forward in our study of mathematics.

If we have a pile of dimes, how would we determine its value? If we count the number of dimes, we'll know how many we have—the *number* of dimes. But this does not tell us the *value* of all the dimes. Say we counted 23 dimes, how much are they worth? Each dime is worth \$0.10—that is the *value* of one dime. To find the total value of the pile of 23 dimes, multiply 23 by \$0.10 to get \$2.30.

The number of dimes times the value of each dime equals the total value of the dimes.

$$\text{number} \cdot \text{value} = \text{total value} \quad 23 \cdot \$0.10 = \$2.30$$

This method leads to the following model.

Total Value of Coins

For the same type of coin, the total value of a number of coins is found by using the model

$$\text{number} \cdot \text{value} = \text{total value}$$

- *number* is the number of coins
- *value* is the value of each coin
- *total value* is the total value of all the coins

If we had several types of coins, we could continue this process for each type of coin, and then we would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

Jesse has $3.02 worth of pennies and nickels in his piggy bank. The number of nickels is three more than eight times the number of pennies. How many nickels and how many pennies does Jesse have?

Step 1. Read the problem. Determine the types of coins involved.		pennies and nickels Pennies are worth

Create a table. Write in the value of each type of coin.	$0.10. Nickels are worth $0.05.
Step 2. Identify what we are looking for.	the number of pennies and nickels
Step 3. Name. Represent the number of each type of coin using variables.	Let $p =$ number of pennies.
The number of nickels is defined in terms of the number of pennies, so start with pennies.	$8p + 3 =$ number of nickels
The number of nickels is three more than eight times the number of pennies.	
In the chart, multiply the number and the value to get the total value of each type of coin.	

Type	Number	• Value ($)	= Total Value ($)
pennies	p	0.01	$0.01p$
nickels	$8p + 3$	0.05	$0.05(8p + 3)$
			$3.02

Step 4. Translate. Write the equation by adding the total value

$$0.01p + 0.05(8p + 3) = 3.02$$

of all the types of coins.

Step 5. Solve the equation.

$$0.01p + 0.40p + 0.15 = 3.02$$
$$0.41p + 0.15 = 3.02$$
$$0.41p = 2.87$$
$$p = 7 \text{ pennies}$$

How many nickels?

$$8p + 3$$
$$8(7) + 3$$
$$59 \text{ nickels}$$

Step 6. Check the answer in the problem and make sure it makes sense.

Jesse has 7 pennies and 59 nickels.

Is the total value \$3.02?

$7(0.01) + 59(0.05) = ?$

$3.023.02 = 3.02$✓

Jesse has \$6.55 worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

Jess has 41 nickels and 18 quarters.

Elane has $7.00 total in dimes and nickels in her coin jar. The number of dimes that Elane has is seven less than three times the number of nickels. How many of each coin does Elane have?

Elane has 22 nickels and 59 dimes.

The steps for solving a coin word problem are summarized below.

Solve coin word problems.

Read the problem. Make sure all the words and ideas are understood.

- Determine the types of coins involved.
- Create a table to organize the information.

- ○ Label the columns "type," "number," "value," and "total value."
- ○ List the types of coins.
- ○ Write in the value of each type of coin.
- ○ Write in the total value of all the coins.

Type	Number	• Value ($) =	Total Value ($)

Identify what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity.

- Use variable expressions to represent the number of each type of coin and write them in the table.
- Multiply the number times the value to get the total value of each type of coin.

Translate into an equation.

- It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
- Write the equation by adding the total values of all the types of coins.

Solve the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

Solve Ticket and Stamp Word Problems

Problems involving tickets or stamps are very much like coin problems. Each type of ticket and stamp has a value, just like each type of coin does. So to solve these problems, we will follow the same steps we used to solve coin problems.

Danny paid $15.75 for stamps. The number of 49-cent stamps was five less than three times the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Danny buy?

Step 1. Determine the types of stamps involved.	49-cent stamps and 35-cent stamps
Step 2. Identify we are looking for.	the number of 49-cent stamps and the number of 35 cent stamps

Step 3. Write variable expressions to represent the number of each type of stamp.

Let x = number of 35-cent stamps.

"The number of 49-cent stamps was five less than three times the number of 35-cent stamps."

$3x-5=$ number of 49-cent stamps

Type	Number	·	Value($)	=	Total Value ($)
49 cent stamps	$3x-5$		0.49		$0.49(3x-5)$
35 cent stamps	x		0.35		$0.35x$
					15.75

Step 4. Write the equation from the total values.

$$0.49(3x-5)+0.35x=15.75$$

Step 5. Solve the equation.

$$1.47x-2.45+0.35x=15.75$$
$$1.82x-2.45=15.75$$
$$1.82x=18.2$$

How many 49-cent stamps?

$x=10$ 35-cent stamps

$$3x-5$$
$$3(10)-5$$
$$25 \quad \text{49-cent stamps}$$

Step 6. Check.

$10(0.35)+25(0.49)\overset{?}{=}15.75$

$3.50+12.25\overset{?}{=}15.75$

$15.75=15.75✓$

Step 7. Answer the question with a

Danny bought ten 35-cent stamps and

complete sentence.	twenty-five 49-cent stamps.

Eric paid \$19.88 for stamps. The number of 49-cent stamps was eight more than twice the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Eric buy?
Eric bought thirty-two 49-cent stamps and twelve 35-cent stamps.

Kailee paid \$14.74 for stamps. The number of 49-cent stamps was four less than three times the number of 20-cent stamps. How many 49-cent stamps and how many 20-cent stamps did Kailee buy?
Kailee bought twenty-six 49-cent stamps and ten 20-cent stamps.

In most of our examples so far, we have been told that one quantity is four more than twice the other, or something similar. In our next example, we have to relate the quantities in a different way.

Suppose Aniket sold a total of 100 tickets. Each ticket was either an adult ticket or a child ticket. If he sold 20 child tickets, how many adult tickets did he sell?

Did you say “80”? How did you figure that out? Did you subtract 20 from 100?

If he sold 45 child tickets, how many adult tickets did he sell?

Did you say “55”? How did you find it? By subtracting 45 from 100?

Now, suppose Aniket sold x child tickets. Then how many adult tickets did he sell? To find out, we would follow the same logic we used above. In each case, we subtracted the number of child tickets from 100 to get the number of adult tickets. We now do the same with x.

We have summarized this in the table.

Child tickets	Adult tickets
20	80
45	55
75	25
x	$100 - x$

We will apply this technique in the next example.

A whale-watching ship had 40 paying passengers on board. The total revenue collected from tickets was $1,196. Full-fare passengers paid $32 each and reduced-fare passengers paid $26 each. How many full-fare passengers and how many reduced-fare passengers were on the ship?

Step 1. Determine the types of tickets involved.	full-fare tickets and reduced-fare tickets
Step 2. Identify what	the number of full-fare

we are looking for. tickets and reduced-fare tickets

Step 3. Name. Represent the number of each type of ticket using variables.

Let $f =$ the number of full-fare tickets.
$40 - f =$ the number of reduced-fare tickets

We know the total number of tickets sold was 40. This means the number of reduced-fare tickets is 40 less the number of full-fare tickets.

Multiply the number times the value to get the total value of each type of ticket.

Type	Number	·	Value ($)	=	Total Value ($)
Full-fare	f		32		$32f$
Reduced-fare	$40 - f$		26		$26(40 - f)$
					1,196

Step 4. Translate. Write the equation by adding the total values of each type of ticket.

$$32f + 26(40 - f) = 1{,}196$$

Step 5. Solve the equation.

$$32f + 1{,}040 - 26f = 1{,}196$$
$$6f = 156$$
$$f = 26 \quad \text{full-fare tickets}$$

How many reduced-fare?

[redacted]

Step 6. Check the answer. There were 26 full-fare tickets at \$32 each and 14 reduced-fare tickets at \$26 each. Is the total value \$116? $26 \cdot 32 = 832$ $14 \cdot 26 = 364$ $1{,}196$✓	
Step 7. Answer the question.	They sold 26 full-fare and 14 reduced-fare tickets.

During her shift at the museum ticket booth, Leah sold 115 tickets for a total of \$1,163. Adult tickets cost \$12 and student tickets cost \$5. How many adult tickets and how many student tickets did Leah sell?

84 adult tickets, 31 student tickets

Galen sold 810 tickets for his church's carnival for a total revenue of $2,820. Children's tickets cost $3 each and adult tickets cost $5 each. How many children's tickets and how many adult tickets did he sell?

615 children's tickets and 195 adult tickets

Solve Mixture Word Problems

Now we'll solve some more general applications of the mixture model. In mixture problems, we are often mixing two quantities, such as raisins and nuts, to create a mixture, such as trail mix. In our tables we will have a row for each item to be mixed as well as one for the final mixture.

Henning is mixing raisins and nuts to make 25 pounds of trail mix. Raisins cost $4.50 a pound and nuts cost $8 a pound. If Henning wants his cost for the trail mix to be $6.60 a pound, how many pounds of raisins and how many pounds

of nuts should he use?

Step 1. Determine what is being mixed.	The 25 pounds of trail mix will come from mixing raisins and nuts.
Step 2. Identify what we are looking for.	the number of pounds of raisins and nuts
Step 3. Represent the number of each type of ticket using variables.	Let $x =$ number of pounds of raisins. $25 - x =$ number of pounds of nuts

As before, we fill in a chart to organize our inf
We
pound for each item.

Type	Number of pounds	·	Price per Pound ($)	=	Total Value ($)
Raisins	x		4.50		$4.5x$
Nuts	$25 - x$		8		$8(25 - x)$
Trail mix	25		6.60		$25(6.60)$

We multiply the number times the value to get the total value.

Notice that the last column in the table gives the information for the total amount of the mixture.

Step 4. Translate into an equation.

The value of the raisins plus the value of the nuts will be the value of the trail mix.

Step 5. Solve the equation.

$$4.5x + 8(25 - x) = 25(6.60)$$

$$4.5x + 200 - 8x = 165$$

$$-3.5x = -35$$

$$x = 10 \quad \text{pounds of raisins}$$

Find the number of pounds of nuts.

$$25 - x$$

$$25 - 10$$

$$15 \quad \text{pounds of nuts}$$

Step 6. Check.

$4.5(10) + 8(15) \stackrel{?}{=} 25(6.60)$

$45 + 120 \stackrel{?}{=} 165$

$165 = 165 \checkmark$

Step 7. Answer the question.

Henning mixed ten pounds of raisins with 15 pounds of nuts.

Orlando is mixing nuts and cereal squares to make a party mix. Nuts sell for $7 a pound and cereal squares sell for $4 a pound. Orlando wants to make 30 pounds of party mix at a

cost of $6.50 a pound, how many pounds of nuts and how many pounds of cereal squares should he use?

Orlando mixed five pounds of cereal squares and 25 pounds of nuts.

Becca wants to mix fruit juice and soda to make a punch. She can buy fruit juice for $3 a gallon and soda for $4 a gallon. If she wants to make 28 gallons of punch at a cost of $3.25 a gallon, how many gallons of fruit juice and how many gallons of soda should she buy?

Becca mixed 21 gallons of fruit punch and seven gallons of soda.

Solve Uniform Motion Applications

When you are driving down the interstate using

your cruise control, the speed of your car stays the same—it is uniform. We call a problem in which the speed of an object is constant a uniform motion application. We will use the distance, rate, and time formula, $D = rt$, to compare two scenarios, such as two vehicles travelling at different rates or in opposite directions.

Our problem solving strategies will still apply here, but we will add to the first step. The first step will include drawing a diagram that shows what is happening in the example. Drawing the diagram helps us understand what is happening so that we will write an appropriate equation. Then we will make a table to organize the information, like we did for the coin, ticket, and stamp applications.

The steps are listed here for easy reference:

Solve a uniform motion application.

Read the problem. Make sure all the words and ideas are understood.

- Draw a diagram to illustrate what is happening.
- Create a table to organize the information.
    - Label the columns rate, time, distance.

- ○ List the two scenarios.
- ○ Write in the information you know.

	Rate •	Time =	Distance

Identify what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity.

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
- Multiply the rate times the time to get the distance.

Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.

Solve the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis’s speed is seven miles per hour faster than Wayne’s speed, so it takes Wayne two hours to ride to the beach while it takes Dennis 1.5 hours for the ride. Find the speed of both bikers.

Step 1. Read the problem. Make sure all the words and ideas are understood.

- Draw a diagram to illustrate what it happening. Shown below is a sketch of what is happening in the example.

- Create a table to organize the information.
    - Label the columns “Rate,” “Time,” and “Distance.”
    - List the two scenarios.
    - Write in the information you know.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Dennis		1.5	
Wayne		2	

Step 2. Identify what you are looking for.

You are asked to find the speed of both bikers.

Notice that the distance formula uses the word “rate,” but it is more common to use “speed”

when we talk about vehicles in everyday English.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart
- Use variable expressions to represent that quantity in each row.
  We are looking for the speed of the bikers. Let’s let *r* represent Wayne’s speed. Since Dennis’ speed is 7 mph faster, we represent that as $r+7$
  $r+7=$ Dennis’ speed$r=$ Wayne’s speed
  Fill in the speeds into the chart.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Dennis	$r + 7$	1.5	
Wayne	r	2	

- Multiply the rate times the time to get the distance.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Dennis	$r + 7$	1.5	$1.5(r + 7)$
Wayne	r	2	$2r$

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
  The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above. How is the distance travelled by Dennis related to the distance travelled by Wayne?
  Since both bikers leave from Riverside and travel to the beach, they travel the same distance. So we write:

	distance traveled by Dennis	=	distance traveled by Wayne
Translate to an equation.	$1.5(r + 7)$	=	$2r$

Step 5. Solve the equation using algebra techniques.

Now solve this equation.

$$\begin{aligned} 1.5(r+7) &= 2r \\ 1.5r + 10.5 &= 2r \\ 10.5 &= 0.5r \\ 21 &= r \end{aligned}$$

So Wayne’s speed is 21 mph.

Find Dennis’ speed.

$$\begin{aligned} &r+7 \\ &21+7 \\ &28 \end{aligned}$$

Dennis’ speed 28 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

$\text{Dennis} 28\text{mph}(1.5\text{hours}) = 42\text{miles} \text{Wayne} 21\text{mph}(2\text{hours}$

Step 7. Answer the question with a complete sentence.

Wayne rode at 21 mph and Dennis rode at 28 mph.

An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

The speed of the local train is 48 mph and the speed of the express train is 60 mph.

Jeromy can drive from his house in Cleveland to his college in Chicago in 4.5 hours. It takes his mother six hours to make the same drive. Jeromy drives 20 miles per hour faster than his mother. Find Jeromy’s speed and his mother’s speed.

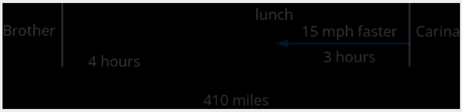
Jeromy drove at a speed of 80 mph and his mother drove 60 mph.

In [link], we had two bikers traveling the same distance. In the next example, two people drive toward each other until they meet.

Carina is driving from her home in Anaheim to Berkeley on the same day her brother is driving from Berkeley to Anaheim, so they decide to meet for lunch along the way in Buttonwillow. The distance from Anaheim to Berkeley is 395 miles. It takes Carina three hours to get to Buttonwillow, while her brother drives four hours to get there. Carina's average speed is 15 miles per hour faster than her brother's average speed. Find Carina's and her brother's average speeds.

Step 1. Read the problem. Make sure all the words and ideas are understood.

- Draw a diagram to illustrate what it happening. Below shows a sketch of what is happening in the example.



- Create a table to organize the information.
    - Label the columns rate, time, distance.
    - List the two scenarios.
    - Write in the information you know.

	Rate (mph)	• Time (hrs)	= Distance (miles)
Carina		3	
Brother		4	
			395

Step 2. Identify what we are looking for.

We are asked to find the average speeds of Carina and her brother.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
  We are looking for their average speeds. Let's let *r* represent the average speed of Carina's brother. Since Carina's speed is

15 mph faster, we represent that as r + 15. Fill in the speeds into the chart.

- Multiply the rate times the time to get the distance.

	Rate (mph)	• Time (hrs)	= Distance (miles)
Carina	$r + 15$	3	$3(r + 15)$
Brother	r	4	$4r$
			395

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
  Again, we need to identify a relationship between the distances in order to write an equation. Look at the diagram we created above and notice the relationship between the distance Carina traveled and the distance her brother traveled.
  The distance Carina traveled plus the distance her brother travel must add up to 410 miles. So we write:

	distance traveled by Carina	+	distance traveled by her brother	= 395
Translate to an equation.	$3(r + 15)$	+	$4r$	= 395

Step 5. Solve the equation using algebra techniques.

Now solve this equation.

$$\begin{aligned} 3(r+15)+4r &= 395 \\ 3r+45+4r &= 395 \\ 7r+45 &= 395 \\ 7r &= 350 \\ r &= 50 \end{aligned}$$

So Carina’s brother's speed was 50 mph.

Carina’s speed is $r+15$.

$$\begin{aligned} &r+15 \\ &50+15 \\ &65 \end{aligned}$$

Carina’s speed was 65 mph.

Step 6. Check the answer in the problem and make sure it makes sense.
Carina drove $65\text{mph}(3\text{hours}) = 195\text{miles}$Her brother drove$50\text{mph}(4\text{hours}) = 200\text{miles}$
$\overline{\qquad\qquad\qquad}395\text{miles}\checkmark$

Step 7. Answer the question with a complete sentence.

Carina drove 65 mph and her brother 50 mph.

Christopher and his parents live 115 miles apart. They met at a restaurant between their homes to celebrate his mother's birthday. Christopher drove one and a half hours while his parents drove one hour to get to the restaurant. Christopher's average speed was ten miles per hour faster than his parents' average speed. What were the average speeds of Christopher and of his parents as they drove to the restaurant?

Christopher's speed was 50 mph and his parents' speed was 40 mph.

Ashley goes to college in Minneapolis, 234 miles from her home in Sioux Falls. She wants her parents to bring her more winter clothes, so they decide to meet at a restaurant on the road between Minneapolis and Sioux Falls. Ashley and her parents both drove two hours to the restaurant. Ashley's average speed was seven miles per hour faster than her parents' average speed. Find Ashley's and her parents' average speed.

Ashley's parents drove 55 mph and Ashley drove 62 mph.

As you read the next example, think about the relationship of the distances traveled. Which of the previous two examples is more similar to this situation?

Two truck drivers leave a rest area on the interstate at the same time. One truck travels east and the other one travels west. The truck traveling west travels at 70 mph and the truck traveling east has an average speed of 60 mph. How long will they travel before they are 325 miles apart?

Step 1. Read the problem. Make all the words and ideas are understood.

- Draw a diagram to illustrate what it happening.

- Create a table to organize the information.
    - Label the columns rate, time, distance.
    - List the two scenarios.
    - Write in the information you know.

	Rate (mph)	·	Time (hrs)	=	Distance (miles)
West	70				
East	60				
					325

Step 2. Identify what we are looking for.

We are asked to find the amount of time the trucks will travel until they are 325 miles apart.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
  We are looking for the time travelled. Both trucks will travel the same amount of time.

Let’s call the time *t*. Since their speeds are different, they will travel different distances.

- Multiply the rate times the time to get the distance.

	Rate (mph)	• Time (hrs)	= Distance (miles)
West	70	t	$70t$
East	60	t	$60t$
			325

Step 4. Translate into an equation.

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
  We need to find a relation between the distances in order to write an equation. Looking at the diagram, what is the relationship between the distances each of the trucks will travel?
  The distance travelled by the truck going west plus the distance travelled by the truck going east must add up to 325 miles. So we write:

	distance traveled by westbound truck	+	distance traveled by eastbound truck	= 325
Translate to an equation.	$70t$	+	$60t$	= 325

Step 5. Solve the equation using algebra techniques.

Now solve this equation

$$70t + 60t = 325$$
$$130t = 325$$
$$t = 2.5$$

So it will take the trucks 2.5 hours to be 325 miles apart.

Step 6. Check the answer in the problem and make sure it makes sense.

$$\begin{array}{ll} \text{Truck going West} & 70\text{mph}(2.5\text{hours}) = 175\text{miles} \\ \text{Truck going East} & \underline{60\text{mph}(2.5\text{hours}) = 150\text{miles}} \\ & 325\text{miles} \checkmark \end{array}$$

Step 7. Answer the question with a complete sentence.
It will take the trucks 2.5 hours to be 325 miles apart.

Pierre and Monique leave their home in Portland at the same time. Pierre drives north on the turnpike at a speed of 75 miles per hour while Monique drives south at a speed of 68 miles per hour. How long will it take them to

be 429 miles apart?

Pierre and Monique will be 429 miles apart in 3 hours.

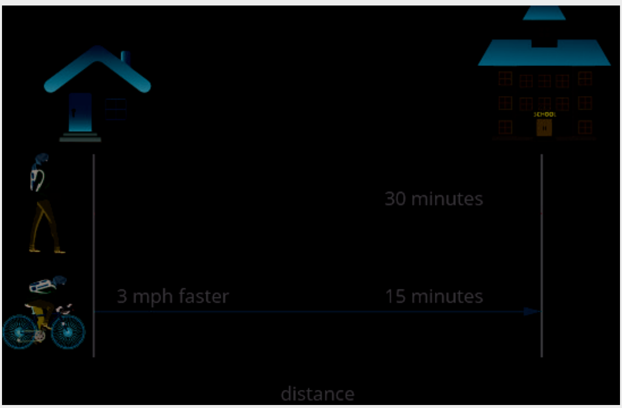
Thanh and Nhat leave their office in Sacramento at the same time. Thanh drives north on I-5 at a speed of 72 miles per hour. Nhat drives south on I-5 at a speed of 76 miles per hour. How long will it take them to be 330 miles apart?

Thanh and Nhat will be 330 miles apart in 2.2 hours.

It is important to make sure that the units match when we use the distance rate and time formula. For instance, if the rate is in miles per hour, then the time must be in hours.

When Naoko walks to school, it takes her 30 minutes. If she rides her bike, it takes her 15 minutes. Her speed is three miles per hour faster when she rides her bike than when she walks. What is her speed walking and her speed riding her bike?

First, we draw a diagram that represents the situation to help us see what is happening.



We are asked to find her speed walking and riding her bike. Let's call her walking speed *r*. Since her biking speed is three miles per hour faster, we will call that speed $r+3$. We write the speeds in the chart.

The speed is in miles per hour, so we need to express the times in hours, too, in order for the units to be the same. Remember, 1 hour is 60 minutes. So:

30 minutes is3060or12hour15 minutes is1560or14hour

We write the times in the chart.

Next, we multiply rate times time to fill in the distance column.

	Rate (mph)	· Time (hrs)	= Distance (miles)
Walk	r	$\frac{1}{2}$	$\frac{1}{2}r$
Bike	$r+3$	$\frac{1}{4}$	$\frac{1}{4}(r+3)$

The equation will come from the fact that the distance from Naoko's home to her school is the same whether she is walking or riding her bike.
So we say:

distance walked	=	distance covered by bike

Translate to an equation.

[redacted]

Solve this equation.

[redacted]

Clear the fractions by multiplying by the LCD of [redacted] the equation.

Simplify.

[redacted]

6

Let’s check if this works.
Walk 3 mph (0.5 hour) = 1.5milesBike 6 mph (0.25 hour) = 1.5miles

Yes, either way Naoko travels 1.5 miles to school.

Naoko’s walking speed is 3 mph and her speed riding her bike is 6 mph.

Suzy takes 50 minutes to hike uphill from the parking lot to the lookout tower. It takes her 30 minutes to hike back down to the parking lot. Her speed going downhill is 1.2 miles per hour faster than her speed going uphill. Find Suzy's uphill and downhill speeds.
Suzy's speed uphill is 1.8 mph and downhill is three mph.

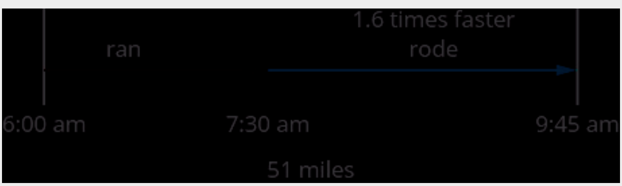
Llewyn takes 45 minutes to drive his boat upstream from the dock to his favorite fishing spot. It takes him 30 minutes to drive the boat back downstream to the dock. The boat's speed going downstream is four miles per hour faster than its speed going upstream. Find the boat's upstream and downstream speeds.
The boat's speed upstream is eight mph and downstream is12 mph.

In the distance, rate and time formula, time represents the actual amount of elapsed time (in

hours, minutes, etc.). If a problem gives us starting and ending times as clock times, we must find the elapsed time in order to use the formula.

Cruz is training to compete in a triathlon. He left his house at 6:00 and ran until 7:30. Then he rode his bike until 9:45. He covered a total distance of 51 miles. His speed when biking was 1.6 times his speed when running. Find Cruz's biking and running speeds.

A diagram will help us model this trip.



Next, we create a table to organize the information. We know the total distance is 51 miles. We are looking for the rate of speed for each part of the trip. The rate while biking is 1.6 times the rate of running. If we let $r =$ the rate running, then the rate biking is $1.6r$.

The times here are given as clock times. Cruz started from home at 6:00 a.m. and started biking at 7:30 a.m. So he spent 1.5 hours running. Then he biked from 7:30 a.m until 9:45 a.m. So he spent 2.25 hours biking.

Now, we multiply the rates by the times.

	Rate (mph) ·	Time (hrs) =	Distance (miles)
run	r	1.5	$1.5r$
bike	$1.6r$	2.25	$2.25(1.6r)$
			51

By looking at the diagram, we can see that the sum of the distance running and the distance biking is 255 miles.

[redacted]

Translate to an equation.

[redacted]

Solve this equation.

$$1.5r + 2.25(1.6r) = 51$$
$$1.5r + 3.6r = 51$$
$$5.1r = 51$$
$$r = 10 \text{ mph running}$$

$1.6r$ biking speed

$1.6(10)$

16 mph

Check.

$\text{Run} 10 \text{mph}(1.5 \text{hours}) = 15 \text{mi} \text{Bike} 16 \text{mph}(2.25 \text{hours}) = 36$

$\overline{\quad\quad} 51 \text{mi}$

Hamilton loves to travel to Las Vegas, 255 miles from his home in Orange County. On his last trip, he left his house at 2:00 p.m. The first part of his trip was on congested city freeways. At 4:00 pm, the traffic cleared and he was able to drive through the desert at a speed 1.75 times as fast as when he drove in the congested area. He arrived in Las Vegas at 6:30 p.m. How fast was he driving during each part of his trip?

Hamilton drove 40 mph in the city and 70 mph in the desert.

Phuong left home on his bicycle at 10:00. He rode on the flat street until 11:15, then rode uphill until 11:45. He rode a total of 31 miles. His speed riding uphill was 0.6 times his speed on the flat street. Find his speed biking uphill and on the flat street.
Phuong rode uphill at a speed of 12 mph and on the flat street at 20 mph.

Key Concepts

- **Total Value of Coins**
  For the same type of coin, the total value of a number of coins is found by using the model
  $\text{number}\cdot\text{value}=\text{totalvalue}$
  - *number* is the number of coins
  - *value* is the value of each coin
  - *total value* is the total value of all the coins
- **How to solve coin word problems.**

Read the problem. Make sure all the words and ideas are understood.
Determine the types of coins involved.
Create a table to organize the information.
Label the columns “type,” “number,” “value,” “total value.”
List the types of coins.
Write in the value of each type of coin.
Write in the total value of all the coins.

Type	Number •	Value ($) =	Total Value ($)

Identify what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity.
Use variable expressions to represent the number of each type of coin and write them in the table.
Multiply the number times the value to get the total value of each type of coin. **Translate** into an equation.
It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
Write the equation by adding the total values of all the types of coins. **Solve** the equation using good algebra techniques. **Check** the

answer in the problem and make sure it makes sense.

Answer the question with a complete sentence.

- **How To Solve a Uniform Motion Application**

  Read the problem. Make sure all the words and ideas are understood.
  Draw a diagram to illustrate what it happening.
  Create a table to organize the information.
  Label the columns rate, time, distance.
  List the two scenarios.
  Write in the information you know.

	Rate •	Time =	Distance

Identify what you are looking for. **Name** what you are looking for. Choose a variable to represent that quantity.
Complete the chart.
Use variable expressions to represent that quantity in each row.
Multiply the rate times the time to get the distance. **Translate** into an equation.
Restate the problem in one sentence with all

the important information.
Then, translate the sentence into an equation. **Solve** the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

Practice Makes Perfect

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

Michaela has $2.05 in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

---

nine nickels, 16 dimes

Liliana has $2.10 in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

In a cash drawer there is $125 in $5 and $10 bills. The number of $10 bills is twice the number of $5 bills. How many of each type of bill is in the drawer?

---

ten $10 bills, five $5 bills

Sumanta has $175 in $5 and $10 bills in his drawer. The number of $5 bills is three times the number of $10 bills. How many of each are in the drawer?

Chi has $11.30 in dimes and quarters. The number of dimes is three more than three times the number of quarters. How many of each are there?

---

63 dimes, 20 quarters

Alison has $9.70 in dimes and quarters. The number of quarters is eight more than four times the number of dimes. How many of each coin does she have?

Mukul has $3.75 in quarters, dimes and nickels in his pocket. He has five more dimes than

quarters and nine more nickels than quarters. How many of each coin are in his pocket?

---

16 nickels, 12 dimes, seven quarters

Vina has $4.70 in quarters, dimes and nickels in her purse. She has eight more dimes than quarters and six more nickels than quarters. How many of each coin are in her purse?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

The first day of a water polo tournament the total value of tickets sold was $17,610. One-day passes sold for $20 and tournament passes sold for $30. The number of tournament passes sold was 37 more than the number of day passes sold. How many day passes and how many tournament passes were sold?

---

330 day passes, 367 tournament passes

At the movie theater, the total value of tickets sold was $2,612.50. Adult tickets sold for $10

each and senior/child tickets sold for $7.50 each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

Julie went to the post office and bought both $0.41 stamps and $0.26 postcards. She spent $51.40. The number of stamps was 20 more than twice the number of postcards. How many of each did she buy?

---

40 postcards, 100 stamps

Jason went to the post office and bought both $0.41 stamps and $0.26 postcards and spent $10.28 The number of stamps was four more than twice the number of postcards. How many of each did he buy?

Hilda has $210 worth of $10 and $12 stock shares. The number of $10 shares is five more than twice the number of $12 shares. How many of each type of share does she have?

---

15 $10 shares, five $12 shares

Mario invested $475 in $45 and $25 stock shares. The number of $25 shares was five less than three times the number of $45 shares. How many of each type of share did he buy?

The ice rink sold 95 tickets for the afternoon skating session, for a total of $828. General admission tickets cost $10 each and youth tickets cost $8 each. How many general admission tickets and how many youth tickets were sold?

---

34 general, 61 youth

For the 7:30 show time, 140 movie tickets were sold. Receipts from the $13 adult tickets and the $10 senior tickets totaled $1,664. How many adult tickets and how many senior tickets were sold?

The box office sold 360 tickets to a concert at the college. The total receipts were $4,170. General admission tickets cost $15 and student tickets cost $10. How many of each kind of ticket was sold?

---

114 general, 246 student

Last Saturday, the museum box office sold 281 tickets for a total of $3,954. Adult tickets cost $15 and student tickets cost $12. How many of each kind of ticket was sold?

Solve Mixture Word Problems

In the following exercises, solve each mixture word problem.

Macario is making 12 pounds of nut mixture with macadamia nuts and almonds. Macadamia nuts cost $9 per pound and almonds cost $5.25 per pound. How many pounds of macadamia nuts and how many pounds of almonds should Macario use for the mixture to cost $6.50 per pound to make?

Four pounds of macadamia nuts, eight pounds almonds

Carmen wants to tile the floor of his house. He will need 1,000 square feet of tile. He will do most of the floor with a tile that costs $1.50 per square foot, but also wants to use an accent tile that costs $9.00 per square foot. How many square feet of each tile should he plan to use if he wants the overall cost to be $3 per square foot?

Riley is planning to plant a lawn in his yard. He will need nine pounds of grass seed. He wants to mix Bermuda seed that costs \$4.80 per pound with Fescue seed that costs \$3.50 per pound. How much of each seed should he buy so that the overall cost will be \$4.02 per pound?

3.6 lbs Bermuda seed, 5.4 lbs Fescue seed

Vartan was paid \$25,000 for a cell phone app that he wrote and wants to invest it to save for his son’s education. He wants to put some of the money into a bond that pays 4% annual interest and the rest into stocks that pay 9% annual interest. If he wants to earn 7.4% annual interest on the total amount, how much money should he invest in each account?

Vern sold his 1964 Ford Mustang for \$55,000 and wants to invest the money to earn him 5.8% interest per year. He will put some of the money into Fund A that earns 3% per year and the rest in Fund B that earns 10% per year. How much should he invest into each fund if he wants to earn 5.8% interest per year on the total amount?

$33,000 in Fund A, $22,000 in Fund B

Dominic pays 7% interest on his $15,000 college loan and 12% interest on his $11,000 car loan. What average interest rate does he pay on the total $26,000 he owes? (Round your answer to the nearest tenth of a percent.)

Liam borrowed a total of $35,000 to pay for college. He pays his parents 3% interest on the $8,000 he borrowed from them and pays the bank 6.8% on the rest. What average interest rate does he pay on the total $35,000? (Round your answer to the nearest tenth of a percent.)

5.9%

Solve Uniform Motion Applications

In the following exercises, solve.

Lilah is moving from Portland to Seattle. It takes her three hours to go by train. Mason leaves the train station in Portland and drives to the train station in Seattle with all Lilah's boxes in his car. It takes him 2.4 hours to get to Seattle, driving at 15 miles per hour faster than the speed of the train. Find Mason's speed and

the speed of the train.

Kathy and Cheryl are walking in a fundraiser. Kathy completes the course in 4.8 hours and Cheryl completes the course in eight hours. Kathy walks two miles per hour faster than Cheryl. Find Kathy's speed and Cheryl's speed.

Kathy 5 mph, Cheryl 3 mph

Two busses go from Sacramento to San Diego. The express bus makes the trip in 6.8 hours and the local bus takes 10.2 hours for the trip. The speed of the express bus is 25 mph faster than the speed of the local bus. Find the speed of both busses.

A commercial jet and a private airplane fly from Denver to Phoenix. It takes the commercial jet 1.6 hours for the flight, and it takes the private airplane 2.6 hours. The speed of the commercial jet is 210 miles per hour faster than the speed of the private airplane. Find the speed of both airplanes to the nearest 10 mph.

---

commercial 550 mph, private plane 340 mph

Saul drove his truck three hours from Dallas towards Kansas City and stopped at a truck stop to get dinner. At the truck stop he met Erwin, who had driven four hours from Kansas City towards Dallas. The distance between Dallas and Kansas City is 542 miles, and Erwin's speed was eight miles per hour slower than Saul's speed. Find the speed of the two truckers.

Charlie and Violet met for lunch at a restaurant between Memphis and New Orleans. Charlie had left Memphis and drove 4.8 hours towards New Orleans. Violet had left New Orleans and drove two hours towards Memphis, at a speed 10 miles per hour faster than Charlie's speed. The distance between Memphis and New Orleans is 394 miles. Find the speed of the two drivers.

Violet 65 mph, Charlie 55 mph

Sisters Helen and Anne live 332 miles apart. For Thanksgiving, they met at their other sister's house partway between their homes. Helen drove 3.2 hours and Anne drove 2.8 hours. Helen's average speed was four miles per hour faster than Anne's. Find Helen's average speed and Anne's average speed.

Ethan and Leo start riding their bikes at the opposite ends of a 65-mile bike path. After Ethan has ridden 1.5 hours and Leo has ridden two hours, they meet on the path. Ethan's speed is six miles per hour faster than Leo's speed. Find the speed of the two bikers.

Ethan 22 mph, Leo 16 mph

Elvira and Aletheia live 3.1 miles apart on the same street. They are in a study group that meets at a coffee shop between their houses. It took Elvira half an hour and Aletheia two-thirds of an hour to walk to the coffee shop. Aletheia's speed is 0.6 miles per hour slower than Elvira's speed. Find both women's walking speeds.

DaMarcus and Fabian live 23 miles apart and play soccer at a park between their homes. DaMarcus rode his bike for three-quarters of an hour and Fabian rode his bike for half an hour to get to the park. Fabian's speed was six miles per hour faster than DaMarcus' speed. Find the speed of both soccer players.

DaMarcus 16 mph, Fabian 22 mph

Cindy and Richard leave their dorm in Charleston at the same time. Cindy rides her bicycle north at a speed of 18 miles per hour. Richard rides his bicycle south at a speed of 14 miles per hour. How long will it take them to be 96 miles apart?

Matt and Chris leave their uncle’s house in Phoenix at the same time. Matt drives west on I-60 at a speed of 76 miles per hour. Chris drives east on I-60 at a speed of 82 miles per hour. How many hours will it take them to be 632 miles apart?

---

four hours

Two busses leave Billings at the same time. The Seattle bus heads west on I-90 at a speed of 73 miles per hour while the Chicago bus heads east at a speed of 79 miles an hour. How many hours will it take them to be 532 miles apart?

Two boats leave the same dock in Cairo at the same time. One heads north on the Mississippi River while the other heads south. The northbound boat travels four miles per hour. The southbound boat goes eight miles per hour. How long will it take them to be 54 miles

apart?

4.5 hours

Lorena walks the path around the park in 30 minutes. If she jogs, it takes her 20 minutes. Her jogging speed is 1.5 miles per hour faster than her walking speed. Find Lorena's walking speed and jogging speed.

Julian rides his bike uphill for 45 minutes, then turns around and rides back downhill. It takes him 15 minutes to get back to where he started. His uphill speed is 3.2 miles per hour slower than his downhill speed. Find Julian's uphill and downhill speed.

---

uphill 1.6 mph, downhill 4.8 mph

Cassius drives his boat upstream for 45 minutes. It takes him 30 minutes to return downstream. His speed going upstream is three miles per hour slower than his speed going downstream. Find his upstream and downstream speeds.

It takes Darline 20 minutes to drive to work in light traffic. To come home, when there is heavy traffic, it takes her 36 minutes. Her speed in light traffic is 24 miles per hour faster than her speed in heavy traffic. Find her speed in light traffic and in heavy traffic.

---

light traffic 54 mph, heavy traffic 30 mph

At 1:30, Marlon left his house to go to the beach, a distance of 7.6 miles. He rode his skateboard until 2:15, and then walked the rest of the way. He arrived at the beach at 3:00. Marlon's speed on his skateboard is 2.5 times his walking speed. Find his speed when skateboarding and when walking.

Aaron left at 9:15 to drive to his mountain cabin 108 miles away. He drove on the freeway until 10:45 and then drove on a mountain road. He arrived at 11:05. His speed on the freeway was three times his speed on the mountain road. Find Aaron's speed on the freeway and on the mountain road.

---

freeway 67 mph, mountain road 22.3 mph

Marisol left Los Angeles at 2:30 to drive to Santa Barbara, a distance of 95 miles. The traffic was heavy until 3:20. She drove the rest of the way in very light traffic and arrived at 4:20. Her speed in heavy traffic was 40 miles per hour slower than her speed in light traffic. Find her speed in heavy traffic and in light traffic.

Lizette is training for a marathon. At 7:00 she left her house and ran until 8:15 then she walked until 11:15. She covered a total distance of 19 miles. Her running speed was five miles per hour faster than her walking speed. Find her running and walking speeds.

---

running eight mph, walking three mph

Everyday Math

John left his house in Irvine at 8:35 a.m. to drive to a meeting in Los Angeles, 45 miles away. He arrived at the meeting at 9:50 a.m.. At 5:30 p.m. he left the meeting and drove home. He arrived home at 7:18 p.m.

ⓐ What was his average speed on the drive

from Irvine to Los Angeles?

ⓑ What was his average speed on the drive from Los Angeles to Irvine?

ⓒ What was the total time he spent driving to and from this meeting?

Sarah wants to arrive at her friend's wedding at 3:00. The distance from Sarah's house to the wedding is 95 miles. Based on usual traffic patterns, Sarah predicts she can drive the first 15 miles at 60 miles per hour, the next 10 miles at 30 miles per hour, and the remainder of the drive at 70 miles per hour.

ⓐ How long will it take Sarah to drive the first 15 miles?

ⓑ How long will it take Sarah to drive the next 10 miles?

ⓒ How long will it take Sarah to drive the rest of the trip?

ⓓ What time should Sarah leave her house?

ⓐ 15 minutes ⓑ 20 minutes ⓒ one hour (d) 1:25

Writing Exercises

Suppose you have six quarters, nine dimes, and four pennies. Explain how you find the total value of all the coins.

Do you find it helpful to use a table when solving coin problems? Why or why not?

Answers will vary.

In the table used to solve coin problems, one column is labeled “number” and another column is labeled “value.” What is the difference between the “number” and the “value”?

When solving a uniform motion problem, how does drawing a diagram of the situation help you?

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve coin word problems.			
solve ticket and stamp word problems.			
solve mixture word problems.			
solve uniform motion applications.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

2.7 Linear Inequalities and Absolute Value Inequalities

In this section you will:

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

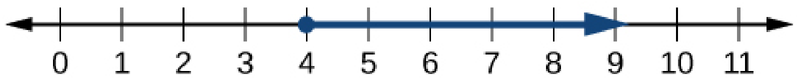
It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value

inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in [link]. The blue ray begins at $x=4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.



We can use set-builder notation: $\{x | x \geq 4\}$, which translates to "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater

than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2,6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1,0)$, all real numbers between, but not including -1 and 0; and $(-\infty,1]$, all real numbers less than and including 1. [link] outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, but not including *a* or *b*	$\{x \mid a<x<b\}$	(a,b)
All real numbers greater than *a*, but not including *a*	$\{x \mid x>a\}$	(a,∞)
All real numbers less than *b*, but not including *b*	$\{x \mid x<b\}$	$(-\infty,b)$
All real numbers greater than *a*,	$\{x \mid x\geq a\}$	$[a,\infty)$

including a		
All real numbers less than b, including b	$\{ x \mid x \le b \}$	$(-\infty, b]$
All real numbers between a and b, including a	$\{ x \mid a \le x < b \}$	$[a, b)$
All real numbers between a and b, including b	$\{ x \mid a < x \le b \}$	$(a, b]$
All real numbers between a and b, including a and b	$\{ x \mid a \le x \le b \}$	$[a, b]$
All real numbers less than a or greater than b	$\{ x \mid x < a \text{ and } x > b \}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{ x \mid x \text{ is all real numbers} \}$	$(-\infty, \infty)$

Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

Use a bracket on the left of -2 and

parentheses after infinity: $[-2,\infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Use interval notation to indicate all real numbers between and including -3 and 5.

$[-3,5]$

Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1,

which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, ∪, between the two intervals.

$(-\infty, -1] \cup [1, \infty)$

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

$(-\infty, -2) \cup [3, \infty)$

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The

one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

Properties of Inequalities
Addition Property If $a < b$, then $a + c < b + c$.
Multiplication Property
If $a < b$ and $c > 0$, then $ac < bc$.
If $a < b$ and $c < 0$, then $ac > bc$.
These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

- (a) $x - 15 < 4$
- (b) $6 \geq x - 1$
- (c) $x + 7 > 9$

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

1.

   $x-15<4$ $x-15+15<4+15$
   Add 15 to both sides. $x<19$
2.

   $6\geq x-1$ $6+1\geq x-1+1$
   Add 1 to both sides. $7\geq x$
3.

   $x+7>9$ $x+7-7>9-7$
   Subtract 7 from both sides. $x>2$

Solve: $3x-2<1$.

$x<1$

Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

1. $3x<6$
2. $-2x-1\geq 5$
3. $5-x>10$

1.

$3x<6 \ \frac{1}{3}(3x)<(6)\frac{1}{3} \ x<2$

2.

$-2x-1\ge 5 \ -2x\ge 6 \ \left(-\frac{1}{2}\right)(-2x)\ge(6)\left(-\frac{1}{2}\right)$ Multiply by $-\frac{1}{2}$. $x\le -3$ Reverse the inequality.

3.

$5-x>10 \ -x>5 \ (-1)(-x)>(5)(-1)$ Multiply by -1. $x<-5$ Reverse the inequality.

Solve: $4x+7\ge 2x-3$.

$x\ge -5$

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just

as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Solving an Inequality Algebraically

Solve the inequality: $13-7x\geq 10x-4$.

Solving this inequality is similar to solving an equation up until the last step.

$13-7x\geq 10x-4$ $13-17x\geq -4$ Move variable terms to one side of the inequality. $-17x\geq -17$ Isolate the variable term. $x\leq 1$ Dividing both sides by -17 reverses the inequality.

The solution set is given by the interval $(-\infty,1]$, or all real numbers less than and including 1.

Solve the inequality and write the answer using interval notation: $-x+4<\frac{1}{2}x+1$.

$(2,\infty)$

Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

We begin solving in the same way we do when solving an equation.

$-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$ $\quad -\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8}$ Put variable terms on one side. $-\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8}$

Write fractions with common denominator. $-\frac{17}{12}x \geq -\frac{5}{8}$ $\quad x \leq -\frac{5}{8}\left(-\frac{12}{17}\right)$

Multiplying by a negative number reverses the inequalit

$x \leq \frac{15}{34}$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

$\left[-\frac{3}{14}, \infty\right)$

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Solving a Compound Inequality Solve the compound inequality: $3 \leq 2x + 2 < 6$.
The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently. $3 \leq 2x + 2$ and $2x + 2 < 6$ $1 \leq 2x$ $2x < 4$ $1\ 2 \leq x$ $x < 2$ Then, we can rewrite the solution as a compound inequality, the same way the problem began. $1\ 2 \leq x < 2$

In interval notation, the solution is written as
[1 2 ,2).

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.
$3 \le 2x + 2 < 6$ $1 \le 2x < 4$
Isolate the variable term, and subtract 2 from all three p
1 2 $\le x < 2$ Divide through all three parts by 2.

We get the same solution: [1 2 ,2).

Solve the compound inequality: $4 < 2x - 8 \le 10$.

$6 < x \le 9$ or (6,9]

Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Let's try the first method. Write two inequalities:

$$3+x>7x-2 \quad \text{and} \quad 7x-2>5x-10$$
$$3>6x-2 \qquad 2x-2>-10$$
$$5>6x \qquad 2x>-8$$
$$\frac{5}{6}>x \qquad x>-4$$
$$x<\frac{5}{6} \qquad -4<x$$

The solution set is $-4<x<\frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See [link].

Solve the compound inequality: $3y<4-5y<5+3y$.

$\left(-\frac{1}{8}, \frac{1}{2}\right)$

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x,0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form
$|A|<B, |A|\leq B, |A|>B,$ or $|A|\geq B,$

Where *A*, and sometimes *B*, represents an algebraic expression dependent on a variable *x*. Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an

investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of *x*-values such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$, and therefore, $|x-600|\le 200$ or

$$-200\le x-600\le 200 \quad -200+600\le x-600+600\le 200+600 \quad 400\le x\le 800$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Absolute Value Inequalities

For an algebraic expression *X,* and $k>0$, an **absolute value inequality** is an inequality of the form

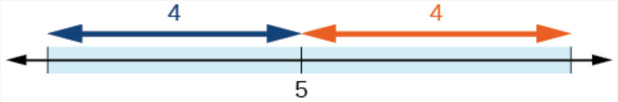
$|X|<k$ is equivalent to $-k<X<k$ $|X|>k$ is equivalent to $X<-k$ or $X>k$

These statements also apply to $|X|\le k$ and $|X|\ge k$.

Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in [link], to represent the condition to be satisfied.



The distance from x to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x-5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$x-5 \leq 4 \text{ and } x-5 \geq -4 \quad x \leq 9 \quad x \geq 1$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \leq 4$ is equivalent to $[1,9]$ in interval notation.

Describe all *x*-values within a distance of 3 from the number 2.

$|x-2| \leq 3$

Solving an Absolute Value Inequality

Solve $|x-1| \leq 3$.

$|x-1| \leq 3$ $-3 \leq x-1 \leq 3$ $-2 \leq x \leq 4$ $[-2,4]$

Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\ 1\ 2\ |4x-5|+3$, determine the *x*-values for which the *y*-values are negative.

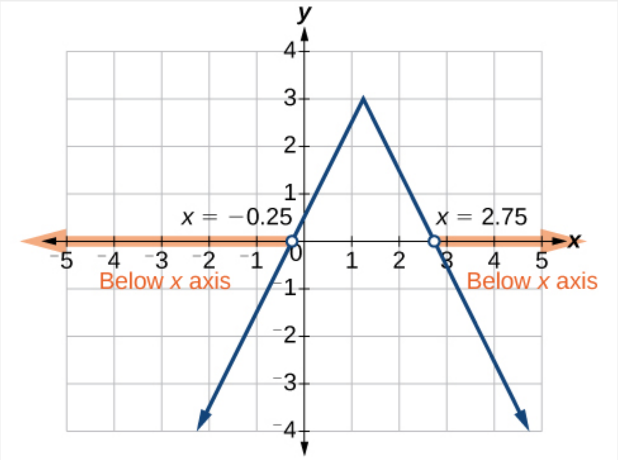
We are trying to determine where $y<0$, which is when $-\ 1\ 2\ |4x-5|+3<0$. We begin by isolating the absolute value.

$-\ 1\ 2\ |4x-5|<-3$ Multiply both sides by –

2, and reverse the inequality. $|4x-5|>6$

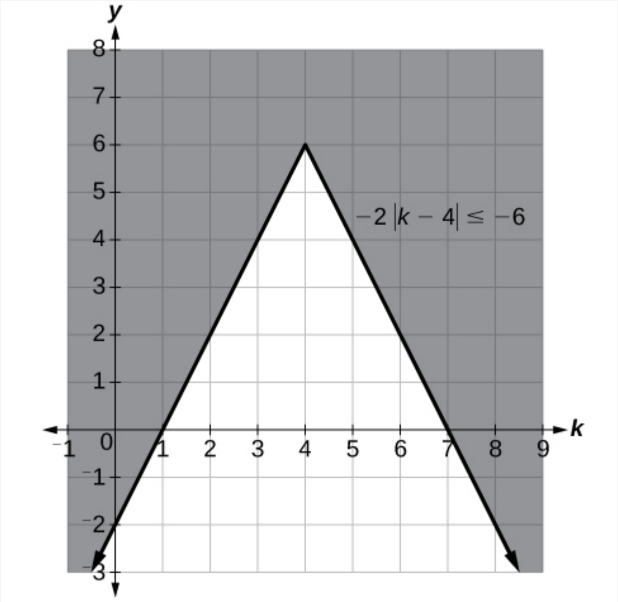
Next, we solve for the equality $|4x-5|=6$.
$4x-5=6$ $4x-5=-6$ $4x=11$ or $4x=-1$ $x=\frac{11}{4}$ $x=-\frac{1}{4}$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$, and that the graph opens downward. See [link].



Solve $-2|k-4|\le -6$.

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$.



Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

- Interval notation
- How to solve linear inequalities
- How to solve an inequality
- Absolute value equations
- Compound inequalities

- Absolute value inequalities

Key Concepts

- Interval notation is a method to indicate the solution set to an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a set and whether the endpoints are included as well. See [link] and [link].
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality. See [link], [link], [link], and [link].
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values, which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities. See [link] and [link].
- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value. See [link] and [link].

- Absolute value inequalities can also be solved by graphing. At least we can check the algebraic solutions by graphing, as we cannot depend on a visual for a precise solution. See [link].

Section Exercises

Verbal

When solving an inequality, explain what happened from Step 1 to Step 2:

$\text{Step 1} \ -2x > 6 \ \text{Step 2} \ x < -3$

When we divide both sides by a negative it changes the sign of both sides so the sense of the inequality sign changes.

When solving an inequality, we arrive at:

$x + 2 < x + 3 \ 2 < 3$

Explain what our solution set is.

When writing our solution in interval notation, how do we represent all the real numbers?

$(-\infty, \infty)$

When solving an inequality, we arrive at:

$x+2>x+3$ $2>3$

Explain what our solution set is.

Describe how to graph $y=|x-3|$

We start by finding the *x*-intercept, or where the function = 0. Once we have that point, which is (3,0), we graph to the right the straight line graph $y=x-3$, and then when we draw it to the left we plot positive *y* values, taking the absolute value of them.

Algebraic

For the following exercises, solve the inequality. Write your final answer in interval notation.

$4x-7 \leq 9$

$3x+2 \geq 7x-1$

$(-\infty, 3\ 4]$

$-2x+3 > x-5$

$4(x+3) \geq 2x-1$

$[-13\ 2 , \infty)$

$-\ 1\ 2\ x \leq\ -5\ 4\ +\ 2\ 5\ x$

$-5(x-1)+3 > 3x-4-4x$

$(-\infty, 3)$

$-3(2x+1) > -2(x+4)$

$x+3\ 8\ -\ x+5\ 5\ \geq\ 3\ 10$

$(-\infty, -\ 37\ 3]$

$\frac{x-1}{3} + \frac{x+2}{5} \le \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

$|x+9| \ge -6$

All real numbers $(-\infty, \infty)$

$|2x+3| < 7$

$|3x-1| > 11$

$(-\infty, -\frac{10}{3}) \cup (4, \infty)$

$|2x+1| + 1 \le 6$

$|x-2| + 4 \ge 10$

$(-\infty, -4] \cup [8, +\infty)$

$|-2x+7| \le 13$

$|x-7|<-4$

No solution

$|x-20|>-1$

$\left| x-\frac{3}{4} \right|<2$

$\left(-5,11 \right)$

For the following exercises, describe all the *x*-values within or including a distance of the given values.

Distance of 5 units from the number 7

Distance of 3 units from the number 9

$[\ 6,12\ ]$

Distance of10 units from the number 4

Distance of 11 units from the number 1

$[\,-10,12\,]$

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

$-4 < 3x + 2 \leq 18$

$3x + 1 > 2x - 5 > x - 7$

$x > -6$ and $x > -2$
Take the intersection of two sets. $x > -2,$
$(-2, +\infty)$

$3y < 5 - 2y < 7 + y$

$2x - 5 < -11 \quad \text{or} \quad 5x + 1 \geq 6$

$x < -3 \quad \text{or} \quad x \geq 1$
Take the union of the two sets. $(-\infty, -3) \cup$
$[1, \infty)$

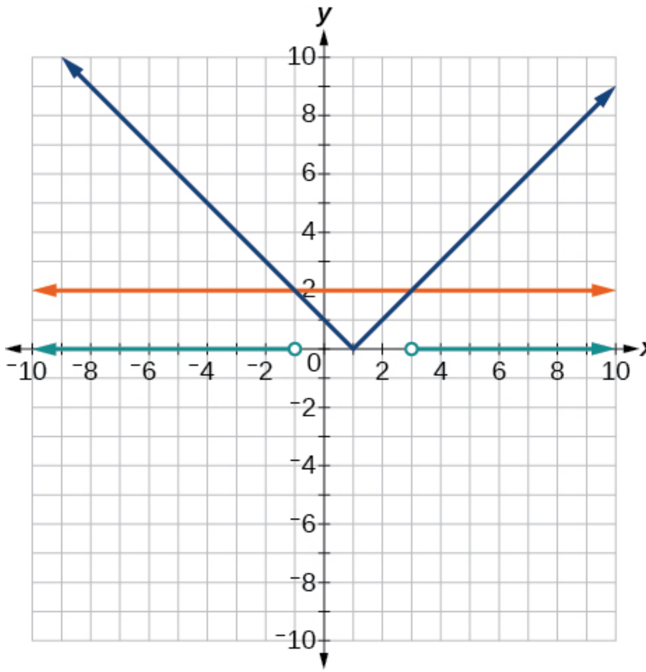
$x + 7 < x + 2$

Graphical

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

$$|x-1|>2$$

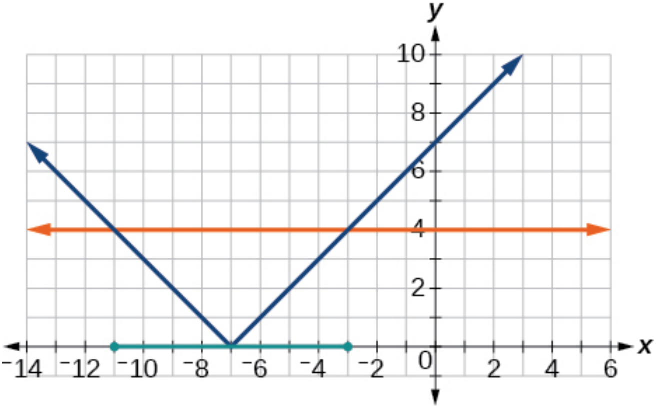
$$(-\infty,-1)\cup(3,\infty)$$



$|x+3| \geq 5$

$|x+7| \leq 4$

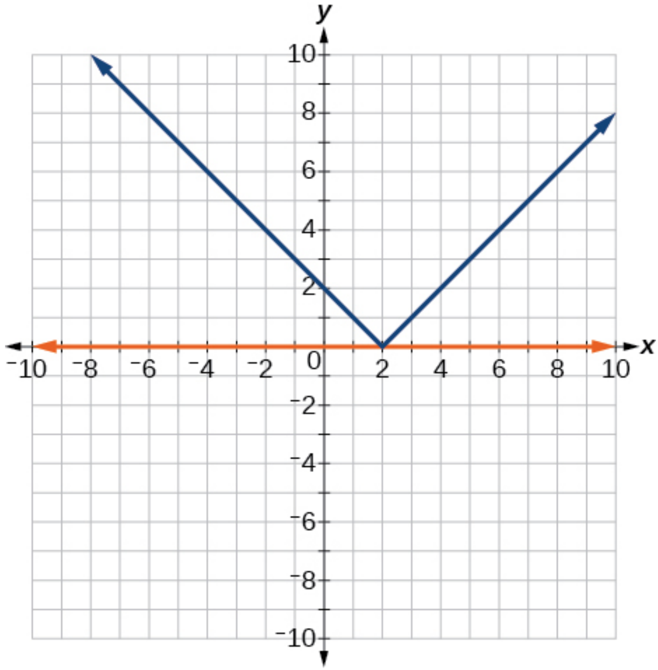
$[-11, -3]$



$|x-2| < 7$

$|x-2| < 0$

It is never less than zero. No solution.



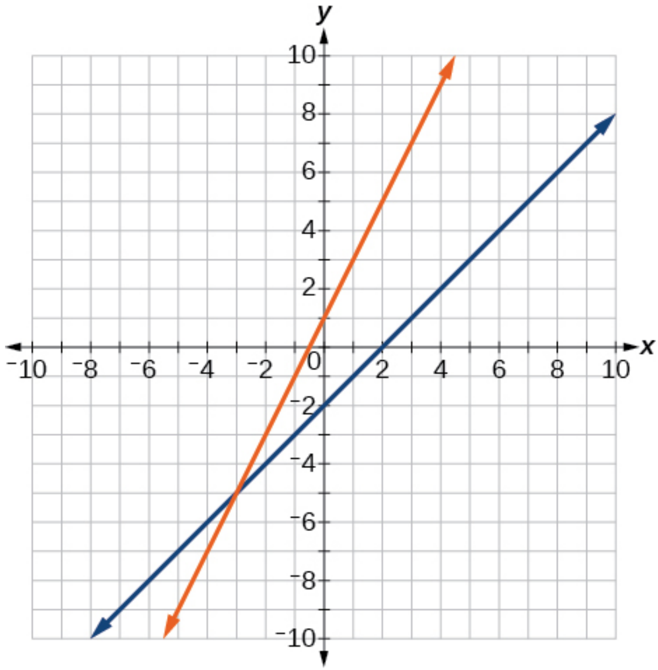
For the following exercises, graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the *y*-values of the lines.

$x+3<3x-4$

$x-2>2x+1$

Where the blue line is above the orange line; point of intersection is $x=-3$.

$(-\infty,-3)$

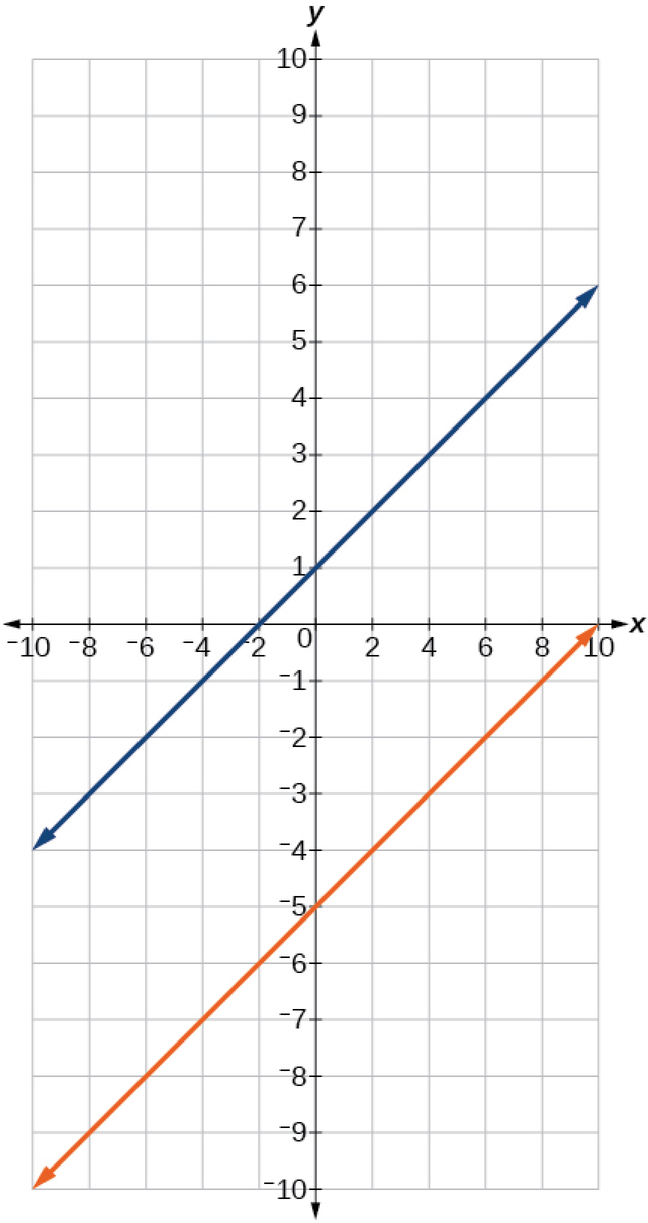


$x+1>x+4$

$1\;2\,x+1>1\;2\,x-5$

Where the blue line is above the orange line; always. All real numbers.

$(-\infty, -\infty)$



$4x + 1 < 1\ 2\ x + 3$

Numeric

For the following exercises, write the set in interval notation.

$\{ x | -1 < x < 3 \}$

$(-1,3)$

$\{ x | x \geq 7 \}$

$\{ x | x < 4 \}$

$(-\infty,4)$

$\{ x | \text{ x is all real numbers} \}$

For the following exercises, write the interval in set-builder notation.

$(-\infty,6)$

$\{ x | x < 6 \}$

$(4, +\infty)$

$[-3,5)$

$\{x \mid -3 \le x < 5\}$

$[-4,1] \cup [9, \infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

$(-2,1]$

$(-\infty, 4]$

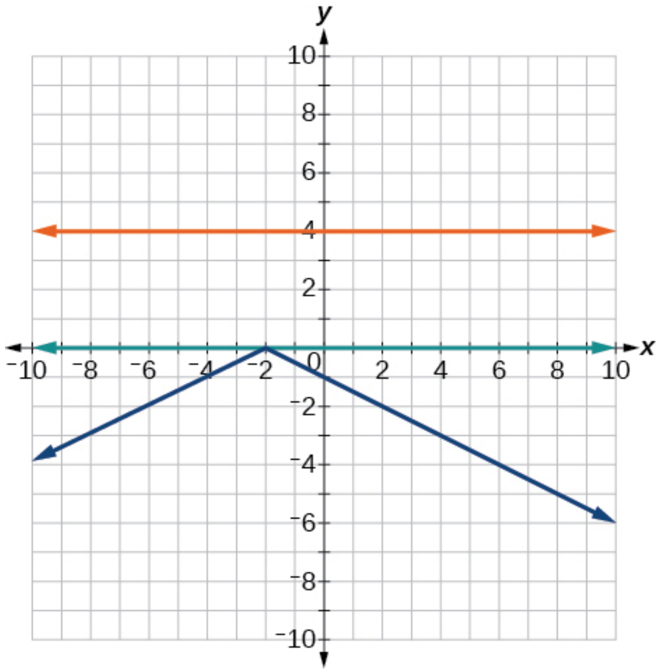
Technology

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter y2 = the right-hand side. Entering the absolute value of an expression is found in the MATH menu, Num, 1:abs(. Find the points of intersection, recall (2nd CALC 5:intersection, 1st curve, enter, 2nd curve, enter, guess, enter). Copy a sketch of the graph and shade the *x*-axis for your solution set to the inequality. Write final answers in interval notation.

$|x+2|-5<2$

$-1\ 2\,|x+2|<4$

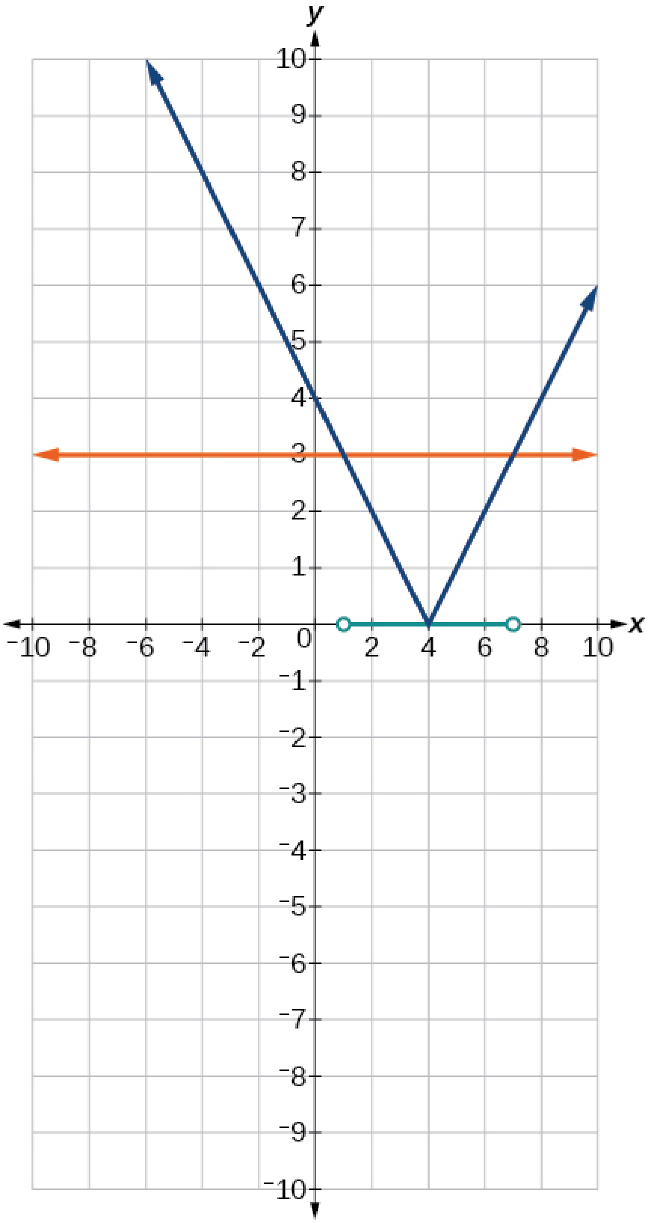
Where the blue is below the orange; always. All real numbers. $(-\infty, +\infty)$.



$$|\ 4x+1\ |-3>2$$

$$|\ x-4\ |<3$$

Where the blue is below the orange; (1,7).



$|\ x+2\ |\geq 5$

Extensions

Solve $\mid 3x+1 \mid = \mid 2x+3 \mid$

$x=2,\ -4\ 5$

Solve $x\ 2\ -x>12$

$x-5\ x+7\ \le 0,\ x \ne -7$

$(\ -7,5\ ]$

$p=-\ x\ 2\ +130x-3000$ is a profit formula for a small business. Find the set of *x*-values that will keep this profit positive.

Real-World Applications

In chemistry the volume for a certain gas is given by $V=20T$, where *V* is measured in cc and *T* is temperature in ºC. If the temperature varies between 80ºC and 120ºC, find the set of volume values.

$80 \leq T \leq 120$ $1{,}600 \leq 20T \leq 2{,}400$

[1,600, 2,400]

A basic cellular package costs $20/mo. for 60 min of calling, with an additional charge of $.30/min beyond that time.. The cost formula would be $C = \$20 + .30(x - 60)$. If you have to keep your bill lower than $50, what is the maximum calling minutes you can use?

Glossary

compound inequality
: a problem or a statement that includes two inequalities

interval
: an interval describes a set of numbers within which a solution falls

interval notation
: a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear inequality
: similar to a linear equation except that the solutions will include sets of numbers

Functions and Function Notation

In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.
 (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first. $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

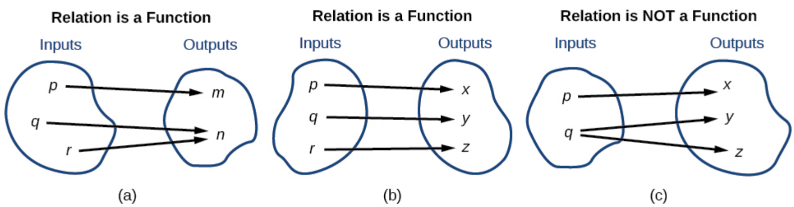
A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1,2,3,4,5\}$, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Now let’s consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as

{ (odd,1),(even,2),(odd,3),(even,4),(odd,5) }

Notice that each element in the domain, {even,odd} is *not* paired with exactly one element in the range, {1,2,3,4,5}. For example, the term "odd" corresponds to three values from the range, {1,3,5} and the term "even" corresponds to two values from the range, {2,4}. This violates the definition of a function, so this relation is not a function.

[link] compares relations that are functions and not functions.



Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

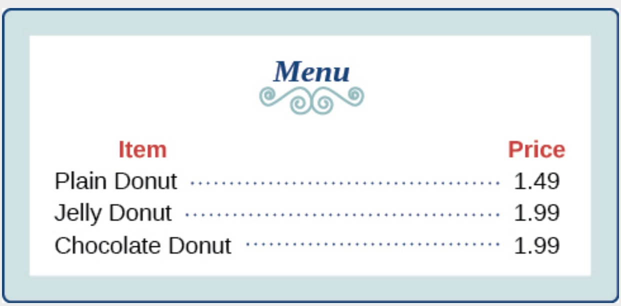
Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in [link] consists of items and their prices.

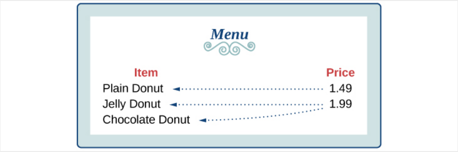
1. Is price a function of the item?
2. Is the item a function of the price?



1. Let's begin by considering the input as the items on the menu. The output values are then the prices. See [link].

Each item on the menu has only one price, so the price is a function of the item.

2. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See [link].



Therefore, the item is a not a function of price.

Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the

grade point average? [link] shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

[link][footnote] lists the five greatest baseball players of all time in order of rank.
http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

ⓐ Is the rank a function of the player name?
ⓑ Is the player name a function of the rank?

ⓐ yes
ⓑ yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional

relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent “height is a function of age,” we start by identifying the descriptive variables h for height and a for age. The letters f,g, and h are often used to represent functions just as we use x,y, and z to represent numbers and A,B, and C to represent sets.

h is f of a

We name the function f; height is a function of age.

$h=f(a)$

We use parentheses to indicate the function input.

$f(a)$

We name the function f; the expression is read as “f of a.”

Remember, we can use any letter to name the function; the notation h(a) shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means “first add *a* and *b*, and the result is the input for the function *f*.” The operations must be performed in this order to obtain the correct result.

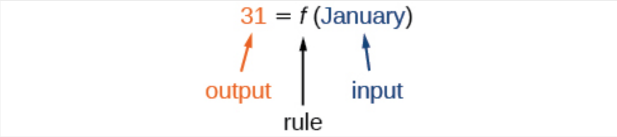
Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.



For example, $f(\text{March}) = 31$, because March

has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

$w = f(d)$

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "*y* is a function of *x*?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we

know about the relationship; other times, the table provides a few select examples from a more complete relationship.

[link] lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D=f(m)$ identifies months by an integer rather than by name.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

[link] defines a function $Q=g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

[link] displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Given a table of input and output values, determine whether the table represents a

function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Identifying Tables that Represent Functions

Which table, [link], [link], or [link], represents a function (if any)?

Input	Output
2	1
5	3
8	6

Input	Output
3	5

0	1
4	5

Input	Output
1	0
5	2
5	4

[link] and [link] define functions. In both, each input value corresponds to exactly one output value. [link] does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by [link] can be represented by writing

$$f(2)=1, f(5)=3, \text{and } f(8)=6$$

Similarly, the statements

$$g(-3)=5, g(0)=1, \text{and } g(4)=5$$

represent the function in [link].

[link] cannot be expressed in a similar way because it does not represent a function.

Does [link] represent a function?

Input	Output
1	10
2	100
3	1000

yes

http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x)=5-3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at

1. 2
2. a
3. $a + h$
4. $\frac{f(a+h) - f(a)}{h}$

Replace the x in the function with each specified value.

1. Because the input value is a number, 2, we can use simple algebra to simplify.
   $f(2) = 2^2 + 3(2) - 4 = 4 + 6 - 4 = 6$
2. In this case, the input value is a letter so we cannot simplify the answer any further.
   $f(a) = a^2 + 3a - 4$
3. With an input value of $a + h$, we must use the distributive property.
   $f(a+h) = (a+h)^2 + 3(a+h) - 4 = a^2 + 2ah + h^2 + 3a + 3h - 4$
4. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
   $f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$

   and we know that

$f(a) = a^2 + 3a - 4$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{2ah+h^2+3h}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out } h. \\ &= 2a+h+3 && \text{Simplify.} \end{aligned}$$

Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Given the function $g(m) = m - 4$, evaluate $g(5)$.

$g(5) = 1$

Solving Functions

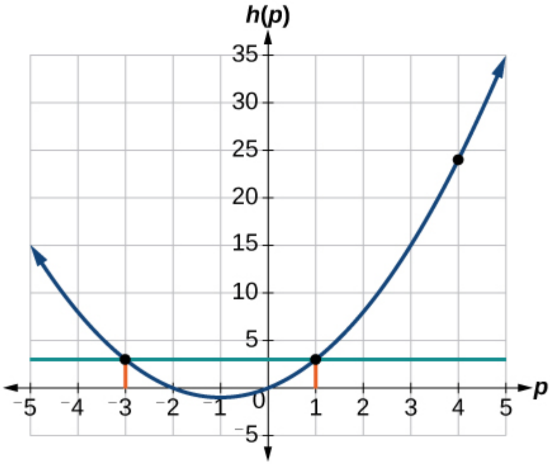
Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

$$\begin{array}{ll} h(p) = 3 & \\ p^2 + 2p = 3 & \text{Substitute the original function } h(p) = p^2 + 2p. \\ p^2 + 2p - 3 = 0 & \text{Subtract 3 from each side.} \\ (p+3)(p-1) = 0 & \text{Factor.} \end{array}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0, \; p = -3 \qquad (p-1) = 0, \; p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in [link]. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
$h(p)$	3	0	0	3	24

Given the function $g(m) = \sqrt{m - 4}$, solve $g(m) = 2$.

$m = 8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a

formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

To express the relationship in this form, we need to be able to write the relationship where

p is a function of n, which means writing it as p = [expressioninvolvingn].

$2n+6p=12$ $6p=12-2n$

Subtract 2n from both sides. $p=$ $12-2n$ 6

Divide both sides by 6 and simplify. $p=$ 12 6 $-$ $2n$ 6 $p=2-$ 1 3 n

Therefore, p as a function of n is written as $p=f(n)=2-$ 1 3 n

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Expressing the Equation of a Circle as a Function

Does the equation x 2 $+$ y 2 $=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

First we subtract x 2 from both sides.

y 2 $=1-$ x 2

We now try to solve for y in this equation.

$y=\pm$ $1-$ x 2 $=+$ $1-$ x 2 and $-$ $1-$ x 2

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(\ x\ )$.

If $x-8\ y\ 3\ =0$, express y as a function of x.

$y=f(\ x\ )=\ x\ 3\ 2$

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+\ 2\ y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See [link]. [footnote]

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Evaluating and Solving a Tabular Function

Using [link],

1. Evaluate $g(3)$.
2. Solve $g(n) = 6$.

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

1. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
2. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. [link] shows two solutions: 2 and 4.

n	1	2	3	4	5
g(n)	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Using [link], evaluate g(1).

$g(1)=8$

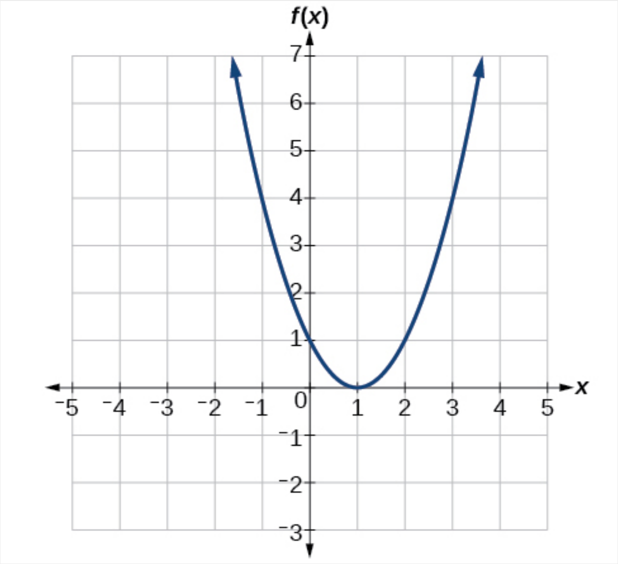
Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

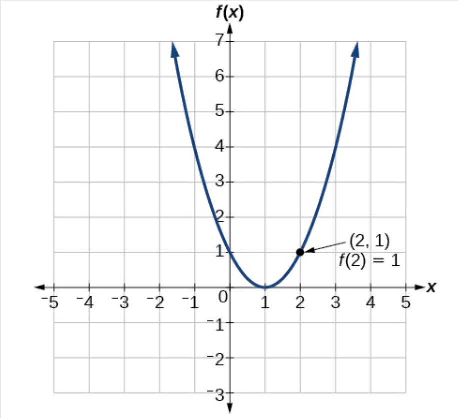
Reading Function Values from a Graph

Given the graph in [link],

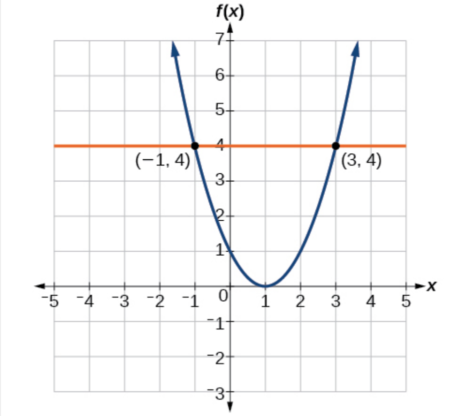
1. Evaluate $f(2)$.
2. Solve $f(x)=4$.



1. To evaluate $f(2)$, locate the point on the curve where $x=2$, then read the *y*-coordinate of that point. The point has coordinates $(2,1)$, so $f(2)=1$. See [link].



2. To solve $f(x)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value 4: $(-1,4)$ and $(3,4)$. These points represent the two solutions to $f(x)=4$: -1 or 3. This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4. See [link].



Using [link], solve $f(x)=1$.
$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in [link] at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in [link].

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let’s look again at the two simple functions sketched in [link]**(a)** and [link]**(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

A circle of radius r has a unique area measure given by $A = \pi r 2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r 2$. Because areas and radii are positive numbers, there is exactly one solution: $A \pi$. So the area of a circle is a one-to-one function of the circle's radius.

ⓐ Is a balance a function of the bank account number?
ⓑ Is a bank account number a function of the balance?
ⓒ Is a balance a one-to-one function of the bank account number?

ⓐ yes, because each bank account has a single balance at any given time
ⓑ no, because several bank account

numbers may have the same balance
ⓒ no, because the same output may correspond to more than one input.

Evaluate the following:

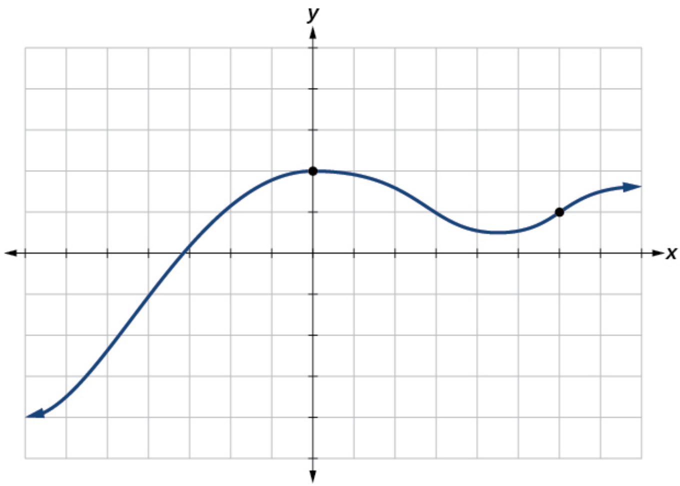
ⓐ If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
ⓑ If so, is the function one-to-one?

ⓐ Yes, letter grade is a function of percent grade;
ⓑ No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

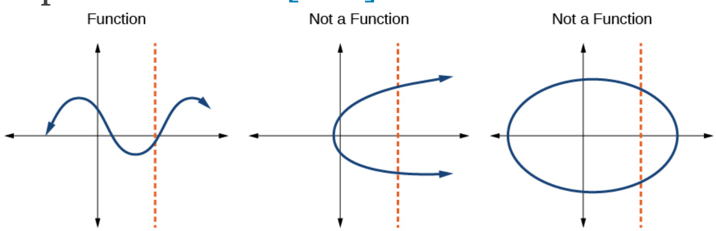
Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y=f(x)$ when the function is named f. The graph of the function is the set of all points (x,y) in the plane that satisfies the equation $y=f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in [link] tell us that $f(0)=2$ and $f(6)=1$. However, the set of all points (x,y) satisfying $y=f(x)$ is a curve. The curve shown includes $(0,2)$ and $(6,1)$ because the curve passes through those points.



The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See [link].



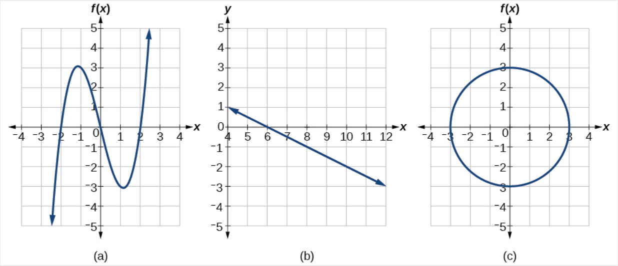
Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.

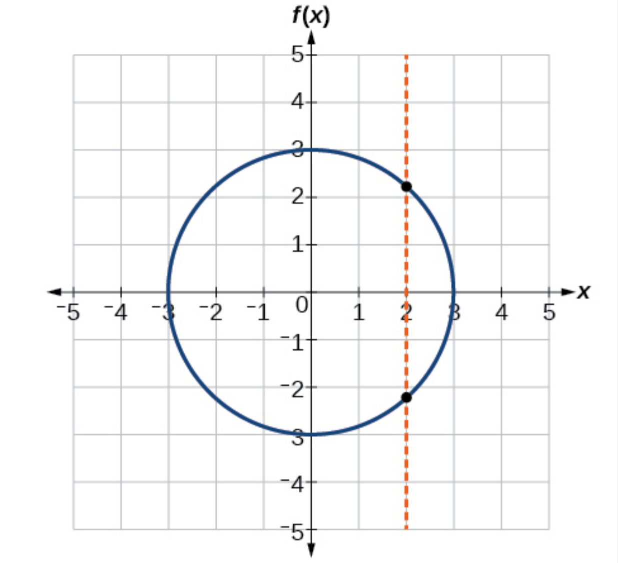
2. If there is any such line, determine that the graph does not represent a function.

Applying the Vertical Line Test

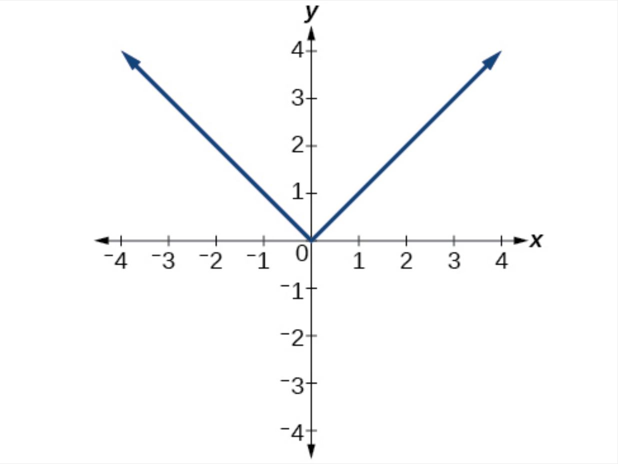
Which of the graphs in [link] represent(s) a function $y=f(x)$?



If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of [link]. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most *x*-values, a vertical line would intersect the graph at more than one point, as shown in [link].



Does the graph in [link] represent a function?



yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

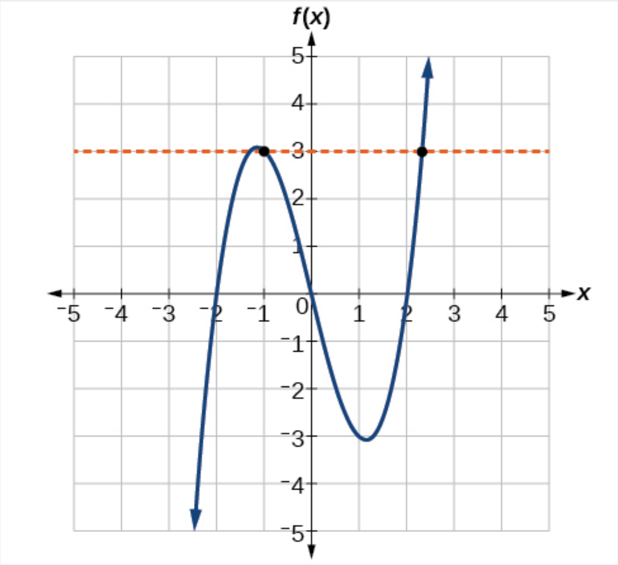
Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Applying the Horizontal Line Test

Consider the functions shown in [link]**(a)** and [link]**(b)**. Are either of the functions one-to-one?

The function in [link]**(a)** is not one-to-one. The horizontal line shown in [link] intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)



The function in [link]**(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Is the graph shown in [link] one-to-one?
No, because it does not pass the horizontal line test.

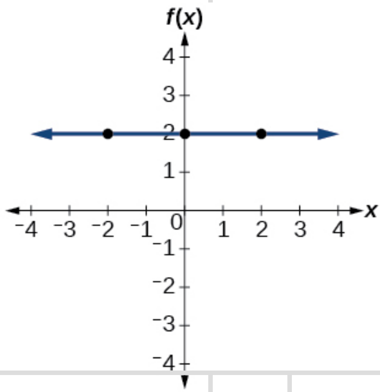
Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y=f(x)$ as the output variable.

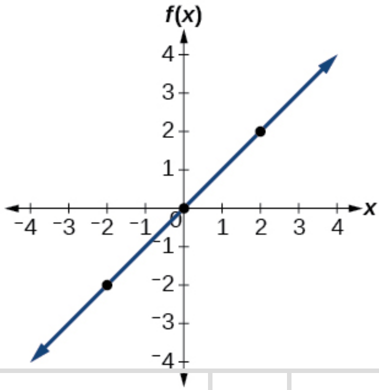
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their

transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in [link].

Toolkit Functions

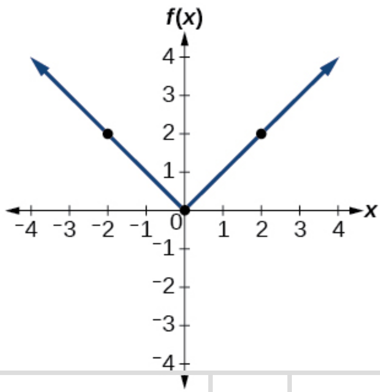
Name	**Function**	**Graph**
Constant	$f(x)=c$, where c is a constant	
Identity	$f(x)=x$	

x	$f(x)$
−2	2
0	2
2	2



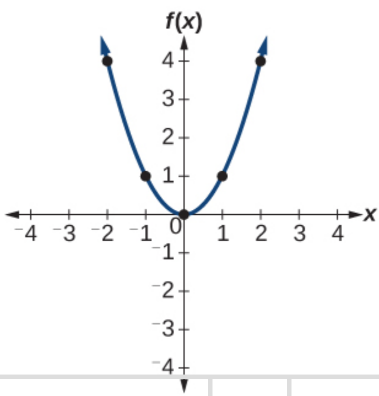
x	$f(x)$
−2	−2
0	0
2	2

Absolute value	$f(x)=\lvert x \rvert$



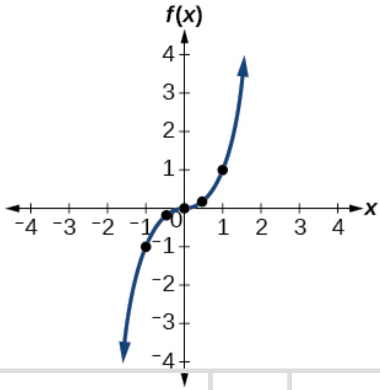
x	$f(x)$
−2	2
0	0
2	2

Quadratic	$f(x)=x^2$



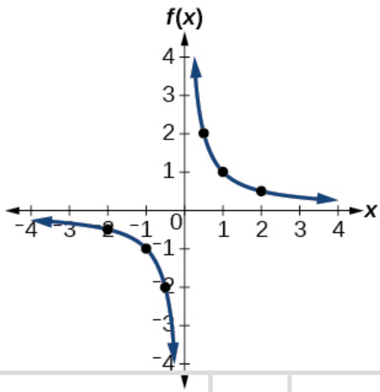
x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

Cubic	$f(x)=x^3$



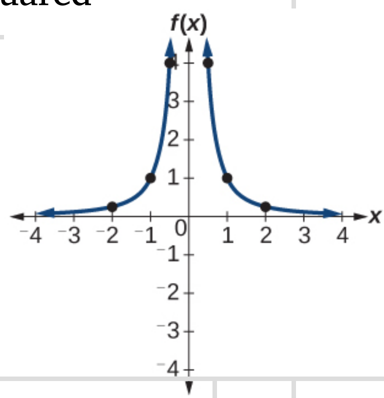
x	f(x)
−1	−1
−0.5	−0.125
0	0
0.5	0.125
1	1

Reciprocal	f(x)= 1 x



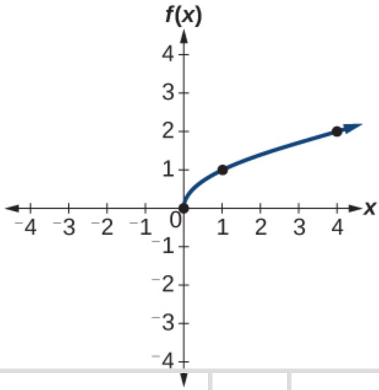
x	f(x)
−2	−0.5
−1	−1
−0.5	−2
0.5	2
1	1
2	0.5

Reciprocal squared	f(x)= 1 x 2



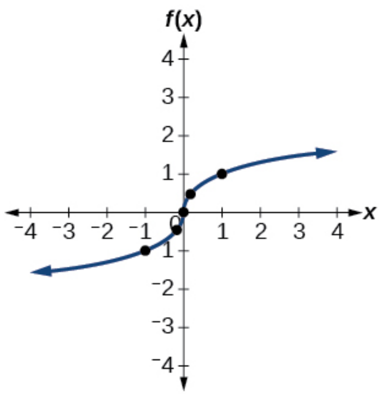
x	f(x)
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

Square root	f(x)= x



x	f(x)
0	0
1	1
4	2

Cube root	$f(x)=\sqrt[3]{x}$



x	f(x)
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1

Access the following online resources for additional instruction and practice with functions.

- Determine if a Relation is a Function
- Vertical Line Test
- Introduction to Functions
- Vertical Line Test on Graph
- One-to-one Functions
- Graphs as One-to-one Functions

Key Equations

Constant function	$f(x)=c$, where c is a constant
Identity function	$f(x)=x$
Absolute value function	$f(x)=\lvert x \rvert$
Quadratic function	$f(x)=x^2$
Cubic function	$f(x)=x^3$
Reciprocal function	$f(x)=\frac{1}{x}$
Reciprocal squared function	$f(x)=\frac{1}{x^2}$
Square root function	$f(x)=\sqrt{x}$
Cube root function	$f(x)=\sqrt[3]{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See [link] and [link].
- Function notation is a shorthand method for

relating the input to the output in the form $y = f(x)$. See [link] and [link].

- In tabular form, a function can be represented by rows or columns that relate to input and output values. See [link].
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See [link] and [link].
- To solve for a specific function value, we determine the input values that yield the specific output value. See [link].
- An algebraic form of a function can be written from an equation. See [link] and [link].
- Input and output values of a function can be identified from a table. See [link].
- Relating input values to output values on a graph is another way to evaluate a function. See [link].
- A function is one-to-one if each output value corresponds to only one input value. See [link].
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See [link].
- The graph of a one-to-one function passes the horizontal line test. See [link].

Section Exercises

Verbal

What is the difference between a relation and a function?

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

What is the difference between the input and the output of a function?

Why does the vertical line test tell us whether the graph of a relation represents a function?

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

How can you determine if a relation is a one-to-one function?

Why does the horizontal line test tell us

whether the graph of a function is one-to-one?

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the following exercises, determine whether the relation represents a function.

$\{ (a,b),(c,d),(a,c) \}$

$\{(a,b),(b,c),(c,c)\}$

---

function

For the following exercises, determine whether the relation represents y as a function of x.

$5x+2y=10$

$y= x 2$

---

function

$x = y\ 2$

$3\ x\ 2\ + y = 14$

function

$2x + y\ 2\ = 6$

$y = -2\ x\ 2\ + 40x$

function

$y = 1\ x$

$x = 3y + 5\ 7y - 1$

function

$x = 1 - y\ 2$

$y = 3x+5 \; 7x-1$

---

function

$x^2 + y^2 = 9$

$2xy = 1$

---

function

$x = y \; 3$

$y = x \; 3$

---

function

$y = 1 - x \; 2$

$x = \pm \; 1 - y$

---

function

$y = \pm\ 1 - x$

$y^2 = x^2$

---

not a function

$y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.

$f(x) = 2x - 5$

$f(-3) = -11;$
$f(2) = -1;$
$f(-a) = -2a - 5;$
$-f(a) = -2a + 5;$
$f(a+h) = 2a + 2h - 5$

$f(x) = -5x^2 + 2x - 1$

$f(x) = \ 2 - x\ + 5$

$f(-3) = \ 5\ + 5;$

$f(2) = 5;$
$f(-a) = 2 + a + 5;$
$-f(a) = -\ 2 - a\ - 5;$
$f(a + h) = 2 - a - h\ + 5$

$f(x) = 6x - 1\ 5x + 2$

$f(x) = | x - 1 | - | x + 1 |$

$f(-3) = 2;\ f(2) = 1 - 3 = -2;$
$f(-a) = | -a - 1 | - | -a + 1 |;$
$-f(a) = -| a - 1 | + | a + 1 |;$
$f(a + h) = | a + h - 1 | - | a + h + 1 |$

Given the function $g(x) = 5 - x\ 2$, evaluate $g(x + h) - g(x)\ h$ $, h \neq 0.$

Given the function $g(x) = x\ 2\ + 2x$, evaluate $g(x) - g(a)\ x - a$ $, x \neq a.$

$g(x) - g(a)\ x - a\ = x + a + 2, x \neq a$

Given the function $k(t) = 2t - 1$:

ⓐ Evaluate $k(2)$.
ⓑ Solve $k(t) = 7$.

Given the function $f(x) = 8 - 3x$:

ⓐ Evaluate $f(-2)$.
ⓑ Solve $f(x) = -1$.

ⓐ $f(-2) = 14$;
ⓑ $x = 3$

Given the function $p(c) = c^2 + c$:

ⓐ Evaluate $p(-3)$.
ⓑ Solve $p(c) = 2$.

Given the function $f(x) = x^2 - 3x$:

ⓐ Evaluate $f(5)$.
ⓑ Solve $f(x) = 4$.

ⓐ $f(5) = 10$;
ⓑ $x = -1$ or $x = 4$

Given the function $f(x) = x + 2$:

ⓐ Evaluate $f(7)$.
ⓑ Solve $f(x)=4$.

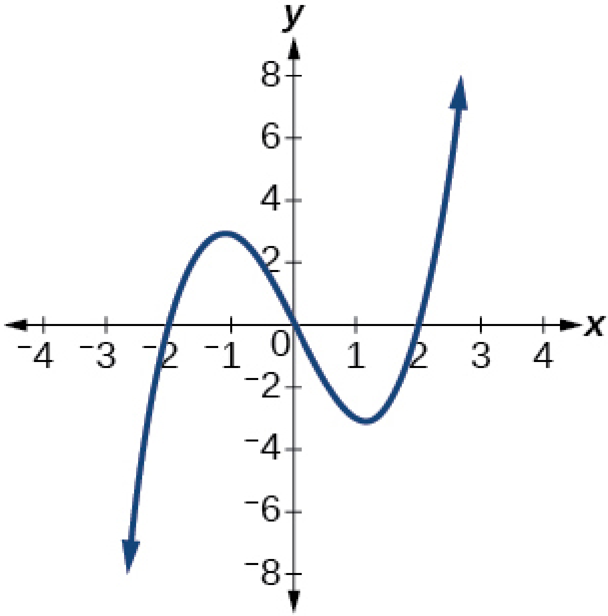
Consider the relationship $3r+2t=18$.

ⓐ Write the relationship as a function $r=f(t)$.
ⓑ Evaluate $f(-3)$.
ⓒ Solve $f(t)=2$.

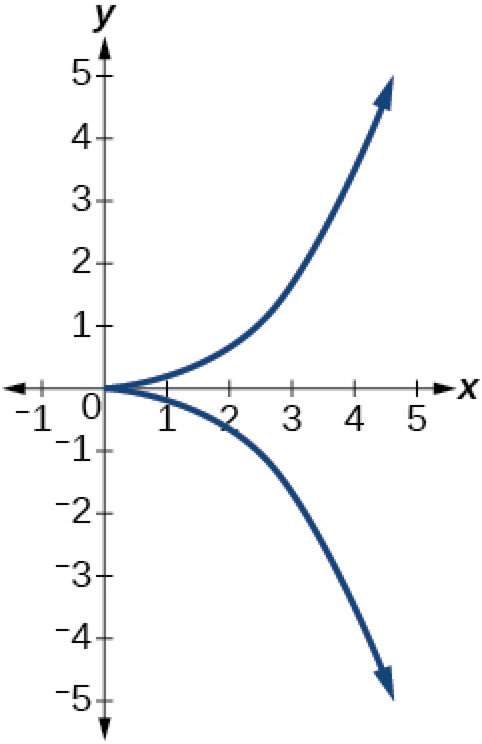
ⓐ $f(t)=6-\ 2\ 3\ t$;
ⓑ $f(-3)=8$;
ⓒ $t=6$

Graphical

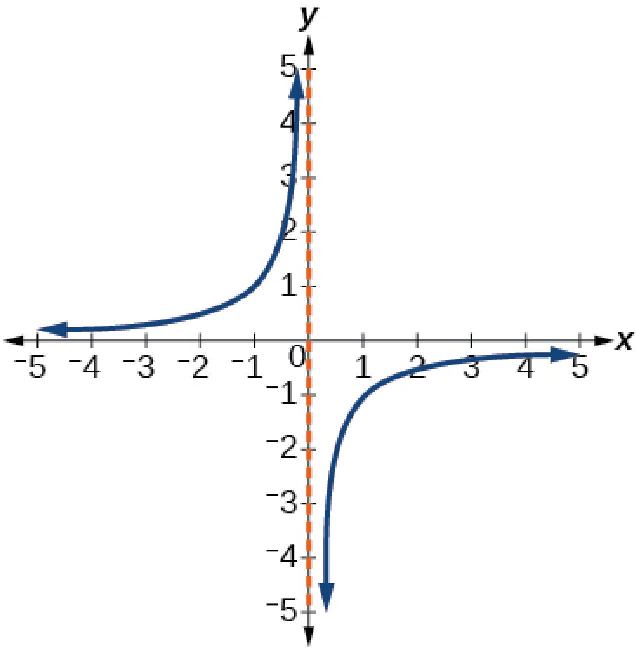
For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

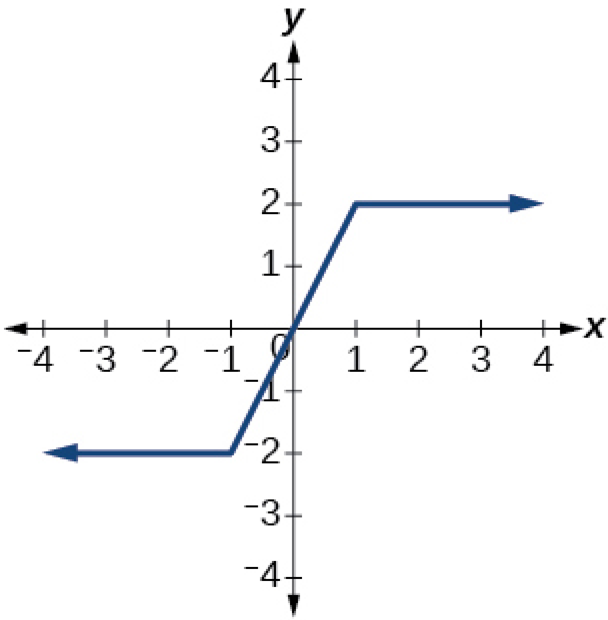


y
8
6
4
2
x
−4
−3
−2
−1
0
1
2
3
4
−2
−4
−6
−8

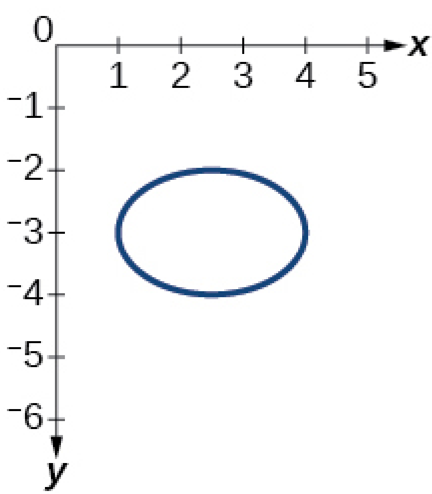


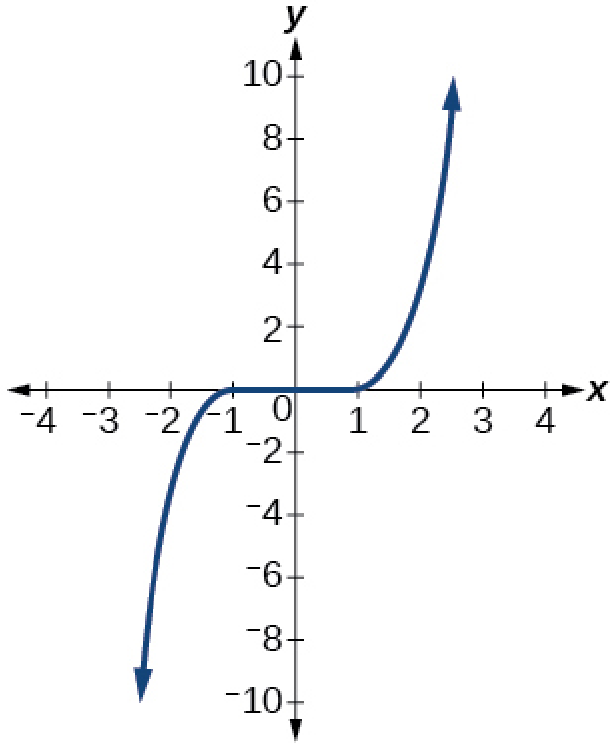
not a function


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
1
2
3
4
5

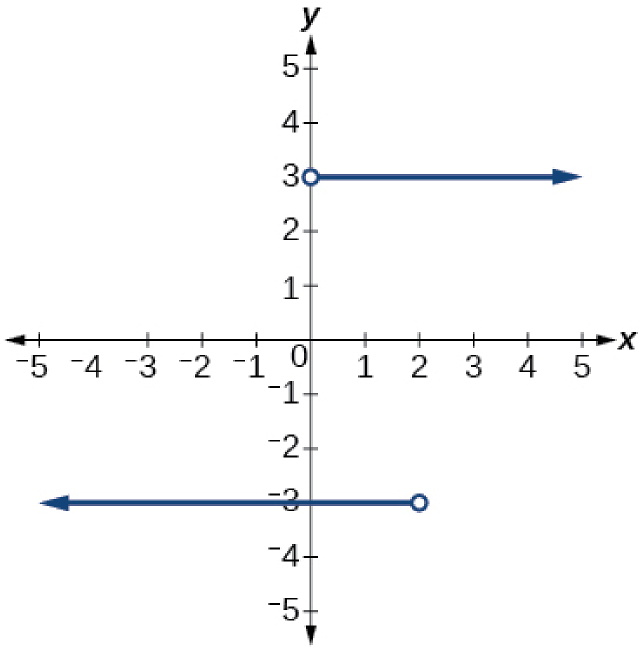


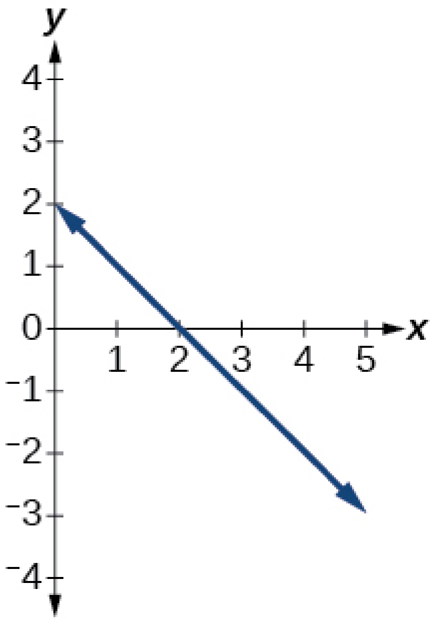
function





function


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
1
2
3
4
5



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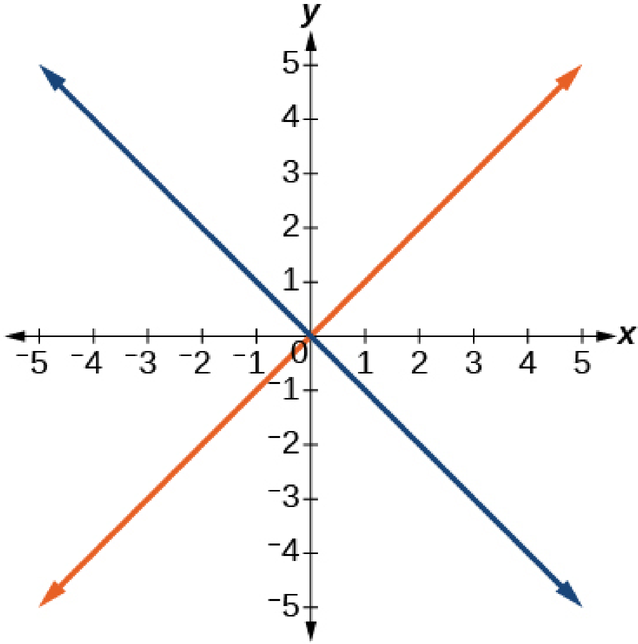
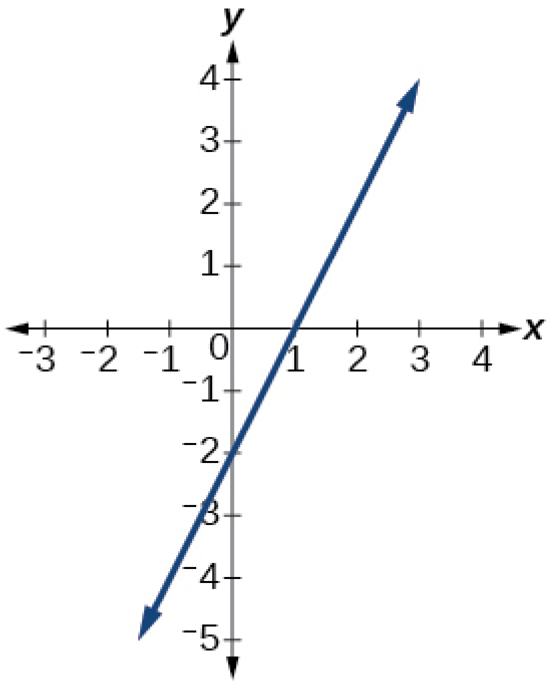
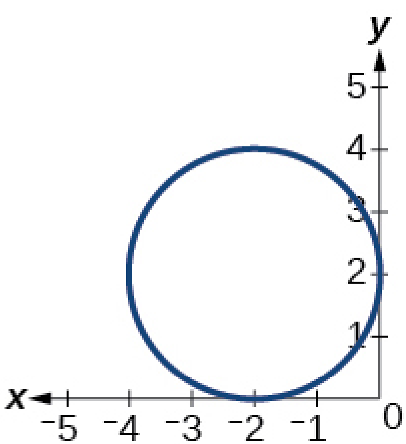
function

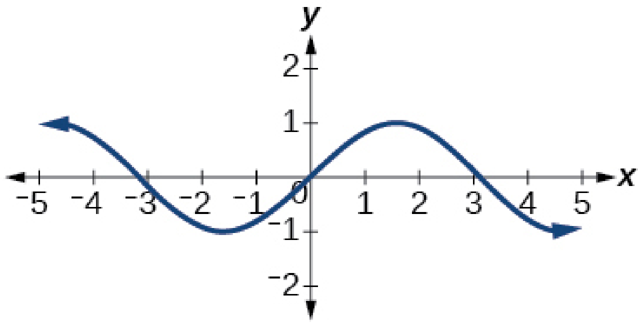


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
1
2
3
4
5



function



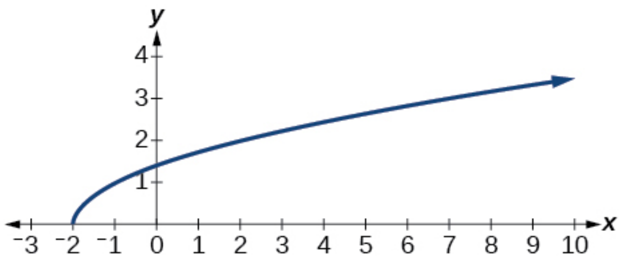


---

function

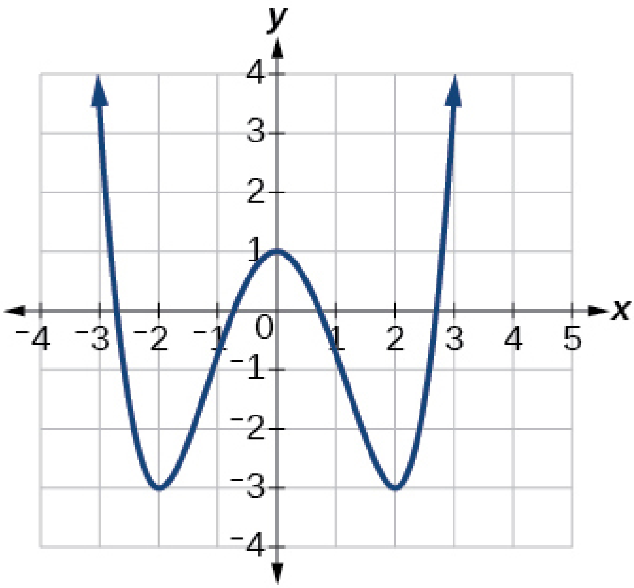
Given the following graph,

ⓐ Evaluate $f(-1)$.
ⓑ Solve for $f(x)=3$.



Given the following graph,

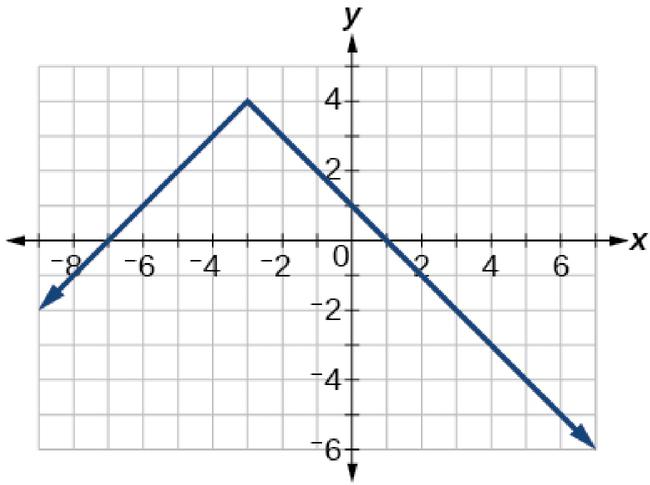
ⓐ Evaluate f(0).
ⓑ Solve for $f(x) = -3$.



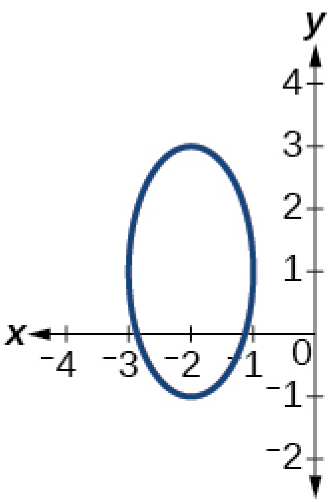
ⓐ $f(0) = 1$;
ⓑ $f(x) = -3, x = -2$ or $x = 2$

Given the following graph,

ⓐ Evaluate f(4).
ⓑ Solve for $f(x) = 1$.



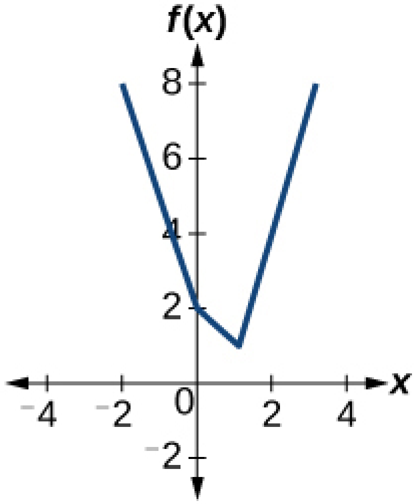
For the following exercises, determine if the given graph is a one-to-one function.

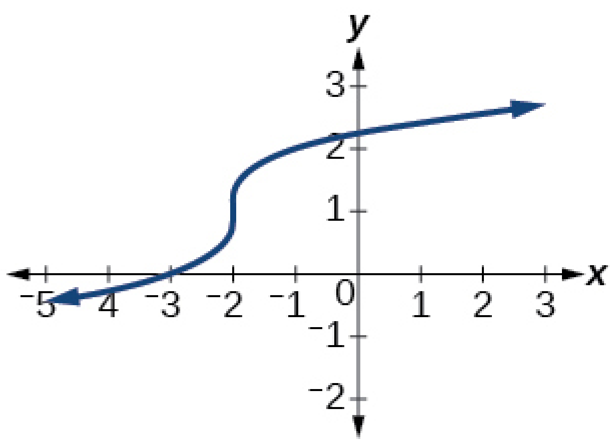


---

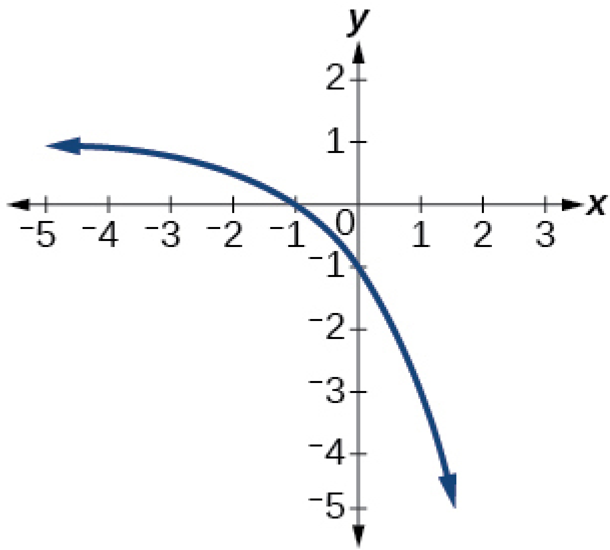
not a function so it is also not a one-to-one

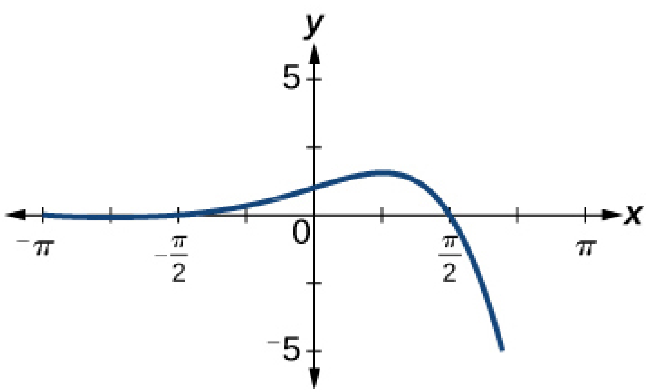
function





one-to-one function





function, but not one-to-one

Numeric

For the following exercises, determine whether the relation represents a function.

$\{(-1,-1),(-2,-2),(-3,-3)\}$

$\{(3,4),(4,5),(5,6)\}$

---

function

$\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

x	5		10		15	
y	3		8		14	

---

function

x	5	10	15
y	3	8	8

x	5	10	10
y	3	8	14

---

not a function

For the following exercises, use the function f represented in [link].

x	**f(x)**
0	74
1	28
2	1
3	53
4	56
5	3

6	36	
7	45	
8	14	
9	47	

Evaluate f(3).

Solve $f(x)=1$.

$f(x)=1, x=2$

For the following exercises, evaluate the function f at the values $f(-2), f(-1), f(0), f(1)$, and $f(2)$.

$f(x)=4-2x$

$f(x)=8-3x$

$f(-2)=14$; $f(-1)=11$; $f(0)=8$; $f(1)=5$; $f(2)=2$

$f(x)=8x^2-7x+3$

$f(x)=3+\sqrt{x+3}$

$f(-2) = 4$; $f(-1) = 4.414$; $f(0) = 4.732$; $f(1) = 5$; $f(2) = 5.236$

$f(x) = x - 2 x + 3$

$f(x) = 3 x$

$f(-2) = 1 9$; $f(-1) = 1 3$; $f(0) = 1$; $f(1) = 3$; $f(2) = 9$

For the following exercises, evaluate the expressions, given functions f,g, and h:

- $f(x) = 3x - 2$
- $g(x) = 5 - x 2$
- $h(x) = -2 x 2 + 3x - 1$

$3f(1) - 4g(-2)$

$f(7 3) - h(-2)$

20

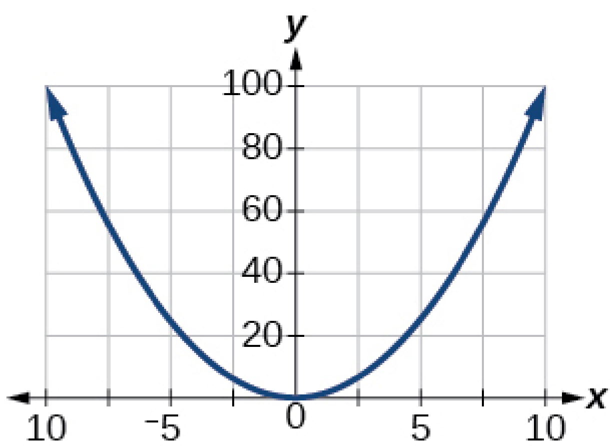
Technology

For the following exercises, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

$[-0.1, 0.1]$

$[-10, 10]$

$[0, 100]$



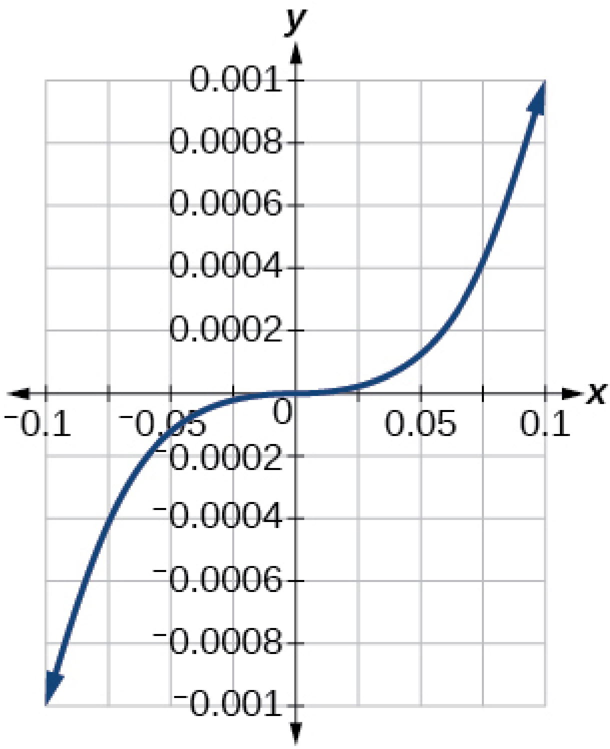
$[-100, 100]$

For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window.

Show each graph.

$[-0.1, 0.1]$

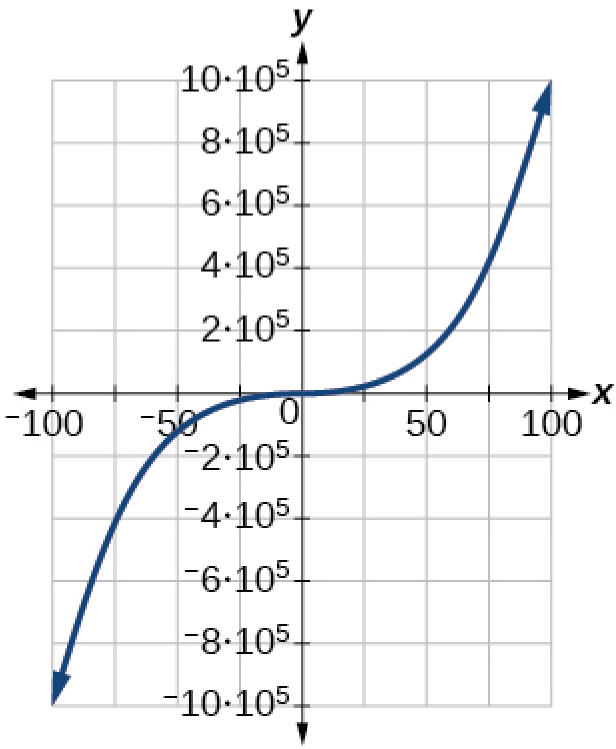
$[-0.001, 0.001]$



$[-10, 10]$

$[-100, 100]$

[− 1,000,000, 1,000,000]

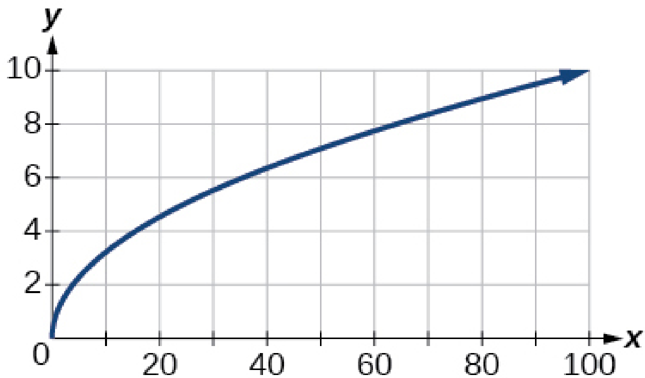


For the following exercises, graph y = x on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

[0, 0.01]

[0, 100]

[0, 10]

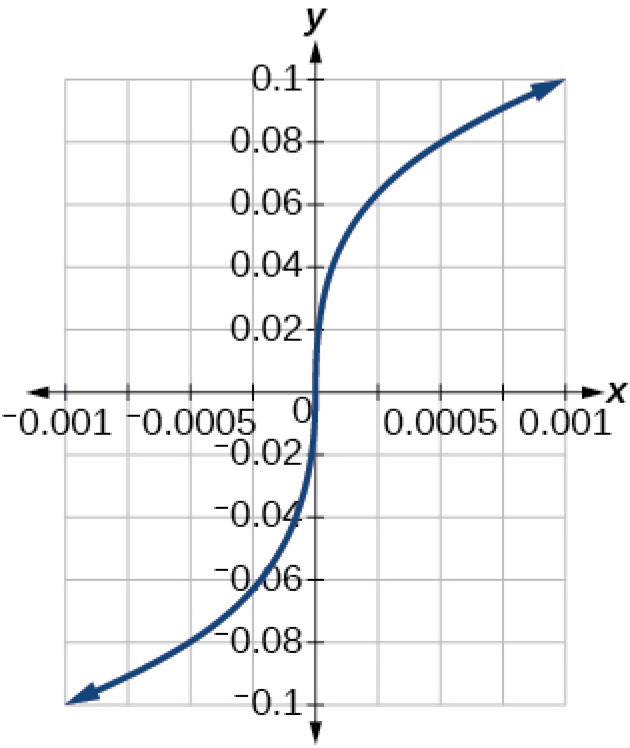


[0, 10,000]

For the following exercises, graph y= x 3 on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

[−0.001,0.001]

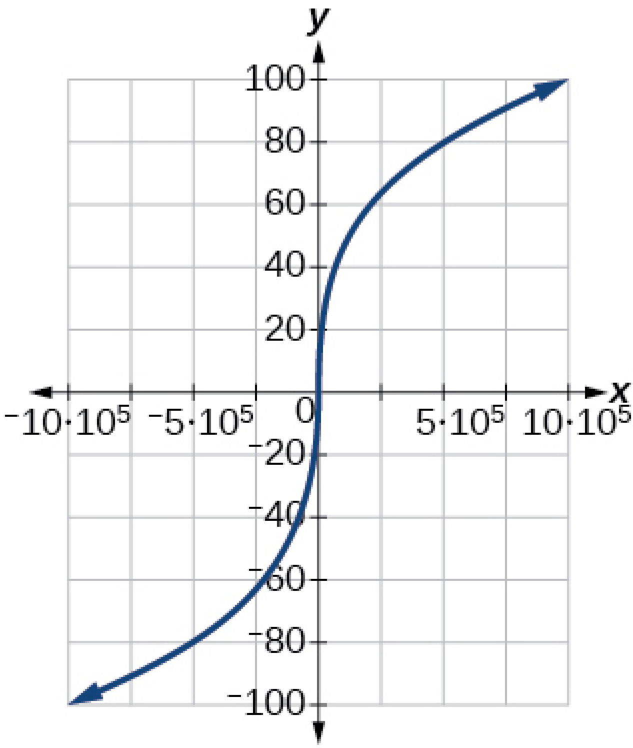
[−0.1,0.1]



$[-1000,1000]$

$[-1,000,000,1,000,000]$

$[-100, 100]$



Real-World Applications

The amount of garbage, G, produced by a city with population p is given by $G=f(p)$. G is measured in tons per week, and p is measured in thousands of people.

ⓐ The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

ⓑ Explain the meaning of the statement f(5)=2.

The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by D=g(a).

ⓐ A garden with area 5000 ft2 requires 50 yd3 of dirt. Express this information in terms of the function g.
ⓑ Explain the meaning of the statement g(100)=1.

ⓐ g(5000)=50;
ⓑ The number of cubic yards of dirt required for a garden of 100 square feet is 1.

Let f(t) be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

ⓐ f(5)=30
ⓑ f(10)=40

Let h(t) be the height above ground, in feet, of

a rocket t seconds after launching. Explain the meaning of each statement:

$h(1) = 200$
$h(2) = 350$

ⓐ The height of a rocket above ground after 1 second is 200 ft.
ⓑ the height of a rocket above ground after 2 seconds is 350 ft.

Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

Glossary

dependent variable
: an output variable

domain
: the set of all possible input values for a relation

function
: a relation in which each input value yields a unique output value

horizontal line test

a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
: an input variable

input
: each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
: a function for which each value of the output is associated with a unique input value

output
: each object or value in the range that is produced when an input value is entered into a function

range
: the set of output values that result from the input values in a relation

relation
: a set of ordered pairs

vertical line test
: a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more

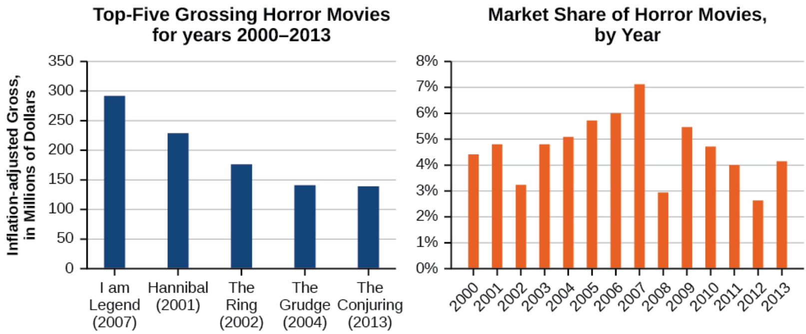
than once

Domain and Range

In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. [link] shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.
Based on data compiled by www.the-numbers.com. [footnote]The Numbers: Where Data and the Movie Business Meet. "Box Office History for Horror Movies." http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

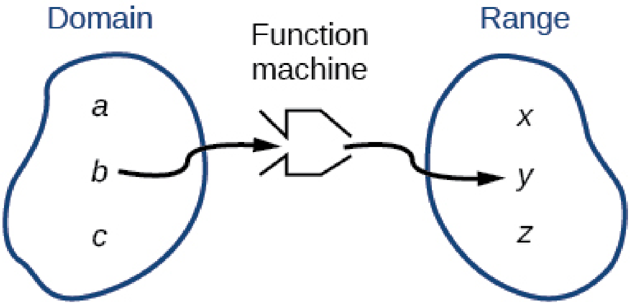


Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function

machine" and the range as another "holding area" for the machine's products. See [link].



We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an odd root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there

is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See [link] for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first

coordinates of the ordered pairs.
$\{2,3,4,5,6\}$

Find the domain of the function:

$\{ (-5,4),(0,0),(5,-4),(10,-8),(15,-12) \}$

$\{-5,0,5,10,15\}$

Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Finding the Domain of a Function

Find the domain of the function $f(x)= x 2 -1$.

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Find the domain of the function: f(x) = 5 − x + x 3 .

$(-\infty, \infty)$

Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to

0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

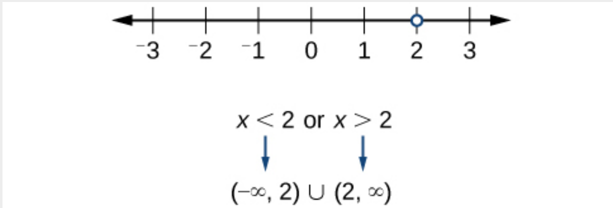
Finding the Domain of a Function Involving a Denominator

Find the domain of the function $f(x)=\frac{x+1}{2-x}$.

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$2-x=0 \quad -x=-2 \quad x=2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty,2)\cup(2,\infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Find the domain of the function: $f(x) = \frac{1+4x}{2x-1}$.

$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x)= 7-x$.

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.
$7-x\geq 0 \quad -x\geq -7 \quad x\leq 7$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty,7]$.

Find the domain of the function $f(x)= 5+2x$.

$[- 5\ 2 ,\infty)$

Can there be functions in which the domain

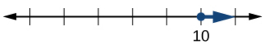
and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{x}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{ x | 10 \le x < 30 \}$ describes the behavior of x in set-builder notation. The braces {} are read as "the set of," and the vertical bar | is read as "such that," so we would read $\{ x | 10 \le x < 30 \}$ as "the set of *x*-values such that 10 is less than or equal to x, and x is less than 30."

[link] compares inequality notation, set-builder notation, and interval notation.

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2,3,5\}$ and $\{4,6\}$ is the set $\{2,3,4,5,6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{ x \mid |x| \geq 3 \} = (-\infty, -3] \cup [3, \infty)$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|\text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{ x|4 < x \leq 12 \}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4,12]$$

Given a line graph, describe the set of values using interval notation.

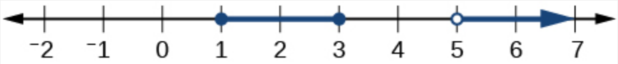
1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled

dot) or) for each excluded end value (open dot).

4. Use the union symbol ∪ to combine all intervals into one set.

Describing Sets on the Real-Number Line

Describe the intervals of values shown in [link] using inequality notation, set-builder notation, and interval notation.



To describe the values, x, included in the intervals shown, we would say, “ x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

Inequality	$1 \le x \le 3 \text{ or } x > 5$
Set-builder notation	$\{x \mid 1 \le x \le 3 \text{ or } x > 5\}$
Interval notation	$[1,3] \cup (5,\infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

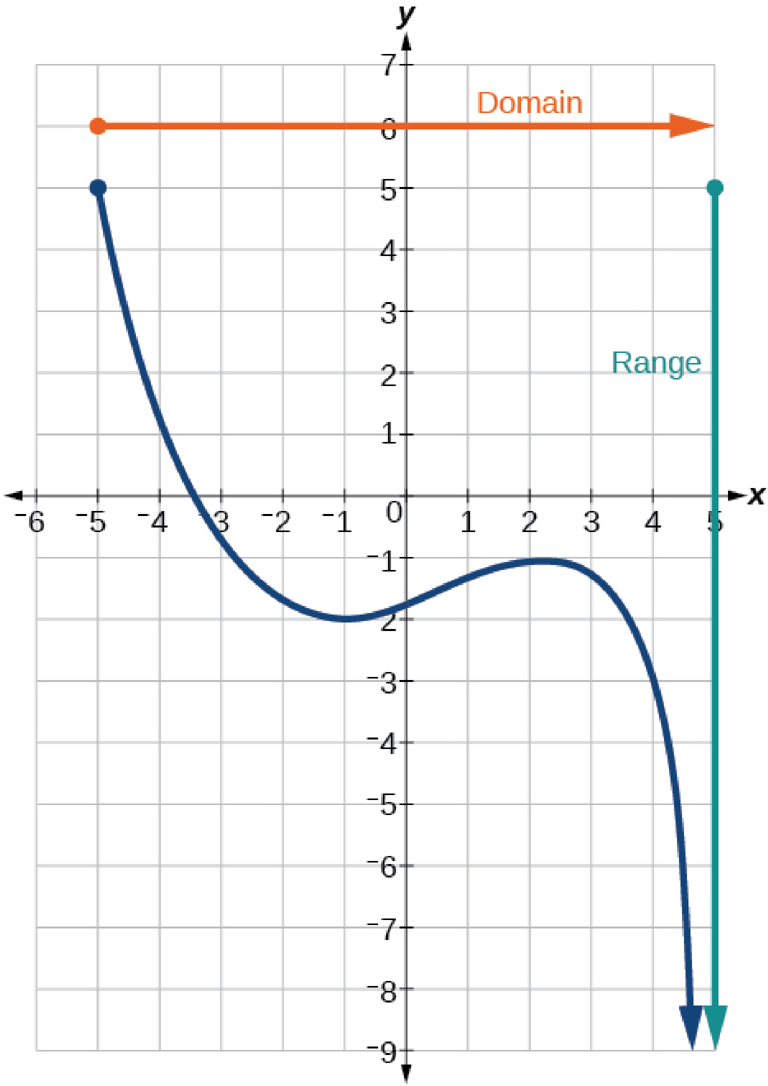
Given [link], specify the graphed set in

1. words
2. set-builder notation
3. interval notation

1. values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;
2. $\{ x|x\le -2 \text{or} -1\le x<3 \}$;
3. $(-\infty,-2]\cup[-1,3)$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the *x*-axis. The range is the set of possible output values, which are shown on the *y*-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See [link].

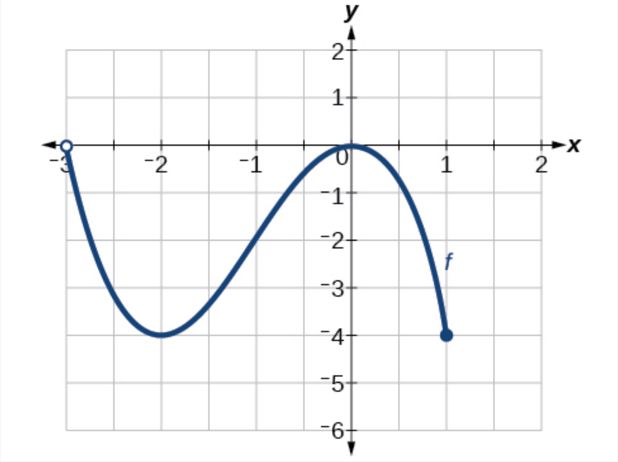


We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right

for domain, and from the bottom of the graph to the top of the graph for range.

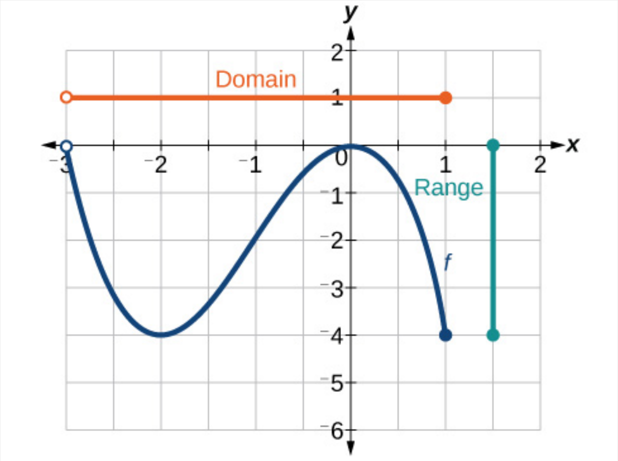
Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in [link].



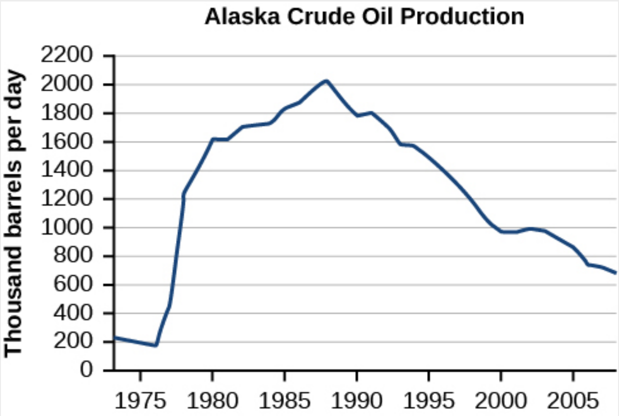
We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $\left(-3,1\right]$.

The vertical extent of the graph is 0 to –4, so the range is $\left[-4,0\right]$. See [link].



Finding Domain and Range from a Graph of Oil Production

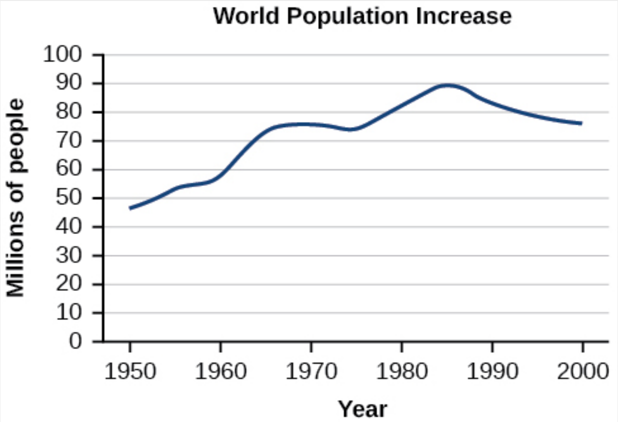
Find the domain and range of the function f whose graph is shown in [link].
(credit: modification of work by the U.S. Energy Information Administration)
[footnote]http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.



The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Given [link], identify the domain and range using interval notation.



domain $=[1950,2002]$ range $=$ $[47{,}000{,}000,89{,}000{,}000]$

Can a function's domain and range be the same?
Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

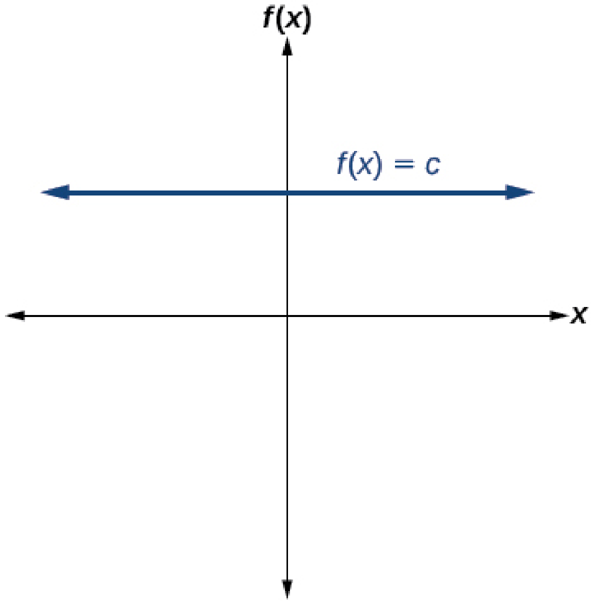
For the **constant function** $f(x)=c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single

element. In interval notation, this is written as [c,c], the interval that both begins and ends with c. For the **identity function** $f(x)=x$, there is no restriction on x. Both the domain and range are the set of all real numbers. For the **absolute value function** $f(x)=|x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0. For the **quadratic function** $f(x)=x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers. For the **cubic function** $f(x)=x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers. For the **reciprocal function** $f(x)=\frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|x\neq 0\}$, the set of all real numbers that are not zero. For the **reciprocal squared function** $f(x)=\frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers. For the **square root**

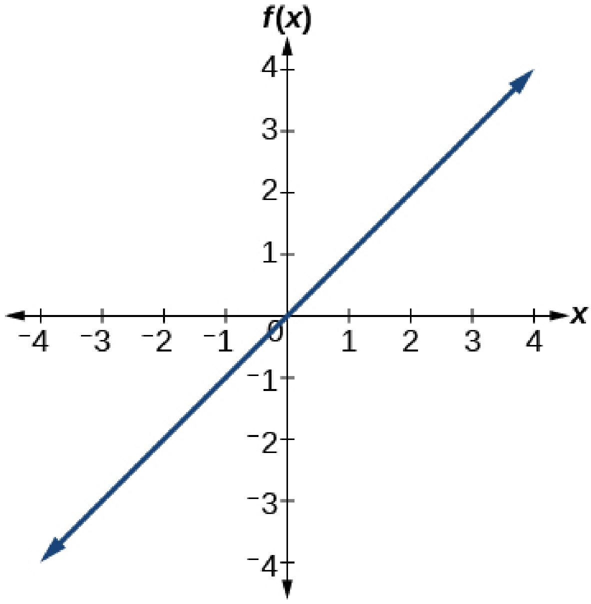
function $f(x)= x$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-x$ also gives us x. For the **cube root function** $f(x)= x\ 3$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

Finding Domains and Ranges of the Toolkit Functions

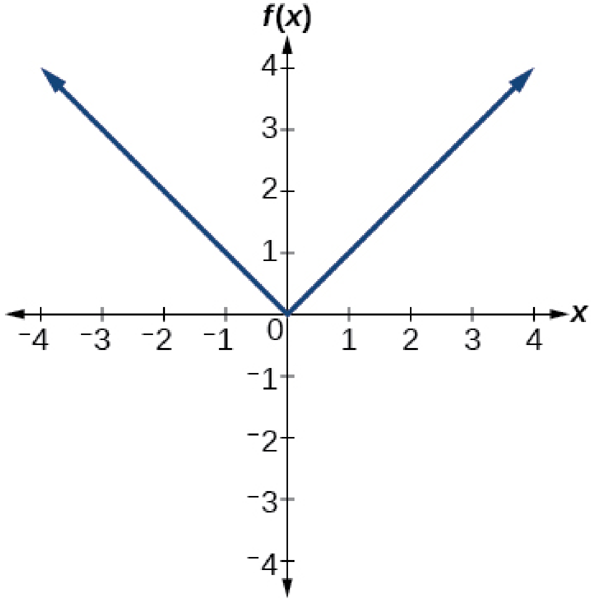
We will now return to our set of toolkit functions to determine the domain and range of each.



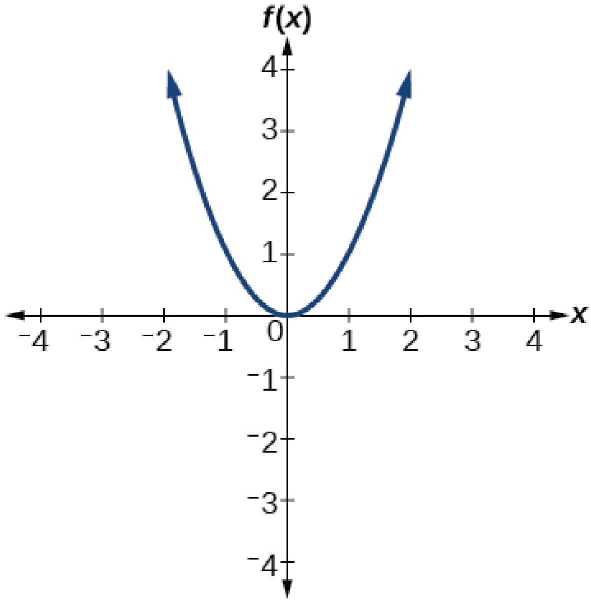
Domain: $(-\infty, \infty)$
Range: $[c, c]$



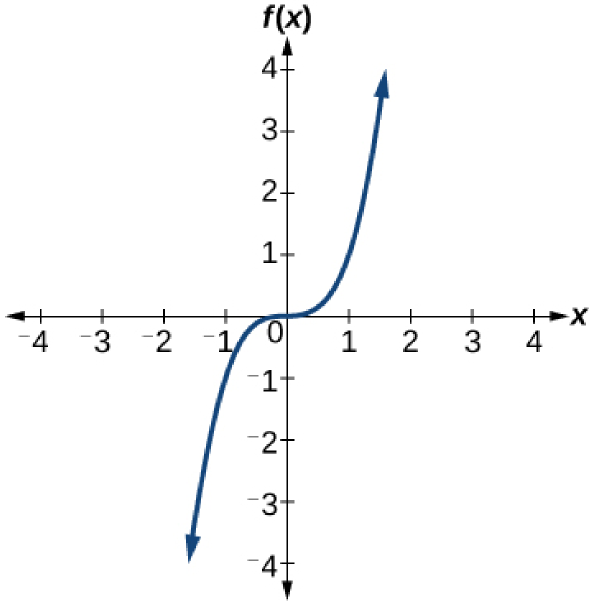
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



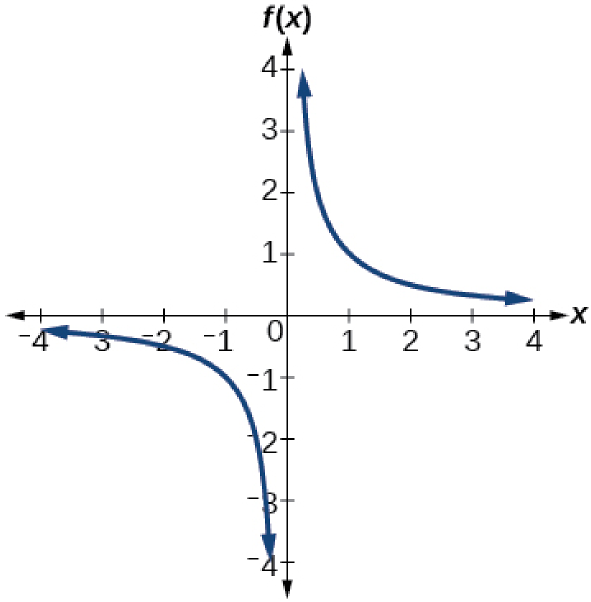
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



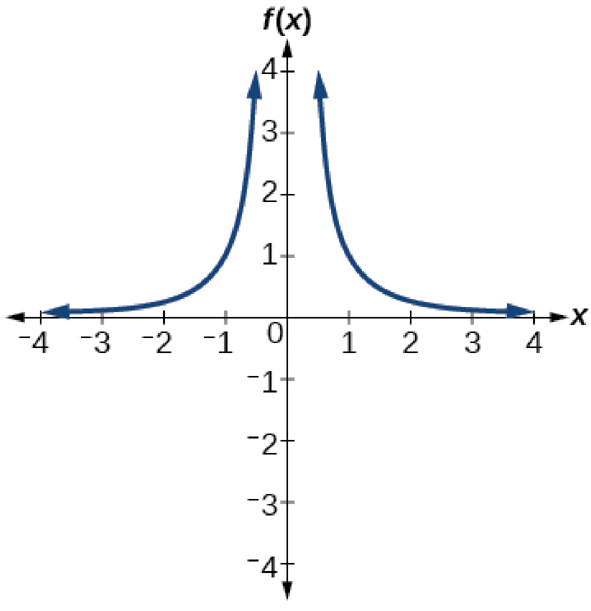
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$



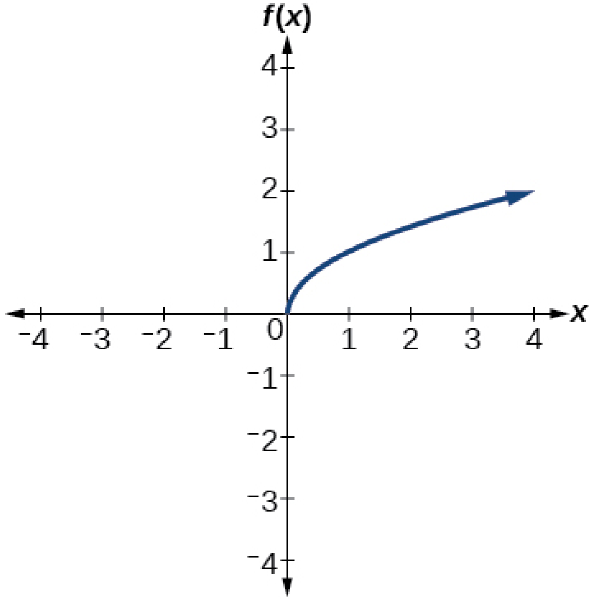
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



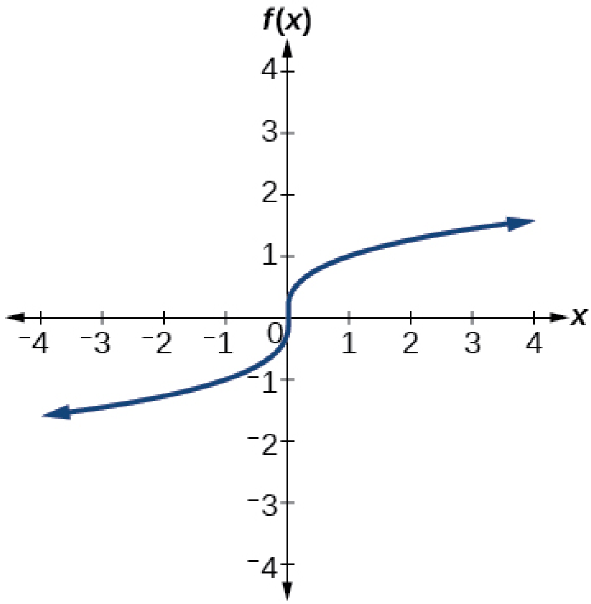
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$



Domain: [0, ∞)
Range: [0, ∞)



Domain: (−∞, ∞)
Range: (−∞, ∞)

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x)=2x^3-x$.

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty,\infty)$ and the range is also $(-\infty,\infty)$.

Finding the Domain and Range

Find the domain and range of $f(x) = 2x+1$.

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Finding the Domain and Range

Find the domain and range of $f(x) = 2x+4$.

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

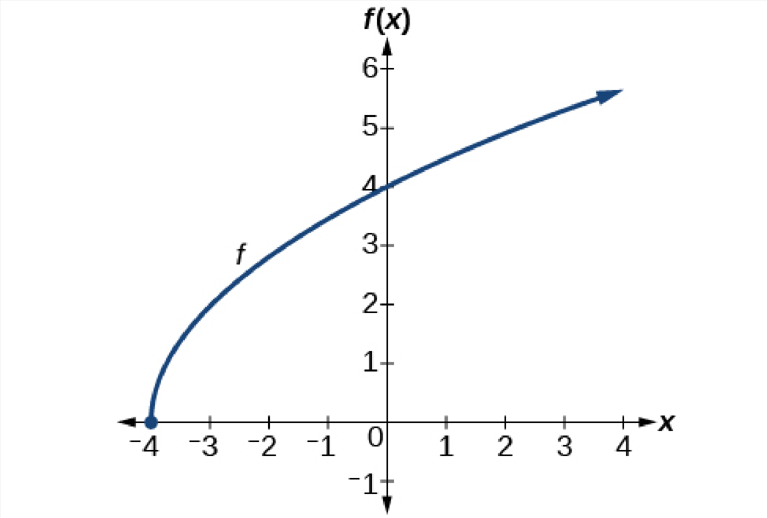
$x+4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

[link] represents the function f.



Find the domain and range of $f(x)=-\ 2-x$.

domain: $(-\infty,2]$; range: $(-\infty,0]$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires

more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$f(x) = x \text{if} x \geq 0$

If we input a negative value, the output is the opposite of the input.

$f(x) = -x \text{if} x < 0$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain “boundaries.” For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax

brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to

calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Writing a Piecewise Function

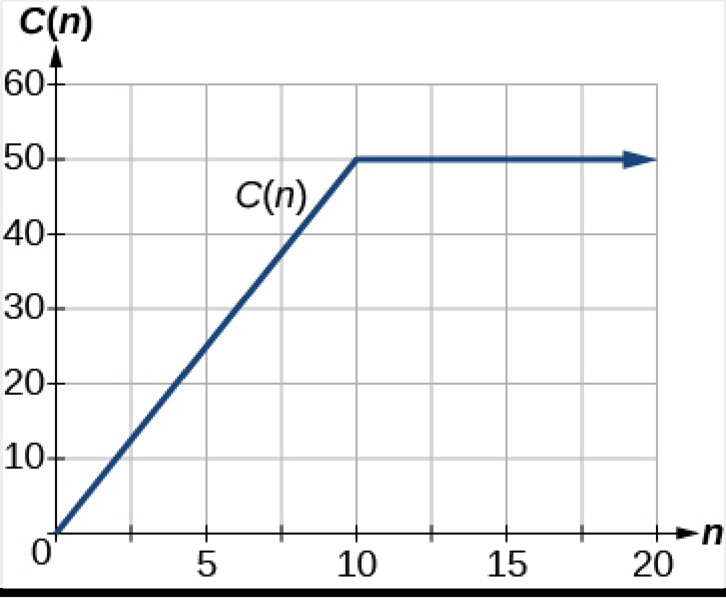
A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Two different formulas will be needed. For *n*-values under 10, $C=5n$. For values of n that are 10 or greater, $C=50$.

$C(n)=\{ 5n \text{ if } 0<n<10 \;\; 50 \text{ if } n\geq 10$

Analysis

The function is represented in [link]. The graph is a diagonal line from $n=0$ to $n=10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n=10$, but not all piecewise functions have this property.



Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$C(g) = \{ 25 \text{ if } 0 < g < 2 \; 25 + 10(g - 2) \text{ if } g \geq 2$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

To find the cost of using 1.5 gigabytes of data, C(1.5), we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

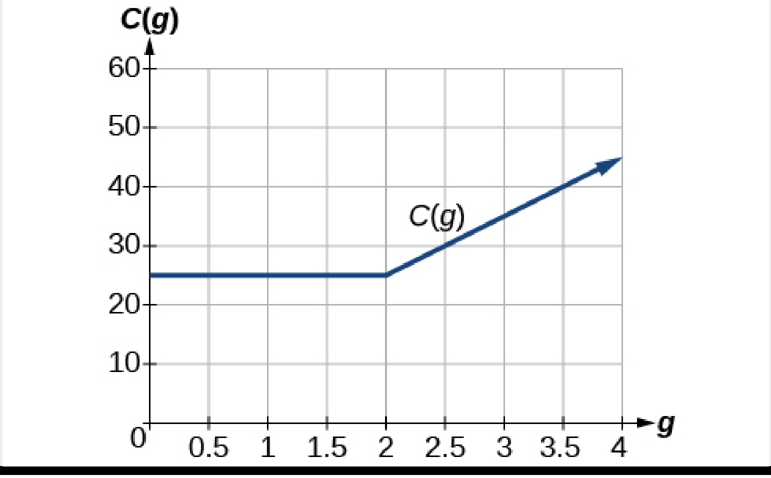
$C(1.5)=\$25$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$C(4)=25+10(4-2)=\$45$

Analysis

The function is represented in [link]. We can see where the function changes from a constant to a shifted and stretched identity at $g=2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

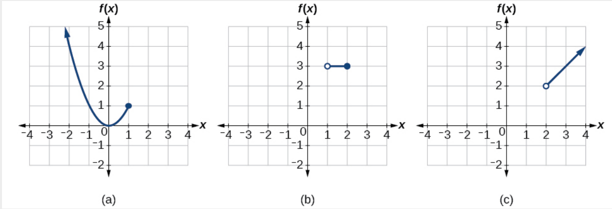
Graphing a Piecewise Function

Sketch a graph of the function.
$f(x)=\{ x^2 \text{ if } x\le 1 \quad 3 \text{ if } 1<x\le 2 \quad x \text{ if } x>2$

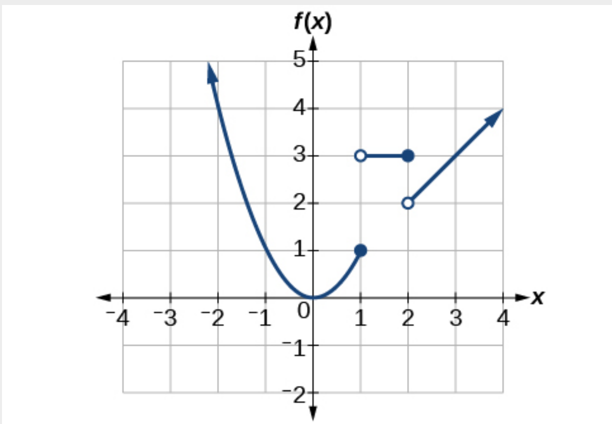
Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

[link] shows the three components of the piecewise function graphed on separate coordinate systems.

(a) $f(x)=x^2$ if $x\leq 1$; (b) $f(x)=3$ if $1<x\leq 2$; (c) $f(x)=x$ if $x>2$



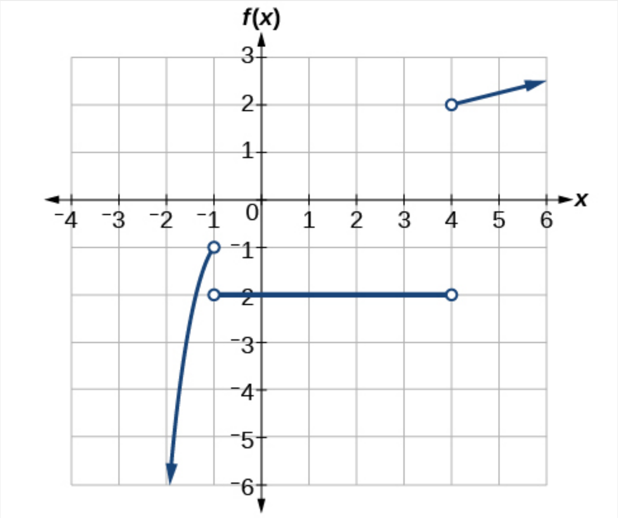
Now that we have sketched each piece individually, we combine them in the same coordinate plane. See [link].



Analysis

Note that the graph does pass the vertical line test even at $x=1$ and $x=2$ because the points $(1,3)$ and $(2,2)$ are not part of the graph of the function, though $(1,1)$ and $(2,3)$ are.

Graph the following piecewise function.
f(x)={ x 3 if x< −1 −2 if −1<x<4 x if x>4



Can more than one formula from a piecewise function be applied to a value in the domain?
No. Each value corresponds to one equation in a piecewise formula.

Access these online resources for additional instruction and practice with domain and range.

- Domain and Range of Square Root Functions
- Determining Domain and Range
- Find Domain and Range Given the Graph
- Find Domain and Range Given a Table
- Find Domain and Range Given Points on a Coordinate Plane

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See [link].
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See [link], [link], and [link].
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See [link].
- For many functions, the domain and range can be determined from a graph. See [link] and

[link].

- An understanding of toolkit functions can be used to find the domain and range of related functions. See [link], [link], and [link].
- A piecewise function is described by more than one formula. See [link] and [link].
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See [link].

Section Exercises

Verbal

Why does the domain differ for different functions?

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

How do we determine the domain of a function defined by an equation?

Explain why the domain of $f(x) = x 3$ is

different from the domain of $f(x) = x$.

There is no restriction on x for $f(x) = x\,3$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x -values are restricted for $f(x) = x$ to nonnegative numbers and the domain is $[0, \infty)$.

When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

How do you graph a piecewise function?

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x -axis and y -axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic

For the following exercises, find the domain of each function using interval notation.

$f(x) = -2x(x-1)(x-2)$

$f(x) = 5 - 2\ x\ 2$

$(-\infty, \infty)$

$f(\ x\ ) = 3\ x - 2$

$f(\ x\ ) = 3 -\ \ 6 - 2x$

$(-\infty, 3]$

$f(x) =\ \ 4 - 3x$

$f(x) =\ \ x\ 2\ \ + 4$

$(-\infty, \infty)$

$f(x) =\ \ 1 - 2x\ 3$

$f(x)= x-1\ 3$

$(-\infty,\infty)$

$f(x)= 9\ x-6$

$f(x)= 3x+1\ 4x+2$

$(-\infty,-\ 1\ 2\ )\cup(-\ 1\ 2\ ,\infty)$

$f(x)= x+4\ x-4$

$f(x)= x-3\ x\ 2\ +9x-22$

$(-\infty,-11)\cup(-11,2)\cup(2,\infty)$

$f(x)= 1\ x\ 2\ -x-6$

$f(x)= 2\ x\ 3\ -250\ x\ 2\ -2x-15$

$(-\infty,-3)\cup(-3,5)\cup(5,\infty)$

$5 x-3$

$2x+1\ 5-x$

$(-\infty,5)$

$f(x)= x-4\ x-6$

$f(x)= x-6\ x-4$

$[6,\infty)$

$f(x)= x\ x$

$f(x)= x\ 2\ -9x\ x\ 2\ -81$

$(-\infty,-9)\cup(-9,9)\cup(9,\infty)$

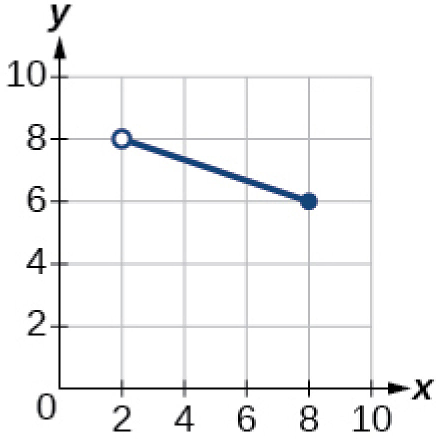
Find the domain of the function $f(x)= 2\ x\ 3\ -50x$ by:

1. using algebra.
2. graphing the function in the radicand and determining intervals on the *x*-axis for

which the radicand is nonnegative.

Graphical

For the following exercises, write the domain and range of each function using interval notation.



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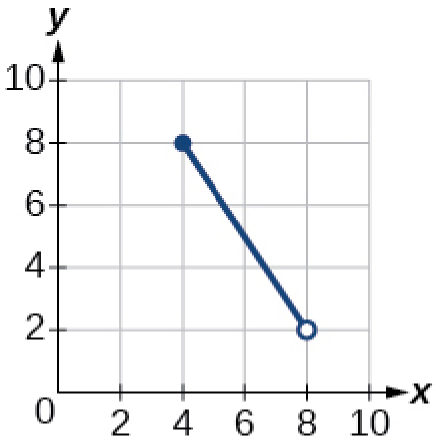
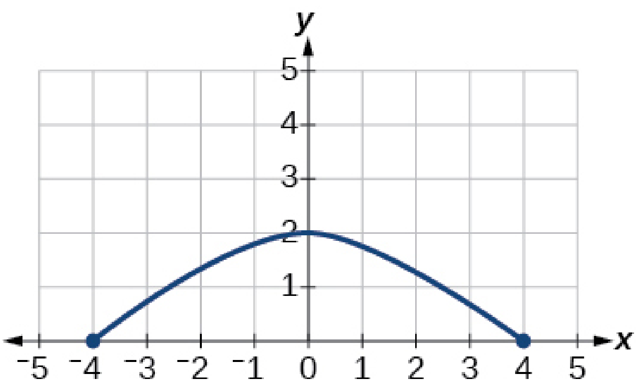
domain: (2,8], range [6,8)





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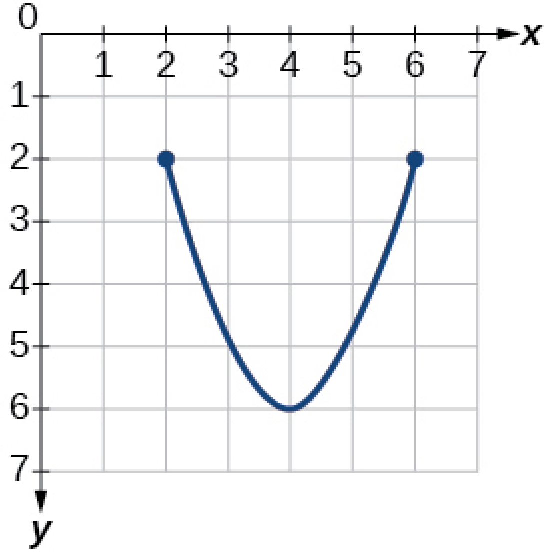
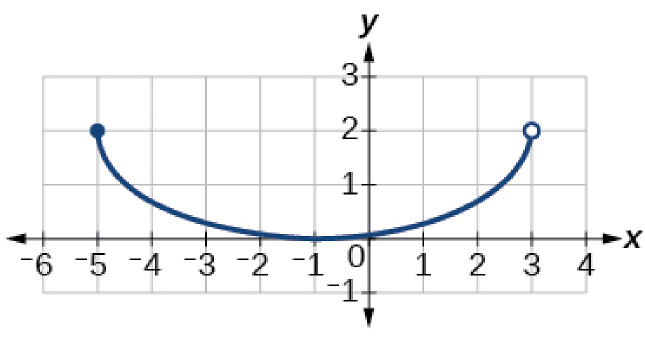
domain: $[-4, 4]$, range: $[0, 2]$





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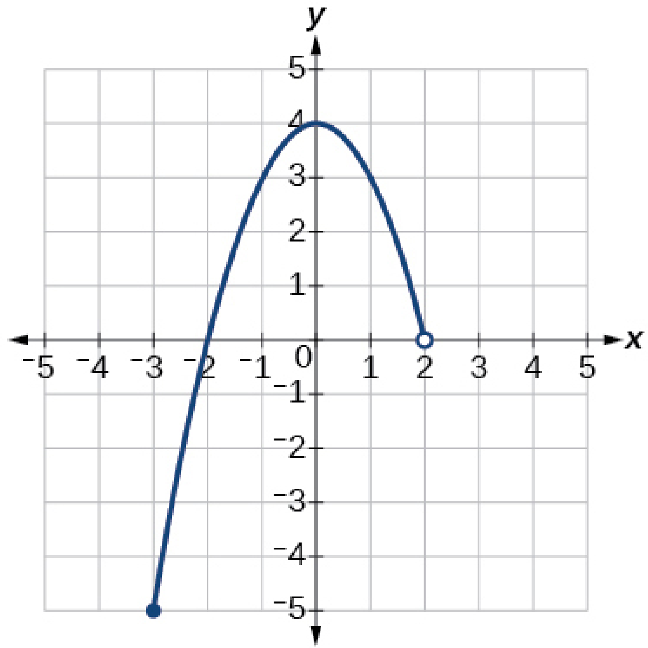
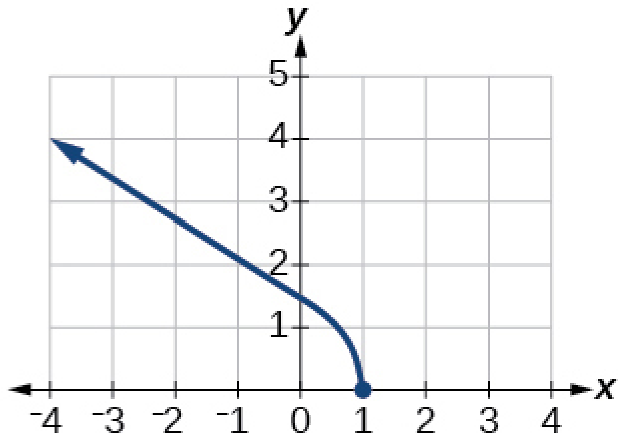
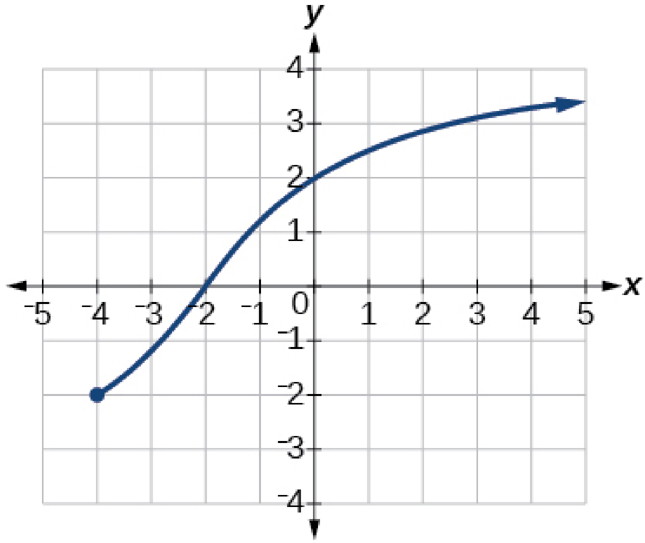
domain: $[-5,3)$, range: $[0,2]$





domain: $(-\infty, 1]$, range: $[0, \infty)$


y
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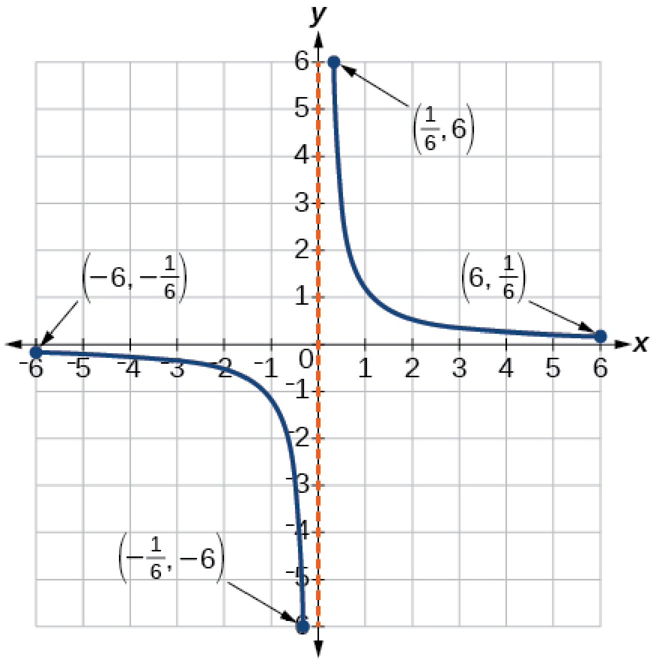

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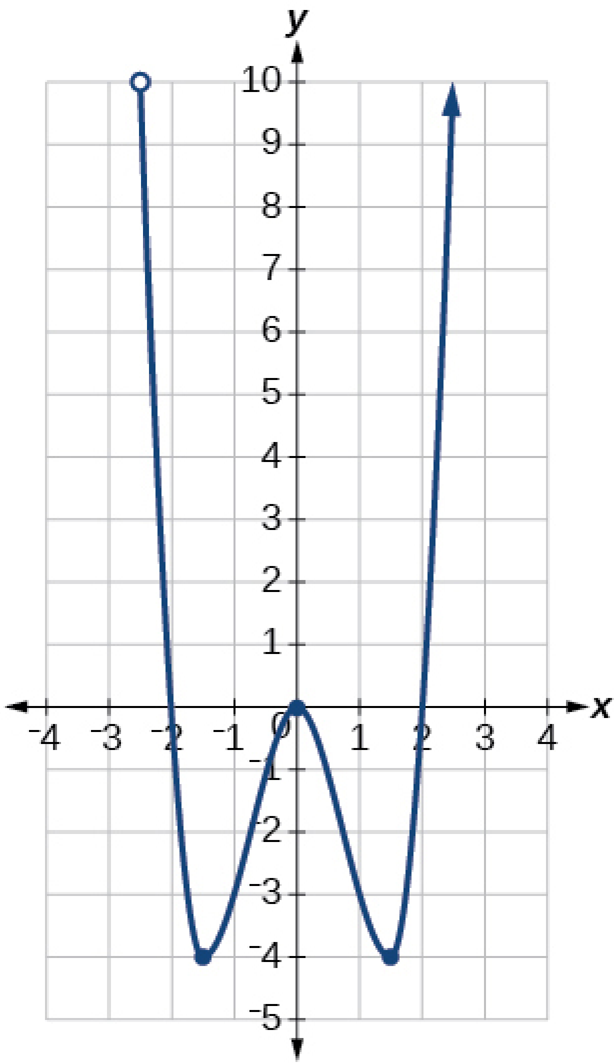
domain: $\left[-6,-\frac{1}{6}\right]\cup\left[\frac{1}{6},6\right]$; range: $\left[-6,-\frac{1}{6}\right]\cup\left[\frac{1}{6},6\right]$



y
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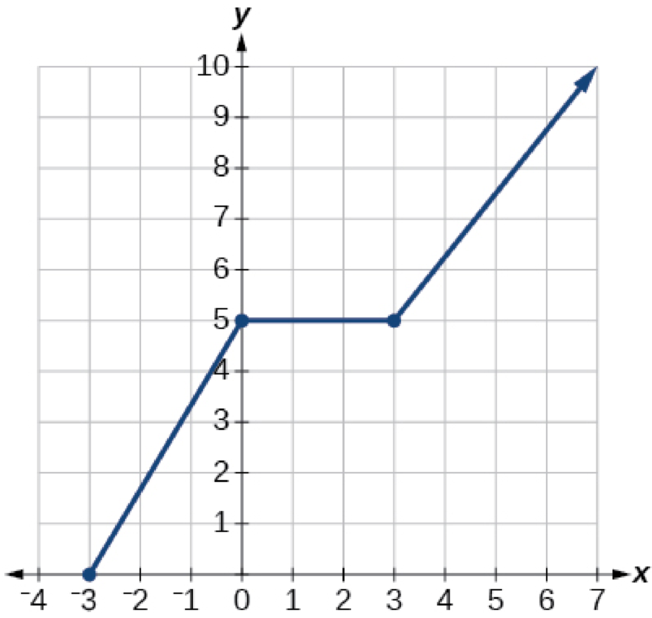


---

domain: $[-3, \infty)$; range: $[0, \infty)$

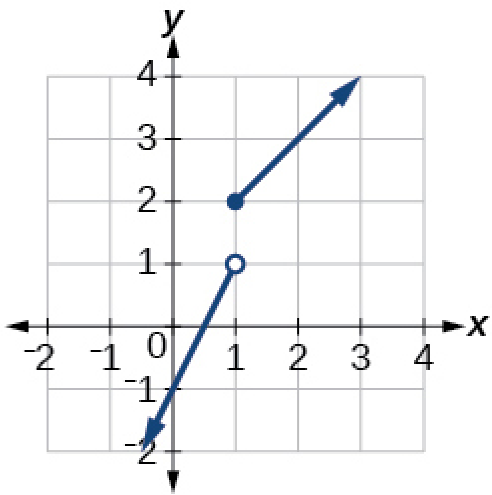
For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

$$f(x) = \{ \; x+1 \text{ if } x < -2 \quad -2x-3 \text{ if } x \geq -2$$

$$f(x) = \{ \; 2x-1 \text{ if } x < 1 \quad 1+x \text{ if } x \geq 1$$

---

domain: $(-\infty, \infty)$

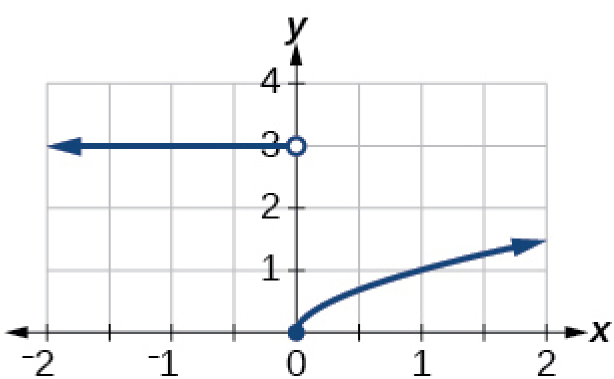


$f(x) = \{ \; x + 1 \text{if} x < 0 \;\; x - 1 \text{if} x > 0$

$f(\; x\;) = \{ \; 3 \text{ if } x < 0 \;\; x \text{ if } x \geq 0$

---

domain: $(-\infty, \infty)$

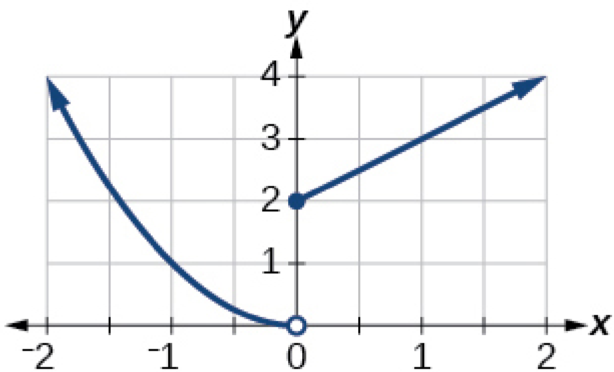


$f(x)=\{\ x\ 2 \qquad \text{if } x<0\ 1-x\ \ \text{if } x>0$

$f(x)=\{\ x\ 2\ x+2\ \text{ifx}<0\ \text{ifx}\geq 0$

---

domain: $(-\infty,\infty)$

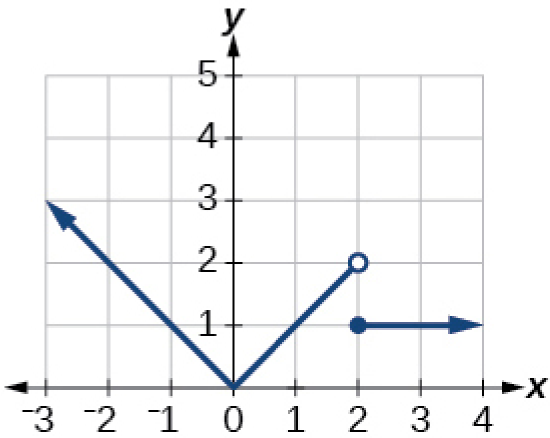


$f(\ x\ )=\{\ x+1\ \text{if } x<1\ x\ 3\ \text{if } x\geq 1$

$f(x)=\{\ |x|\ 1\ \text{ifx}<2\ \text{ifx}\geq 2$

---

domain: $(-\infty,\infty)$



Numeric

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

$$f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\geq -2 \end{cases}$$

$$f(x)=\begin{cases} 1 & \text{if } x\leq -3 \\ 0 & \text{if } x>-3 \end{cases}$$

$f(-3)=1;\ f(-2)=0;\ f(-1)=0;\ f(0)=0$

$$f(x)=\begin{cases} -2x^2+3 & \text{if } x\leq -1 \\ 5x-7 & \text{if } x>-1 \end{cases}$$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

$f(x)=\{ 7x+3 \text{ if } x<0 \ 7x+6 \text{ if } x\geq 0$

$f(-1)=-4$; $f(0)=6$; $f(2)=20$; $f(4)=34$

$f(x)=\{ x^2-2 \text{ if } x<2 \ 4+|x-5| \text{ if } x\geq 2$

$f(x)=\{ 5x \text{ if } x<0 \ 3 \text{ if } 0\leq x\leq 3 \ x^2 \text{ if } x>3$

$f(-1)=-5$; $f(0)=3$; $f(2)=3$; $f(4)=16$

For the following exercises, write the domain for the piecewise function in interval notation.

$f(x)=\{ x+1 \text{ if } x<-2 \ -2x-3 \text{ if } x\geq -2$

$f(x)=\{ x^2-2 \text{ if } x<1 \ -x^2+2 \text{ if } x>1$

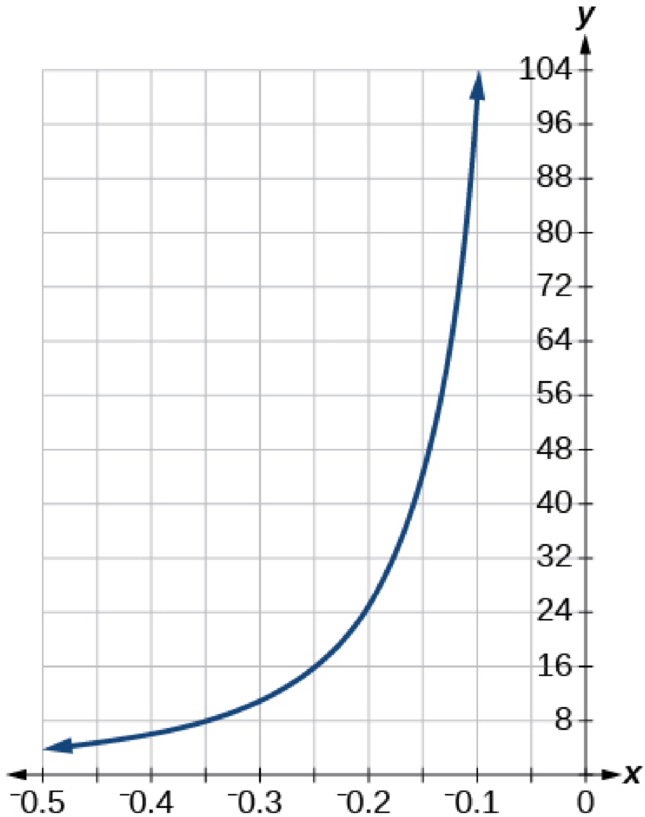
---

domain: $(-\infty,1)\cup(1,\infty)$

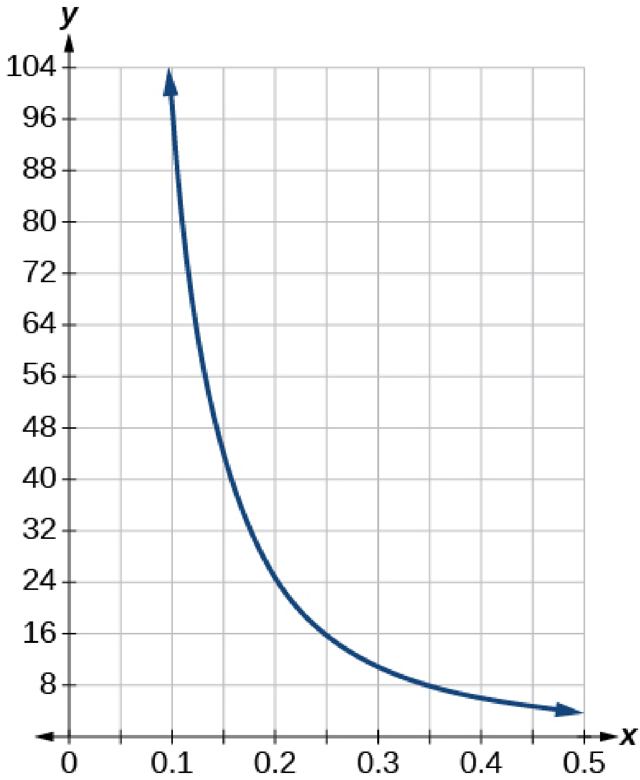
$f(x)=\{ 2x-3 \ -3x^2 \text{ if } x<0 \text{ if } x\geq 2$

Technology

Graph y= 1 x 2 on the viewing window [−0.5, −0.1] and [0.1,0.5]. Determine the corresponding range for the viewing window. Show the graphs.



window: $[-0.5,-0.1]$; range: $[4,100]$



window: [0.1,0.5]; range: [4,100]

Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

Suppose the range of a function f is $[-5,8]$.

What is the range of $|f(x)|$?

$[0,8]$

Create a function in which the range is all nonnegative real numbers.

Create a function in which the domain is $x>2$.

Many answers. One function is $f(x)= 1 x-2$.

Real-World Applications

The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t)=-16 t 2 +96t$. What is the domain of the function? What does the domain mean in the context of the problem?

The cost in dollars of making x items is given by the function $C(x)=10x+500$.

1. The fixed cost is determined when zero items are produced. Find the fixed cost for

this item.
2. What is the cost of making 25 items?
3. Suppose the maximum cost allowed is \$1500. What are the domain and range of the cost function, C(x)?

Glossary

interval notation
: a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function
: a function in which more than one formula is used to define the output

set-builder notation
: a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

Rates of Change and Behavior of Graphs

In this section, you will:

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. [link] [footnote] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in [link] did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-*y* over delta-*x*" or "the change in y divided by the change in x. " Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average

rate of change was
$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

Rate of Change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are “output units per input units.”

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input

values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Computing an Average Rate of Change

Using the data in [link], find the average rate of change of the price of gasoline between 2007 and 2009.

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\$2.41 - \$2.84}{2009 - 2007} = \frac{-\$0.43}{2 \text{ years}} = -\$0.22 \text{ per year}$$

Analysis

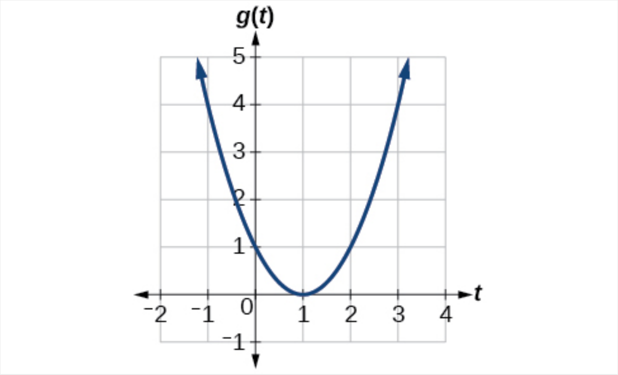
Note that a decrease is expressed by a negative change or “negative increase.” A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Using the data in [link], find the average rate of change between 2005 and 2010.

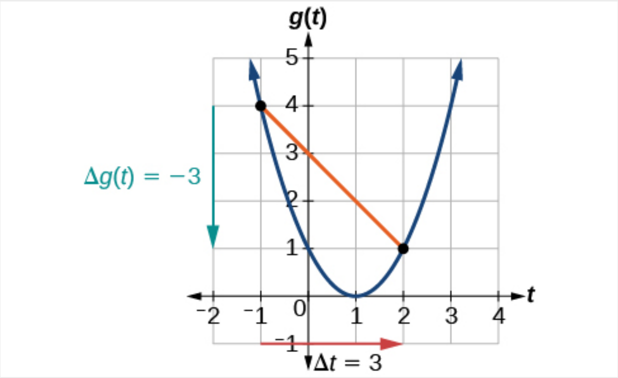
$\frac{\$2.84-\$2.31}{5\text{ years}}=\frac{\$0.53}{5\text{ years}}=$ \$0.106 per year.

Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in [link], find the average rate of change on the interval $[-1,2]$.



At $t=-1$, [link] shows $g(-1)=4$. At $t=2$, the graph shows $g(2)=1$.



The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in [link]. Find her average speed over the first 6 hours.

t **(hours)**	0	1	2	3	4	5	6	7
***D*(*t*) (miles)**	10	55	90	153	214	240	292	300

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47$$

The average speed is 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval [2,3], the average speed is 63 miles per hour.

Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval [2,4].

We can start by computing the function values at each endpoint of the interval.

$$f(2)=2^2-\frac{1}{2} \qquad f(4)=4^2-\frac{1}{4}$$
$$=4-\frac{1}{2} \qquad =16-\frac{1}{4}$$
$$=\frac{7}{2} \qquad =\frac{63}{4}$$

Now we compute the average rate of change.

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4)-f(2)}{4-2}\\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2}\\ &= \frac{\frac{49}{4}}{2}\\ &= \frac{49}{8}\end{aligned}$$

Find the average rate of change of $f(x)=x-2$ x on the interval [1,9].

1 2

Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d)=$ 2 d 2 . Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

We are computing the average rate of change of $F(d)=$ 2 d 2 on the interval [2,6].
Average rate of change $=$ $F(6)-F(2)$ $6-2$ $=$ 2 6 2 $-$ 2 2 2 $6-2$ Simplify. $=$ 2 36 $-$ 2 4 4 $=$ $-$ 16 36 4 Combine numerator terms. $=-$ 1 9 Simplify

The average rate of change is $-$ 1 9 newton per centimeter.

Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.} \\
&= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.} \\
&= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.} \\
&= \frac{a(a + 3)}{a} && \text{Divide by the common factor } a. \\
&= a + 3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

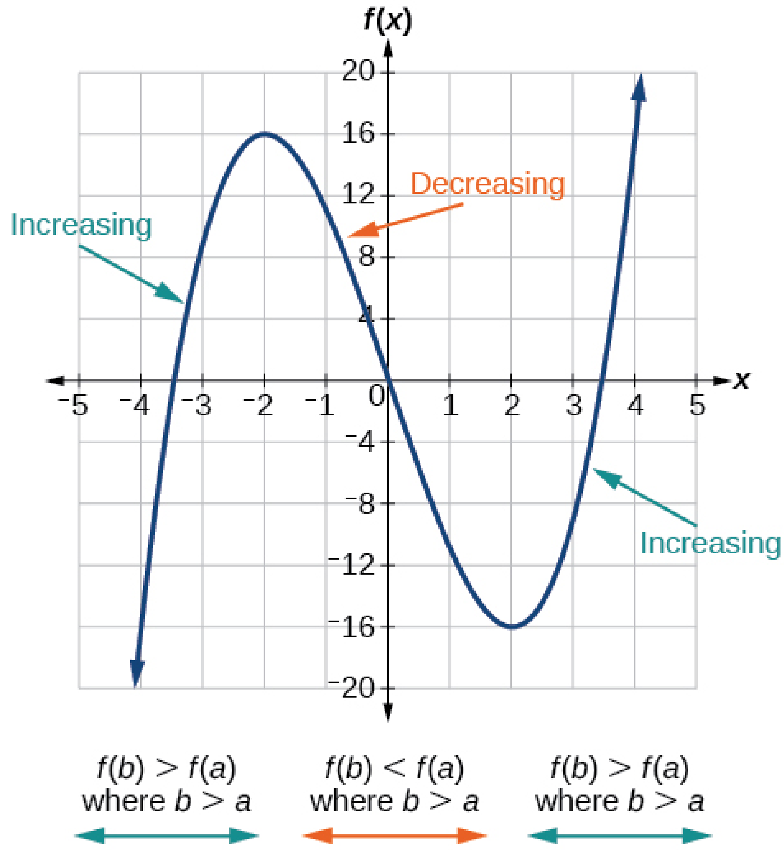
Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

$a + 7$

The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2,2)$.Definition of a local maximum

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. [link] shows examples of increasing and decreasing intervals on a function.

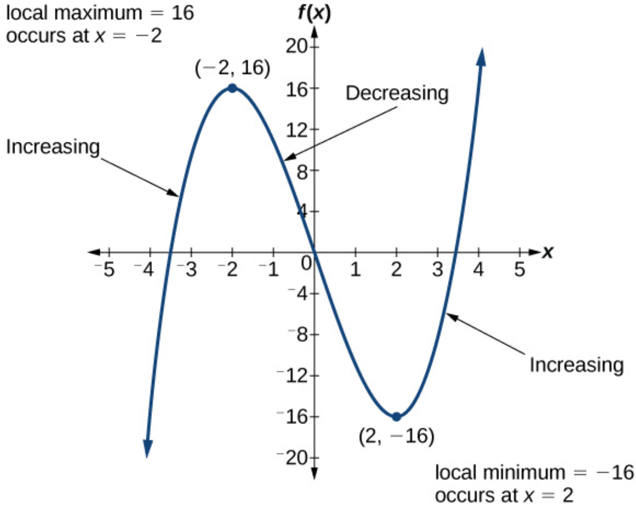


While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is the location of a **local maximum**. The function value at that point is the local maximum. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is the location of a **local**

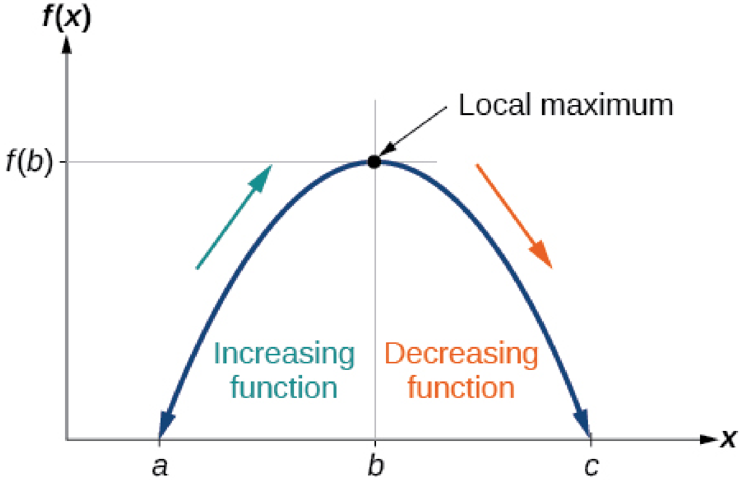
minimum. The function value at that point is the local minimum. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in [link], the local maximum is 16, and it occurs at $x=-2$. The local minimum is -16 and it occurs at $x=2$.



To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. [link] illustrates these ideas for a local maximum.



These observations lead us to a formal definition of local extrema.

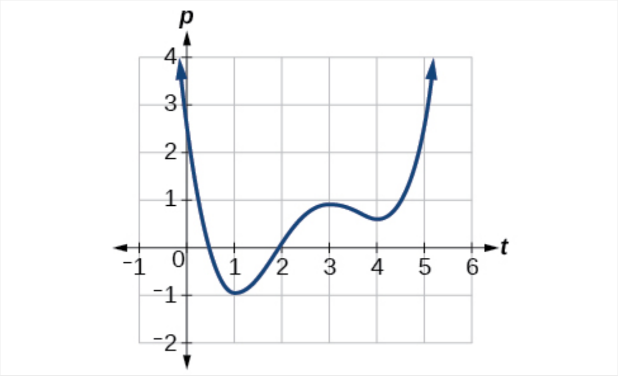
Local Minima and Local Maxima

A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.

A function f is a **decreasing function** on an open interval if f(b)<f(a) for every a , b interval where b>a.
A function f has a local maximum at a point b in an open interval (a,c) if f(b) is greater than or equal to f(x) for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a,c) if f(b) is less than or equal to f(x) for every x (x does not equal b) in the interval.

Finding Increasing and Decreasing Intervals on a Graph

Given the function p(t) in [link], identify the intervals on which the function appears to be increasing.



We see that the function is not constant on any

interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t=1$ to $t=3$ and from $t=4$ on.

In interval notation, we would say the function appears to be increasing on the interval (1,3) and the interval $(4,\infty)$.

Analysis

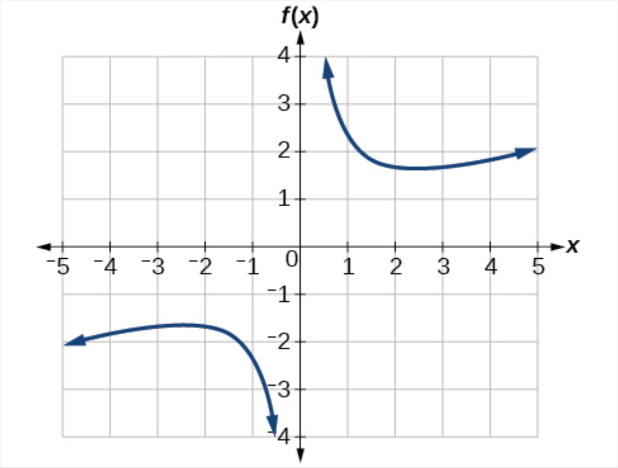
Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t=1$, $t=3$, and $t=4$. These points are the local extrema (two minima and a maximum).

Finding Local Extrema from a Graph

Graph the function $f(x)=\frac{2}{x}+\frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

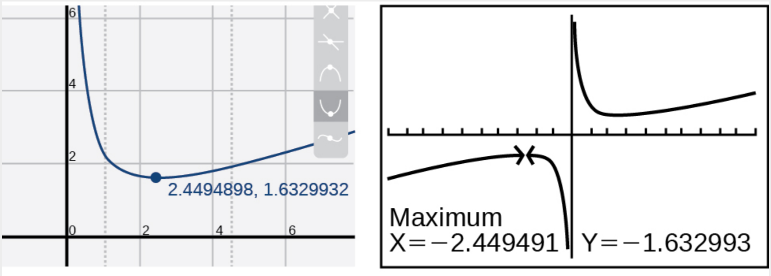
Using technology, we find that the graph of the function looks like that in [link]. It appears

there is a low point, or local minimum, between $x=2$ and $x=3$, and a mirror-image high point, or local maximum, somewhere between $x=-3$ and $x=-2$.



Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. [link] provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

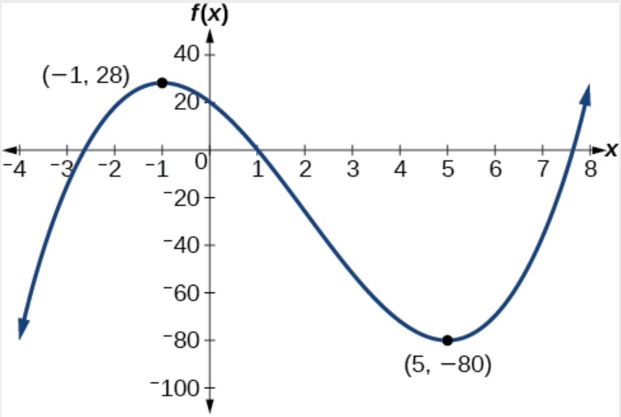


(a) (b)

Based on these estimates, the function is increasing on the interval $(-\infty,-2.449)$ and $(2.449,\infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

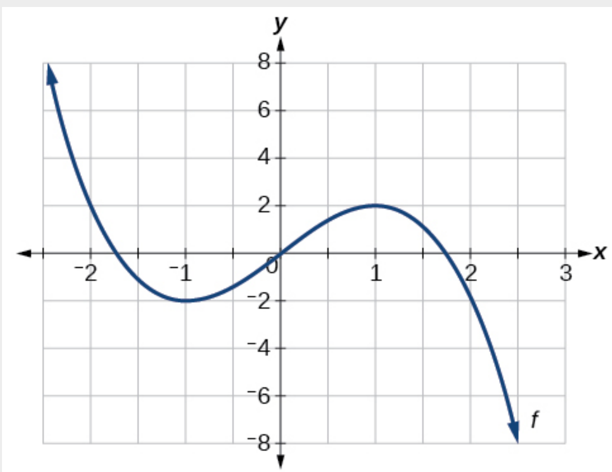
Graph the function $f(x)=x^3-6x^2-15x+20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

The local maximum appears to occur at $(-1,28)$, and the local minimum occurs at $(5,-80)$. The function is increasing on $(-\infty,-1)\cup(5,\infty)$ and decreasing on $(-1,5)$.



Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in [link], find all local maxima and minima.

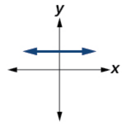
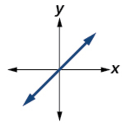
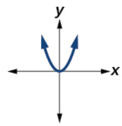
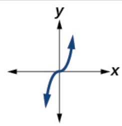
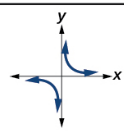
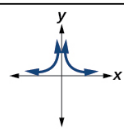
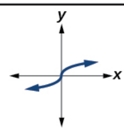
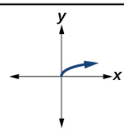
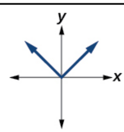


Observe the graph of f. The graph attains a local maximum at $x=1$ because it is the highest point in an open interval around $x=1$. The local maximum is the y -coordinate at $x=1$, which is 2.

The graph attains a local minimum at $x=-1$ because it is the lowest point in an open interval around $x=-1$. The local minimum is the y-coordinate at $x=-1$, which is -2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

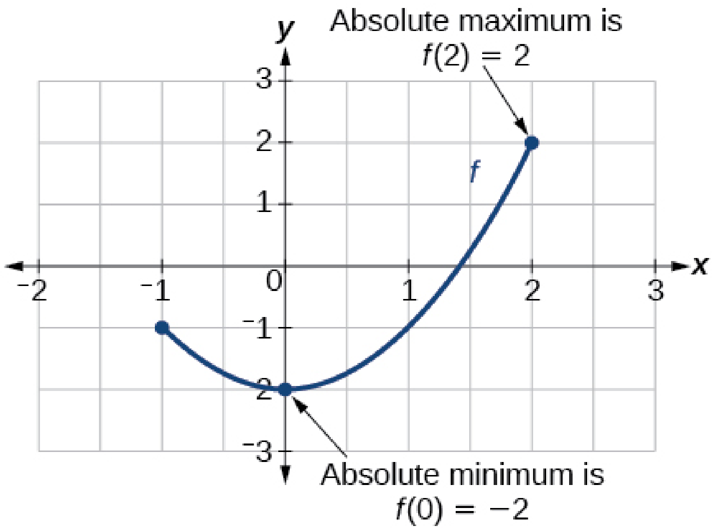
We will now return to our toolkit functions and discuss their graphical behavior in [link], [link], and [link].

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0)\cup(0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y- coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function. See [link].



Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

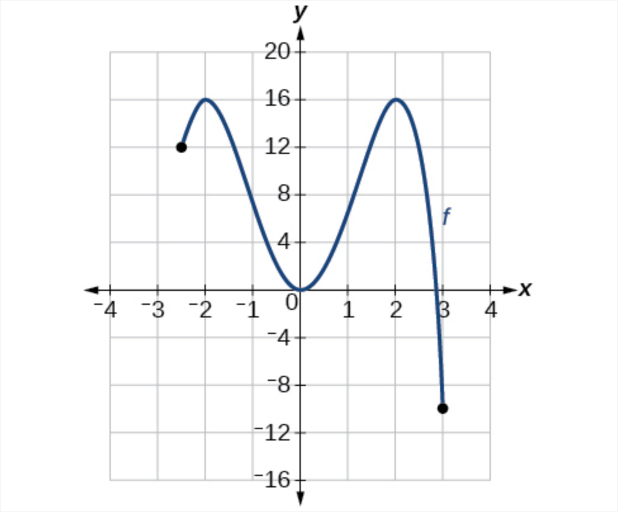
Absolute Maxima and Minima

The **absolute maximum** of f at $x=c$ is $f(c)$ where $f(c)\geq f(x)$ for all x in the domain of f.

The **absolute minimum** of f at $x=d$ is $f(d)$ where $f(d)\leq f(x)$ for all x in the domain of f.

Finding Absolute Maxima and Minima from a Graph

For the function f shown in [link], find all absolute maxima and minima.



Observe the graph of f. The graph attains an absolute maximum in two locations, $x=-2$ and $x=2$, because at these locations, the graph

attains its highest point on the domain of the function. The absolute maximum is the *y*-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the *y*-coordinate at $x = 3$, which is -10.

Access this online resource for additional instruction and practice with rates of change.

- Average Rate of Change

Key Equations

Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See [link].
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See [link].
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See [link].
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See [link] and [link].
- The average rate of change can sometimes be determined as an expression. See [link].
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See [link].
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller

(more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See [link] and [link].
- The highest and lowest points on a graph indicate the maxima and minima. See [link].

Section Exercises

Verbal

Can the average rate of change of a function be constant?

Yes, the average rate of change of all linear functions is constant.

If a function f is increasing on (a,b) and decreasing on (b,c), then what can be said about the local extremum of f on (a,c)?

How are the absolute maximum and minimum similar to and different from the local extrema?

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

How does the graph of the absolute value function compare to the graph of the quadratic function, $y=x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h.

$f(x)=4x^2-7$ on $[1,b]$

$4(b+1)$

$g(x)=2x^2-9$ on $[4,b]$

$p(x)=3x+4$ on $[2,2+h]$

3

$k(x)=4x-2$ on $[3,3+h]$

$f(x)=2x^2+1$ on $[x,x+h]$

$4x+2h$

$g(x)=3x^2-2$ on $[x,x+h]$

$a(t)= 1\ t+4$ on $[9,9+h]$

$-1\ 13(13+h)$

$b(x)= 1\ x+3$ on $[1,1+h]$

$j(x)=3x^3$ on $[1,1+h]$

$3h^2+9h+9$

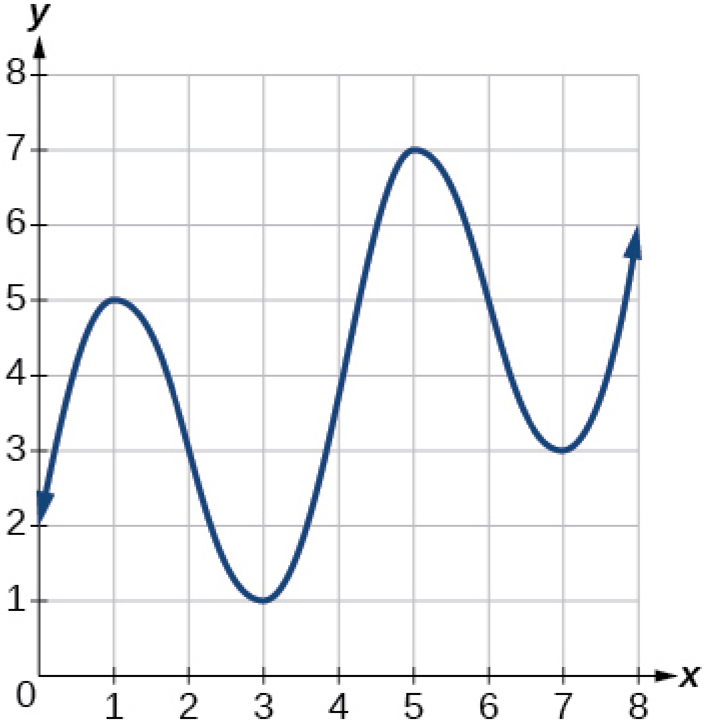
$r(t)=4t^3$ on $[2,2+h]$

$f(x)=2x^2-3x$ on $[x,x+h]$

$4x + 2h - 3$

Graphical

For the following exercises, consider the graph of f shown in [link].

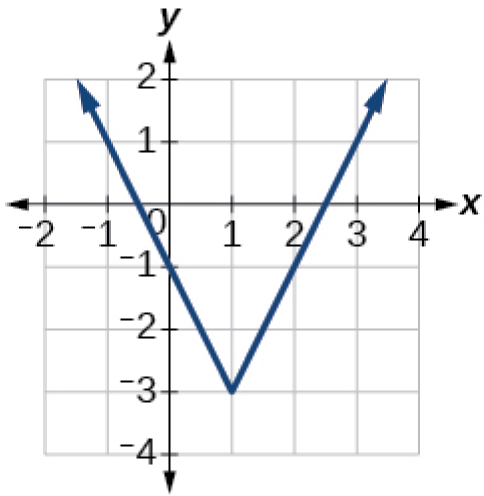


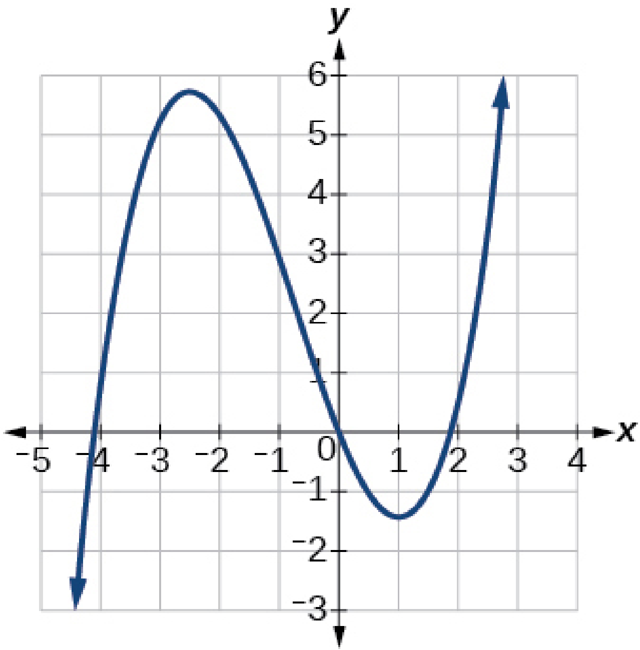
Estimate the average rate of change from $x = 1$ to $x = 4$.

Estimate the average rate of change from $x = 2$ to $x = 5$.

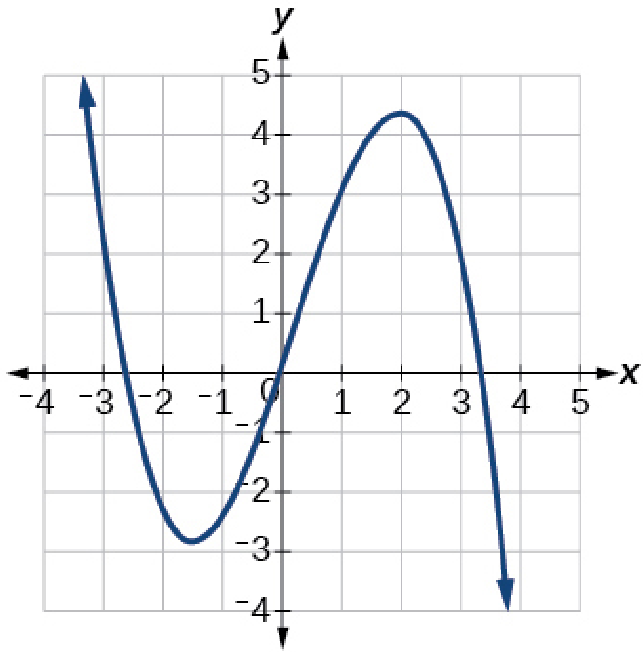
43

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

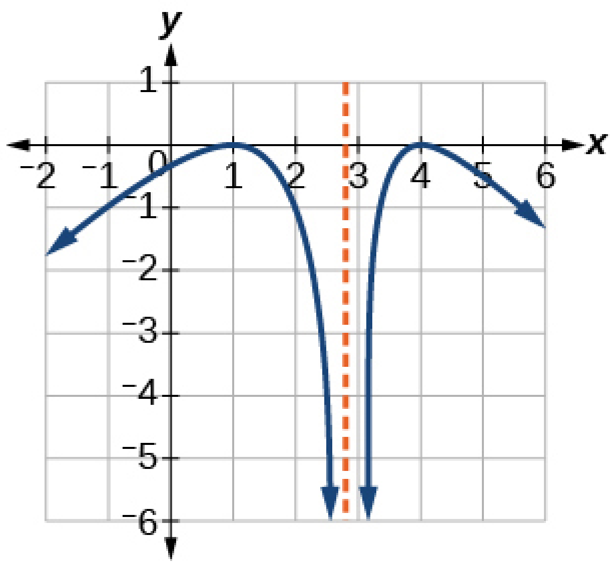




increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$



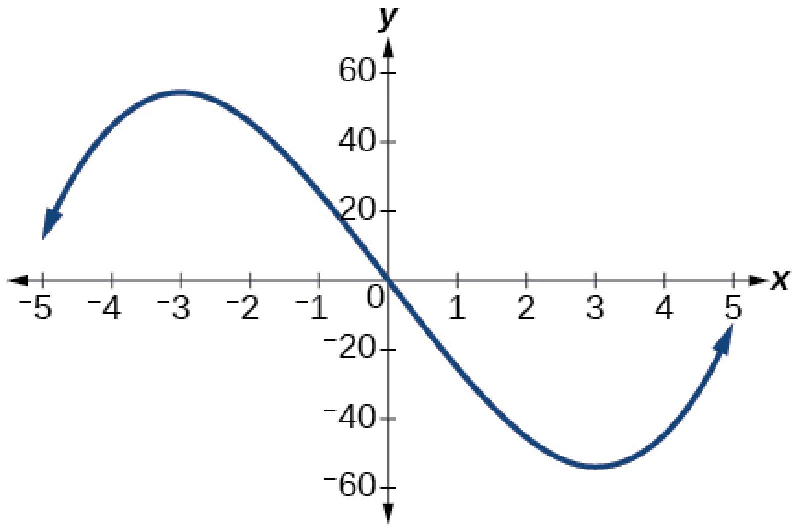
y
x
5
4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4
5



y
x
1
0
−1
−2
−3
−4
−5
−6
−2
−1
1
2
3
4
5
6

increasing on $(-\infty,1)\cup(3,4)$, decreasing on $(1,3)\cup(4,\infty)$

For the following exercises, consider the graph shown in [link].



Estimate the intervals where the function is increasing or decreasing.

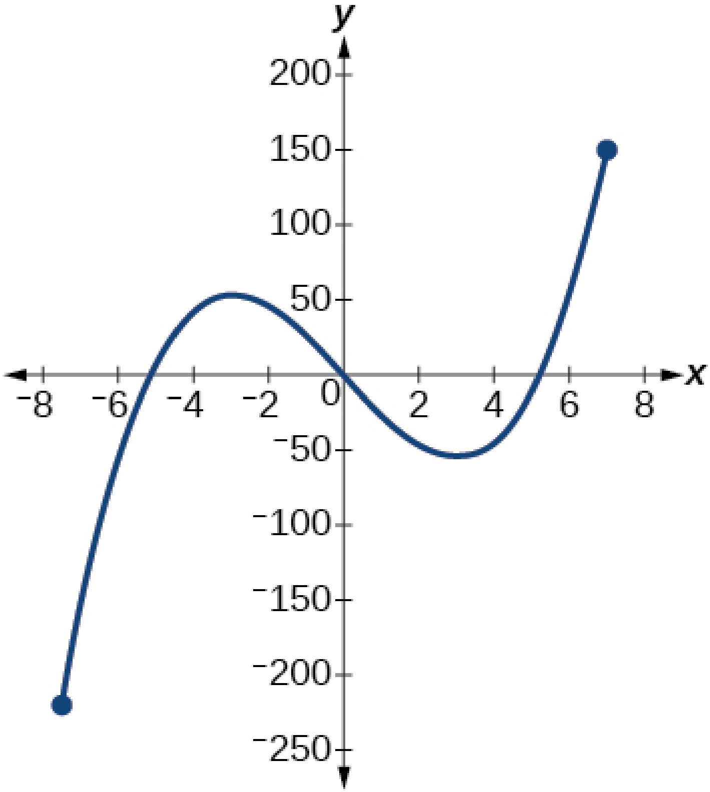
Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

---

local maximum: $(-3,50)$, local minimum: $(3,-50)$

For the following exercises, consider the graph in

[link].



If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

---

absolute maximum at approximately (7,150), absolute minimum at approximately $(-7.5,$

$-220)$

Numeric

[link] gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

[link] gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

---

a. –3000; b. –1250

For the following exercises, find the average rate of change of each function on the interval specified.

$f(x) = x^2$ on [1,5]

$h(x)=5-2x^2$ on $[-2,4]$

-4

$q(x)=x^3$ on $[-4,2]$

$g(x)=3x^3-1$ on $[-3,3]$

27

$y=\frac{1}{x}$ on $[1, 3]$

$p(t)=\frac{(t^2-4)(t+1)}{t^2+3}$ on $[-3,1]$

–0.167

$k(t)=6t^2+\frac{4}{t^3}$ on $[-1,3]$

Technology

For the following exercises, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

$f(x) = x^4 - 4x^3 + 5$

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

$h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

$g(t) = t\, t+3$

Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$

$k(t) = 3t^{\frac{2}{3}} - t$

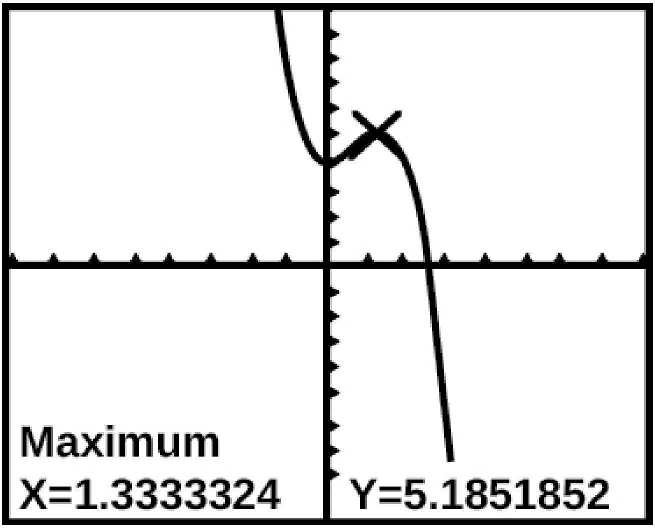
$m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

Local maximum at $(-0.5, 6)$, local minima at $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$ and $(2.1, \infty)$

$n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Extension

The graph of the function f is shown in [link].



Based on the calculator screen shot, the point (1.333,5.185) is which of the following?

1. a relative (local) maximum of the function
2. the vertex of the function
3. the absolute maximum of the function
4. a zero of the function

A

Let f(x)= 1 x . Find a number c such that the average rate of change of the function f on the interval (1,c) is − 1 4 .

Let f(x)= 1 x . Find the number b such that the average rate of change of f on the interval (2,b) is − 1 10 .

---

b=5

Real-World Applications

At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

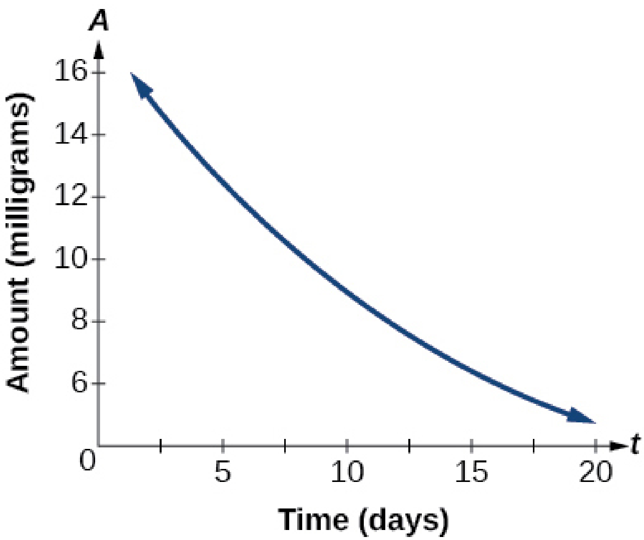
A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

2.7 gallons per minute

Near the surface of the moon, the distance that an object falls is a function of time. It is given

by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

The graph in [link] illustrates the decay of a radioactive substance over t days.



Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

---

approximately –0.6 milligrams per day

Glossary

absolute maximum

the greatest value of a function over an interval

absolute minimum
: the lowest value of a function over an interval

average rate of change
: the difference in the output values of a function found for two values of the input divided by the difference between the inputs

decreasing function
: a function is decreasing in some open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$

increasing function
: a function is increasing in some open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$

local extrema
: collectively, all of a function's local maxima and minima

local maximum
: a value of the input where a function changes from increasing to decreasing as the input value increases.

local minimum
: a value of the input where a function changes

from decreasing to increasing as the input value increases.

rate of change
: the change of an output quantity relative to the change of the input quantity

Composition of Functions

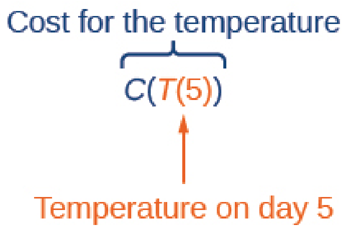
In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $\text{Cost} = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine

the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write C(T(5)).



By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife’s separate annual incomes over a period of years, with the result being their total household income. We want to do this for

every year, adding only that year's incomes and then collecting all the data in a new column. If w(y) is the wife's income and h(y) is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y)=h(y)+w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T=h+w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g, f-g, fg$, and f g by the relations

$$(f+g)(x)=f(x)+g(x) \quad (f-g)(x)=f(x)-g(x) \quad (fg)(x)=f(x)g(x) \quad (f\ g)(x)= f(x)\ g(x)$$

Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $(g\ f)(x)$, given $f(x)=x-1$ and $g(x)=x\ 2\ -1$. Are they the same function?

Begin by writing the general form, and then substitute the given functions.

$(g-f)(x)=g(x)-f(x)$ $(g-f)(x)=\ x\ 2\ -1-(x-1)$ $=\ x\ 2\ -x$ $=x(x-1)$

$(g\ f)(x)=\ g(x)\ f(x)$ $(g\ f)(x)=\ x\ 2\ -1\ x-1$ $=\ (x+1)(x-1)\ x-1$ where $x\neq 1$ $=x+1$

No, the functions are not the same.

Note: For $(g\ f)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$f(x)=x-1$ and $g(x)=\ x\ 2\ -1$

Are they the same function?

$$(fg)(x) = f(x)g(x) = (x-1)(x^2-1) = x^3 - x^2 - x + 1 \quad (f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$$

No, the functions are not the same.

Create a Function by Composition of Functions

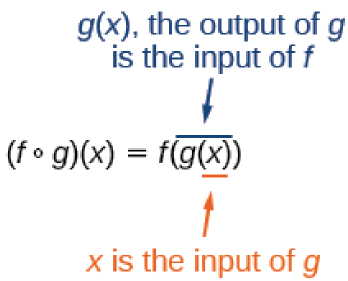
Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as “f composed with g at x,” and the right-hand side as “f of g of x.” The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is

called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output g(x). Then the function f takes g(x) as an input and yields an output f(g(x)).



In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions

can be composed only in one specific order.

For example, if $f(x)=x^2$ and $g(x)=x+2$, then

$$f(g(x)) = f(x+2) = (x+2)^2 = x^2+4x+4$$

but

$$g(f(x)) = g(x^2) = x^2+2$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f\circ g$ such that

$$(f\circ g)(x)=f(g(x))$$

The domain of the composite function $f\circ g$ is all x such that x is in the domain of g and $g(x)$ is in the

domain of f.
It is important to realize that the product of functions fg is not the same as the function composition f(g(x)), because, in general, f(x)g(x) $\neq$ f(g(x)).

Determining whether Composition of Functions is Commutative

Using the functions provided, find f(g(x)) and g(f(x)). Determine whether the composition of the functions is commutative.

$f(x)=2x+1 \quad g(x)=3-x$

Let's begin by substituting g(x) into f(x).

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute f(x) into g(x).

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= -2x+2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Interpreting Composite Functions

The function c(s) gives the number of calories burned completing s sit-ups, and s(t) gives the number of sit-ups a person can complete in t minutes. Interpret c(s(3)).

The inside expression in the composition is s(3). Because the input to the *s*-function is time, t = 3 represents 3 minutes, and s(3) is the number of sit-ups completed in 3 minutes.

Using s(3) as the input to the function c(s) gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Investigating the Order of Function Composition

Suppose f(x) gives miles that can be driven in x hours and g(y) gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: f(g(y)) or g(f(x))?

The function y = f(x) is a function whose

output is the number of miles driven corresponding to the number of hours driven.
number of miles $= f(\text{number of hours})$

The function g(y) is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:
number of gallons $= g(\text{number of miles})$

The expression g(y) takes miles as the input and a number of gallons as the output. The function f(x) requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression f(g(y)) is meaningless.

The expression f(x) takes hours as input and a number of miles driven as the output. The function g(y) requires a number of miles as the input. Using f(x) (miles driven) as an input value for g(y), where gallons of gas depends on miles driven, does make sense. The expression g(f(x)) makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, f(x), in x hours.

Are there any situations where f(g(y)) and g(f(x)) would both be meaningful or useful expressions?
Yes. For many pure mathematical functions, both

compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

The gravitational force on a planet a distance *r* from the sun is given by the function G(r). The acceleration of a planet subjected to any force F is given by the function a(F). Form a meaningful composition of these two functions, and explain what it means.

A gravitational force is still a force, so a(G(r)) makes sense as the acceleration of a planet at a distance *r* from the Sun (due to gravity), but G(a(F)) does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any

input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Using a Table to Evaluate a Composite Function

Using [link], evaluate f(g(3)) and g(f(3)).

x	f(x)	g(x)
1	6	3

2	8		5	
3	3		2	
4	1		7	

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3)=2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2)=8$.

$$g(3)=2 \quad f(g(3))=f(2)=8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3)=3$. Then, using the table for g, we can evaluate

$$g(f(3))=g(3)=2$$

[link] shows the composite functions $f\circ g$ and $g\circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Using [link], evaluate $f(g(1))$ and $g(f(4))$.

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y- axes of the graphs.

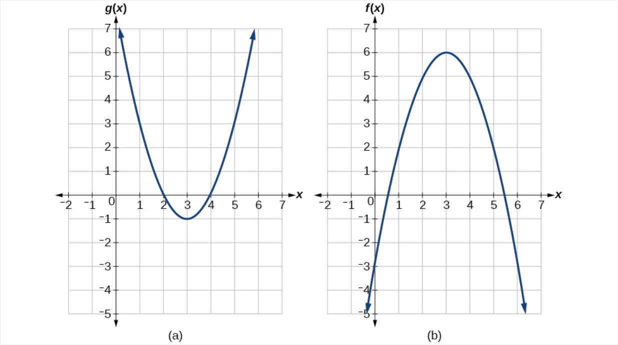
Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x- axis of its graph.
2. Read off the output of the inner function from the y- axis of its graph.
3. Locate the inner function output on the x- axis of the graph of the outer function.
4. Read the output of the outer function from the y- axis of its graph. This is the output of the

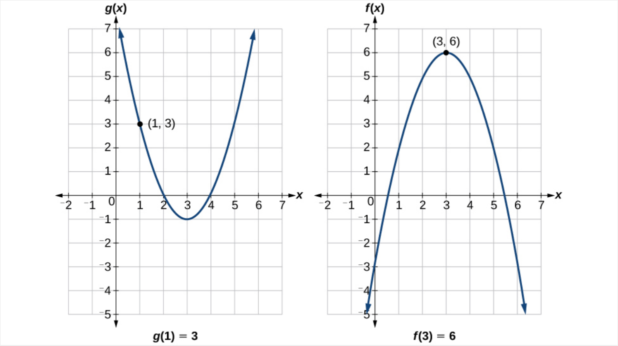
composite function.

Using a Graph to Evaluate a Composite Function

Using [link], evaluate f(g(1)).



To evaluate f(g(1)), we start with the inside evaluation. See [link].



We evaluate g(1) using the graph of g(x),

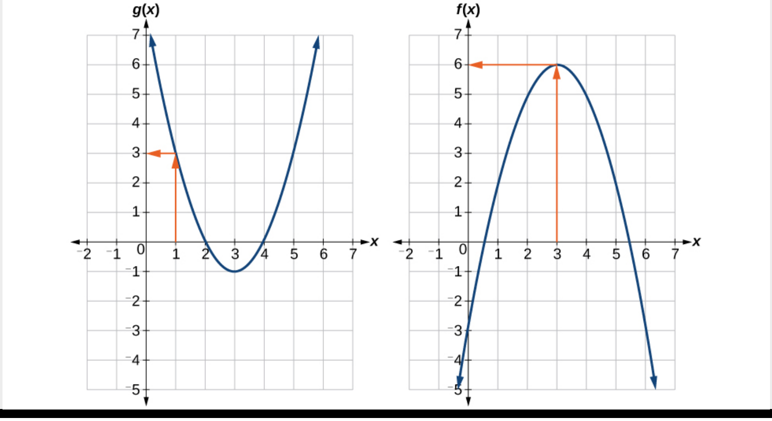
finding the input of 1 on the x- axis and finding the output value of the graph at that input. Here, $g(1)=3$. We use this value as the input to the function f.

$f(g(1))=f(3)$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x- axis and reading the output value of the graph at this input. Here, $f(3)=6$, so $f(g(1))=6$.

Analysis

[link] shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



Using [link], evaluate g(f(2)).

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$h(1) = 3(1) + 2$ $h(1) = 5$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$f(h(1)) = f(5)$ $f(h(1)) = 5^2 - 5$ $f(h(1)) = 20$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

1. $h(f(2))$
2. $h(f(-2))$

a. 8; b. 20

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still

undefined. Thus the domain of f∘g consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and g(x) is in the domain of f.

Domain of a Composite Function

The domain of a composite function f(g(x)) is the set of those inputs x in the domain of g for which g(x) is in the domain of f.

Given a function composition f(g(x)), determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which g(x) is in the domain of f. That is, exclude those inputs x from the domain of g for which g(x) is not in the domain of f. The resulting set is the domain of f∘g.

Finding the Domain of a Composite

Function

Find the domain of
$(f\circ g)(x)$ where $f(x)=\ 5\ x-1$ and $g(x)=\ 4\ 3x\ -2$

The domain of $g(x)$ consists of all real numbers except $x=\ 2\ 3$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x)=1$.
$4\ 3x-2\ =1\ 4=3x-2\ 6=3x\ x=2$

So the domain of $f\circ g$ is the set of all real numbers except 2 3 and 2. This means that
$x\neq\ 2\ 3$ or $x\neq 2$

We can write this in interval notation as
$(-\infty,\ 2\ 3\ )\cup(\ 2\ 3\ ,2\ )\cup(\ 2,\infty\ )$

Finding the Domain of a Composite Function Involving Radicals

Find the domain of
$(f\circ g)(x)$ where $f(x)=\ x+2$ and $g(x)=\ 3-x$

Because we cannot take the square root of a negative number, the domain of g is $(-\infty,3]$. Now we check the domain of the composite function

$(f\circ g)(x) = 3-x+2$

For $(f\circ g)(x) = 3-x+2$, $3-x+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $3-x \geq 0$, or, $3-x \geq 0$, which gives a domain of $(-\infty,3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f\circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Find the domain of

$(f\circ g)(x)$ where $f(x) = 1\ x-2$ and $g(x) = x+4$

$[-4,0)\cup(0,\infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Decomposing a Function

Write $f(x) = 5 - x^2$ as the composition of two functions.

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$h(x) = 5 - x^2$ and $g(x) = x$

We can check our answer by recomposing the

functions.
$g(h(x))=g(\ 5-\ x\ 2\ )=\ 5-\ x\ 2$

Write $f(x)=\ 4\ 3-\ 4+\ x\ 2$ as the composition of two functions.

Possible answer:

$g(\ x\ )=\ 4+\ x\ 2$
$h(\ x\ )=\ 4\ 3-x$
$f=h\circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions
- Composite Function Notation Application
- Composite Functions Using Graphs
- Decompose Functions
- Composite Function Values

Key Equation

Composite function	$(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See [link].
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See [link] and [link].
- The order of function composition must be considered when interpreting the meaning of composite functions. See [link].
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See [link].

- A composite function can be evaluated from a graph. See [link].
- A composite function can be evaluated from a formula. See [link].
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See [link] and [link].
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See [link].

Section Exercises

Verbal

How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

What is the composition of two functions, $f \circ g$?

If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

How do you find the domain for the composition of two functions, $f \circ g$?

Algebraic

Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g, f - g, fg$, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

$(f + g)(x) = 2x + 6$, domain: $(-\infty, \infty)$

$(f - g)(x) = 2x^2 + 2x - 6$, domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x$, domain:

$(-\infty,\infty)$

$(\ f\ g\ )(\ x\ )=\ x\ 2\ +2x\ 6-\ x\ 2$, domain: $(-\infty,-\ 6\ )\cup(-\ 6\ ,\ 6\ )\cup(\ 6\ ,\infty)$

Given $f(x)=-3\ x\ 2\ +x$ and $g(x)=5$, find $f+g,f-g,fg$, and $f\ g$. Determine the domain for each function in interval notation.

Given $f(x)=2\ x\ 2\ +4x$ and $g(x)=\ 1\ 2x$, find $f+g,f-g,fg$, and $f\ g$. Determine the domain for each function in interval notation.

$(f+g)(\ x\ )=\ 4\ x\ 3\ +8\ x\ 2\ +1\ 2x$, domain: $(-\infty,0)\cup(0,\infty)$

$(f-g)(\ x\ )=\ 4\ x\ 3\ +8\ x\ 2\ -1\ 2x$, domain: $(-\infty,0)\cup(0,\infty)$

$(fg)(\ x\ )=x+2$, domain: $(-\infty,0)\cup(0,\infty)$

$(\ f\ g\ )(\ x\ )=4\ x\ 3\ +8\ x\ 2$, domain: $(-\infty,0)\cup(0,\infty)$

Given $f(x)=\ 1\ x-4$ and $g(x)=\ 1\ 6-x$, find $f+g,f-g,fg$, and $f\ g$. Determine the domain for each function in interval notation.

Given f(x) = 3 x 2 and g(x) = x − 5 , find f + g, f − g, fg, and f g . Determine the domain for each function in interval notation.

(f + g)(x) = 3 x 2 + x − 5 , domain: [5, ∞)

(f − g)(x) = 3 x 2 − x − 5 , domain: [5, ∞)

(fg)(x) = 3 x 2 x − 5 , domain: [5, ∞)

(f g)(x) = 3 x 2 x − 5 , domain: (5, ∞)

Given f(x) = x and g(x) = |x − 3|, find g f . Determine the domain of the function in interval notation.

Given f(x) = 2 x 2 + 1 and g(x) = 3x − 5, find the following:

1. f(g(2))
2. f(g(x))
3. g(f(x))
4. (g∘g)(x)
5. (f∘f)(−2)

---

a. 3; b. f(g(x)) = 2 (3x − 5) 2 + 1; c. g(f)(x)) = 6 x 2 − 2; d. (g∘g)(x) = 3(3x − 5) − 5 = 9x − 20; e. (f∘f)(−2) = 163

For the following exercises, use each pair of functions to find f(g(x)) and g(f(x)). Simplify your answers.

f(x)= x 2 +1,g(x)= x+2

f(x)= x +2,g(x)= x 2 +3

---

f(g(x))= x 2 +3 +2,g(f(x))=x+4 x +7

f(x)=| x |,g(x)=5x+1

f(x)= x 3 ,g(x)= x+1 x 3

---

f(g(x))= x+1 x 3 3 = x+1 3 x ,g(f(x))= x 3 +1 x

f(x)= 1 x−6 ,g(x)= 7 x +6

f(x)= 1 x−4 ,g(x)= 2 x +4

(f∘g)(x)= 1 2 x +4−4 = x 2 ,(g∘f)(x)=2x −4

For the following exercises, use each set of functions to find f(g(h(x))). Simplify your answers.

$f(x)= x 4 +6$, $g(x)=x-6$, and $h(x)= x$

$f(x)= x 2 +1$, $g(x)= 1 x$, and $h(x)=x+3$

$f(g(h(x)))= (1 x+3) 2 +1$

Given $f(x)= 1 x$ and $g(x)=x-3$, find the following:

1. $(f\circ g)(x)$
2. the domain of $(f\circ g)(x)$ in interval notation
3. $(g\circ f)(x)$
4. the domain of $(g\circ f)(x)$
5. $(f g)x$

Given $f(x)= 2-4x$ and $g(x)= - 3 x$, find the following:

1. $(g\circ f)(x)$
2. the domain of $(g\circ f)(x)$ in interval notation

---

a. $(g\circ f)(x)= - 3 2-4x$; b. $(-\infty , 1 2)$

Given the functions $f(x) = 1 - x\ x$ and $g(x) = 1$
$1 + x\ 2$, find the following:

1. $(g \circ f)(x)$
2. $(g \circ f)(2)$

Given functions $p(x) = 1\ x$ and $m(x) = x\ 2\ - 4$, state the domain of each of the following functions using interval notation:

1. $p(x)\ m(x)$
2. $p(m(x))$
3. $m(p(x))$

---

a. $(0,2) \cup (2, \infty)$; b. $(-\infty, -2) \cup (2, \infty)$; c. $(0, \infty)$

Given functions $q(x) = 1\ x$ and $h(x) = x\ 2\ - 9$, state the domain of each of the following functions using interval notation.

1. $q(x)\ h(x)$
2. $q(\ h(x)\ )$
3. $h(\ q(x)\ )$

For $f(x) = 1\ x$ and $g(x) = x - 1$, write the domain of $(f \circ g)(x)$ in interval notation.

$(1, \infty)$

For the following exercises, find functions f(x) and g(x) so the given function can be expressed as $h(x)=f(\ g(x)\ )$.

$h(x)=\ (x+2)\ 2$

$h(x)=\ (x-5)\ 3$

---

sample: $f(x)=\ x\ 3$ $g(x)=x-5$

$h(x)=\ 3\ x-5$

$h(x)=\ 4\ (x+2)\ 2$

---

sample: $f(x)=\ 4\ x$ $g(x)=\ (x+2)\ 2$

$h(x)=4+\ x\ 3$

$h(x)=\ 1\ 2x-3\ 3$

---

sample: $f(x)=\ x\ 3$ $g(x)=\ 1\ 2x-3$

$h(x) = 1\ (3\ x\ 2\ - 4)\ - 3$

$h(x) = 3x - 2\ x + 5\ 4$

---

sample: $f(x) = x\ 4$ $g(x) = 3x - 2\ x + 5$

$h(x) = (\ 8 + x\ 3\ 8 - x\ 3\ )\ 4$

$h(x) = 2x + 6$

---

sample: $f(x) = x$
$g(x) = 2x + 6$

$h(x) = (5x - 1)\ 3$

$h(x) = x - 1\ 3$

---

sample: $f(x) = x\ 3$
$g(x) = (x - 1)$

$h(x) = |\ x\ 2\ + 7\ |$

$h(x) = 1\ (x - 2)\ 3$

sample: $f(x) = x\ 3$
$g(x) = 1\ x - 2$

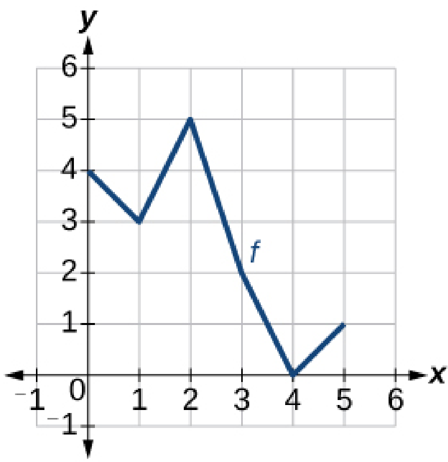
$h(x) = (\ 1\ 2x - 3\ )\ 2$

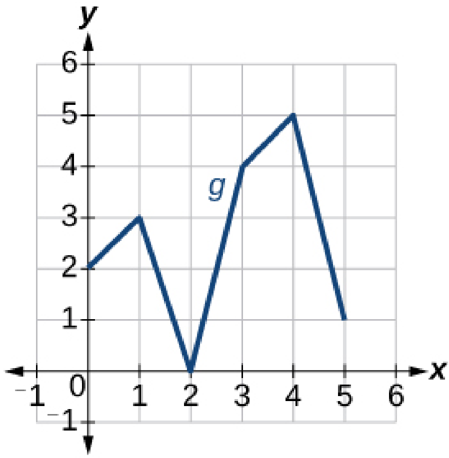
$h(x) = 2x - 1\ 3x + 4$

sample: $f(x) = x$
$g(x) = 2x - 1\ 3x + 4$

Graphical

For the following exercises, use the graphs of f, shown in [link], and g, shown in [link], to evaluate the expressions.





f(g(3))

f(g(1))

2

g(f(1))

g(f(0))

5

f(f(5))

f(f(4))

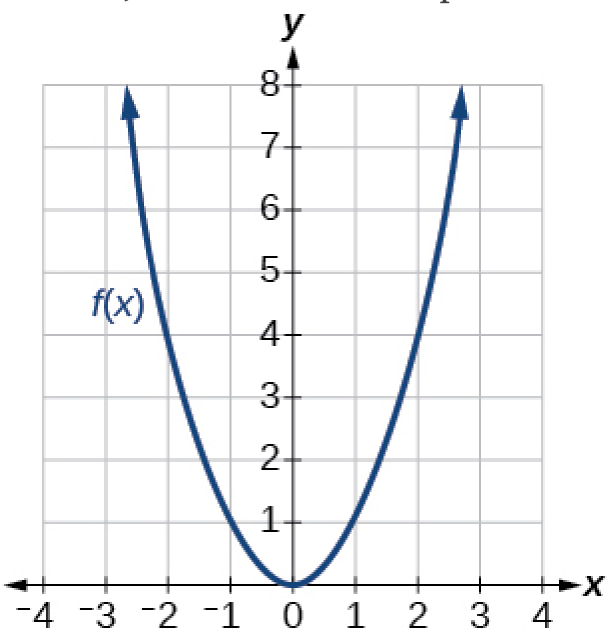
4

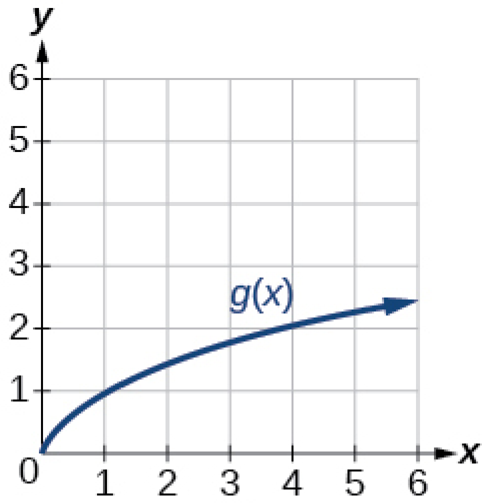
g(g(2))

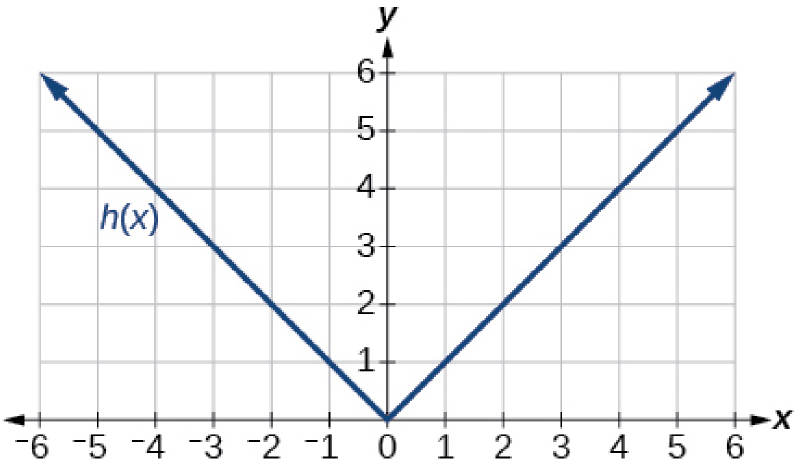
g(g(0))

0

For the following exercises, use graphs of f(x), shown in [link], g(x), shown in [link], and h(x), shown in [link], to evaluate the expressions.







g(f(1))

g(f(2))

2

$f(g(4))$

$f(g(1))$

1

$f(h(2))$

$h(f(2))$

4

$f(g(h(4)))$

$f(g(f(-2)))$

4

Numeric

For the following exercises, use the function values for f and g shown in [link] to evaluate each expression.

x	f(x)	g(x)
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

f(g(8))

f(g(5))

9

g(f(5))

g(f(3))

4

f(f(4))

f(f(1))

2

g(g(2))

g(g(6))

3

For the following exercises, use the function values for f and g shown in [link] to evaluate the expressions.

x	f(x)	g(x)
3	11	8
2	9	3
1	7	0
0	5	1
1	3	0
2	1	3

3	-1		-8	

(f∘g)(1)

(f∘g)(2)

11

(g∘f)(2)

(g∘f)(3)

0

(g∘g)(1)

(f∘f)(3)

7

For the following exercises, use each pair of functions to find f(g(0)) and g(f(0)).

$f(x)=4x+8, g(x)=7-x^2$

$f(x)=5x+7, g(x)=4-2x^2$

$f(g(0))=27, g(f(0))=-94$

$f(x)=\ x+4\ , g(x)=12-x^3$

$f(x)=\ 1\ x+2\ , g(x)=4x+3$

$f(g(0))=\ 1\ 5\ , g(f(0))=5$

For the following exercises, use the functions $f(x)=2x^2+1$ and $g(x)=3x+5$ to evaluate or find the composite function as indicated.

$f(g(2))$

$f(g(x))$

$18x^2+60x+51$

$g(f(-3))$

$(g \circ g)(x)$

$g \circ g(x) = 9x + 20$

Extensions

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

2

What is the domain of $(g \circ f)(x)$?

What is the domain of $(f \circ g)(x)$?

$(-\infty, \infty)$

Let $f(x) = \frac{1}{x}$.

1. Find $(f \circ f)(x)$.
2. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x+1$.

True or False: $(g \circ f)(x) = F(x)$.

False

True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

$(f \circ g)(6); (g \circ f)(6)$

$(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

$(g \circ f)(a); (f \circ g)(a)$

$(f \circ g)(11); (g \circ f)(11)$

$(f \circ g)(11) = 11, (g \circ f)(11) = 11$

Real-World Applications

The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $6, you would do which of the following?

1. Evaluate $D(\ C(6)\ )$.
2. Evaluate $C(\ D(6)\ )$.
3. Solve $D(\ C(x)\ ) = 6$.
4. Solve $C(\ D(p)\ ) = 6$.

The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

1. Evaluate $A(\ m(4)\ )$.
2. Evaluate $m(\ A(4)\ )$.

3. Solve $A(\, m(t)\,) = 4$.
4. Solve $m(\, A(d)\,) = 4$.

---

c

A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function P(x) that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)

A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25 \, t + 2$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

$A(t) = \pi \, (\, 25 \, t + 2 \,) \, 2$ and $A(2) = \pi \, (\, 25 \; 4 \,) \, 2$ $= 2500\pi$ square inches

A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula

$r(t)=2t+1$, express the area burned as a function of time, t (minutes).

Use the function you found in the previous exercise to find the total area burned after 5 minutes.

$A(5)=\pi\left(2(5)+1\right)^2=121\pi$ square units

The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V)=\sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t)=10+20t$.

1. Find the composite function $r(V(t))$.
2. Find the *exact* time when the radius reaches 10 inches.

The number of bacteria in a refrigerated food product is given by $N(T)=23T^2-56T+1$, $3<T<33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t)=5t+1.5$, where t is the time in hours.

1. Find the composite function $N(T(t))$.
2. Find the time (round to two decimal

places) when the bacteria count reaches 6752.

---

a. $N(T(t))=23(5t+1.5)^2-56(5t+1.5)+1$; b. 3.38 hours

Glossary

composite function
: the new function formed by function composition, when the output of one function is used as the input of another

Transformation of Functions

In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x -axis and the y -axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

(credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may

shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Vertical shift by k=1 of the cube root function f(x)= x 3 . Horizontal shift of the function f(x)= x 3 . Note that (x+1) means h=-1 which shifts the graph to the left, that is, towards *negative* values of x.

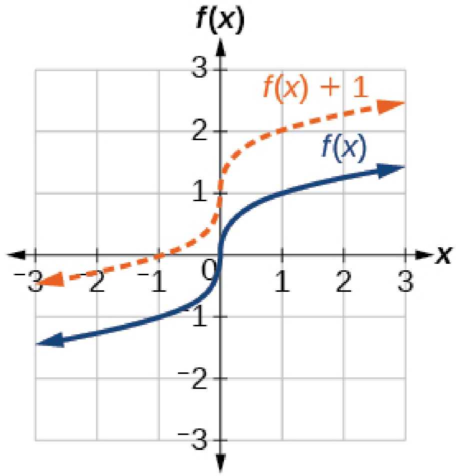
Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting

the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x)=f(x)+k$, the function $f(x)$ is shifted vertically k units. See [link] for an example.



To help you visualize the concept of a vertical shift, consider that $y=f(x)$. Therefore, $f(x)+k$ is equivalent to $y+k$. Every unit of y is replaced by $y+k$, so the y- value increases or decreases depending on the value of k. The result is a shift upward or downward.

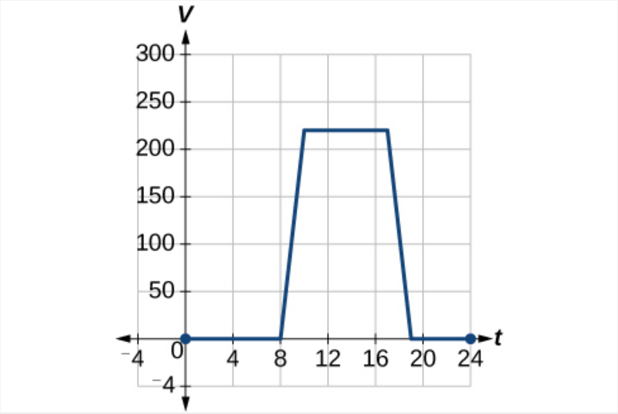
Vertical Shift

Given a function $f(x)$, a new function

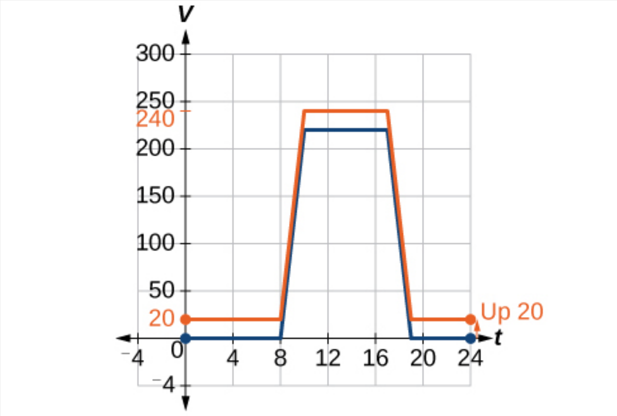
$g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function f(x). All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. [link] shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.



We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in [link].



Notice that in [link], for each input value, the output value has increased by 20, so if we call the new function S(t), we could write
$S(t) = V(t) + 20$

This notation tells us that, for any value of t,S(t) can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See [link].

t	0	8	10	17	19	24
V(t)	0	0	220	220	0	0
S(t)	20	20	240	240	20	20

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Shifting a Tabular Function Vertically

A function f(x) is given in [link]. Create a table for the function $g(x)=f(x)-3$.

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$f(2) = 1$ Given $g(x) = f(x) - 3$

Given transformation $g(2) = f(2) - 3 = 1 - 3 = -2$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in [link].

x	2		4		6		8		
f(x)	1		3		7		11		
g(x)	-2		0		4		8		

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

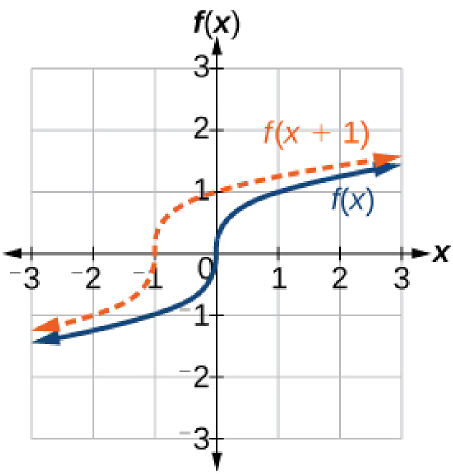
The function $h(t) = -4.9t^2 + 30t$ gives the

height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function b(t) to h(t), and then find a formula for b(t).

$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in [link].



For example, if $f(x)= x^2$, then $g(x)= (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x)=f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Adding a Constant to an Input

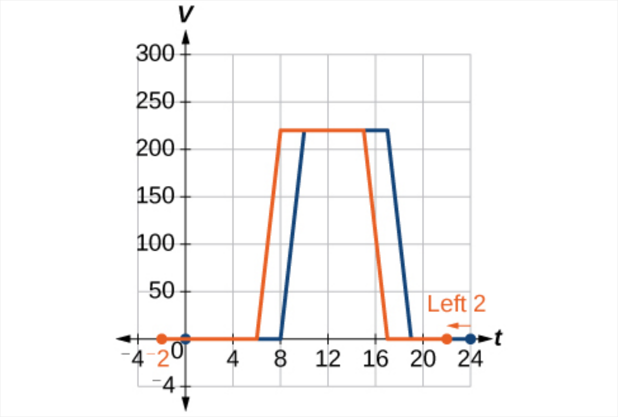
Returning to our building airflow example from [link], suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$V(t)=$ the original venting plan $F(t)=$starting 2 hrs sooner

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8)=F(6)$. See [link]. Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10)=F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h=-2$. That means that the same output values are reached when $F(t)=V(t-(-2))=V(t+2)$.



Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the *left*.

Horizontal changes or “inside changes” affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function F(t) uses the same outputs as V(t), but matches those outputs to inputs 2 hours earlier than those of V(t). Said another way, we must add 2 hours to the input of V to find the corresponding output for F: $F(t)=V(t+2)$.

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Shifting a Tabular Function Horizontally

A function f(x) is given in [link]. Create a table for the function $g(x)=f(x-3)$.

x	2		4		6		8	

f(x)	1	3	7	11

The formula $g(x)=f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2)=1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$g(5)=f(5-3) \quad =f(2) \quad =1$$

We continue with the other values to create [link].

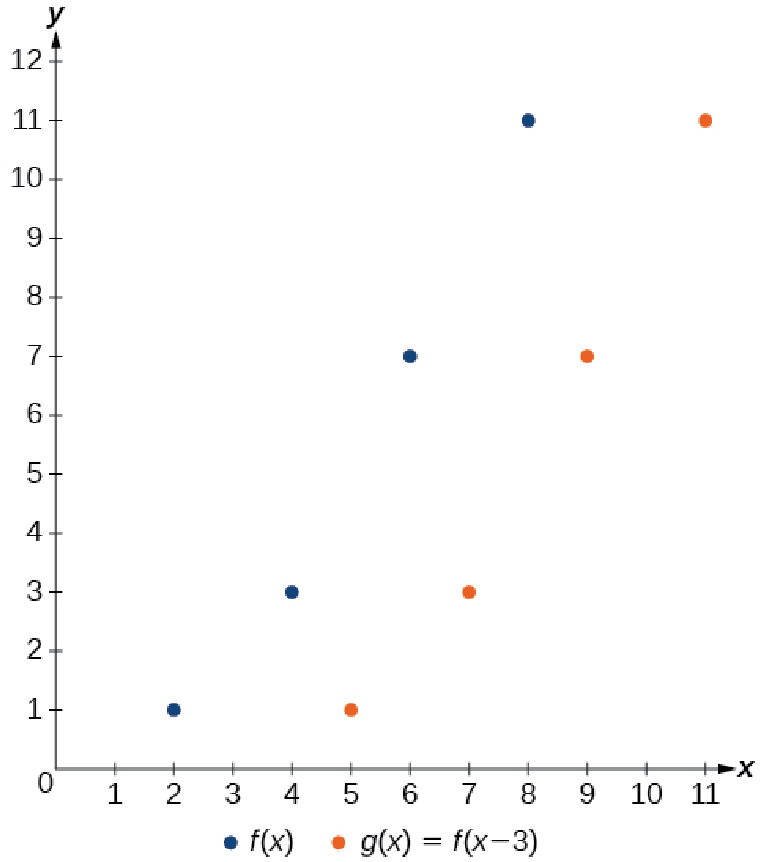
x	5	7	9	11
x − 3	2	4	6	8
f(x − 3)	1	3	7	11
g(x)	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input

values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

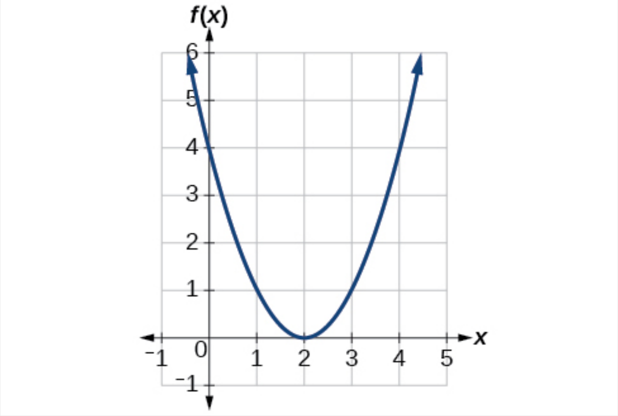
Analysis

[link] represents both of the functions. We can see the horizontal shift in each point.



Identifying a Horizontal Shift of a Toolkit Function

[link] represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the *x*-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the *x*-values must be 2 units larger because of the shift to the right by 2 units. We can then use the

definition of the f(x) function to write a formula for g(x) by evaluating $f(x-2)$.

$f(x) = x^2$ $g(x) = f(x-2)$ $g(x) = f(x-2) = (x-2)^2$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x-2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

Interpreting Horizontal versus Vertical Shifts

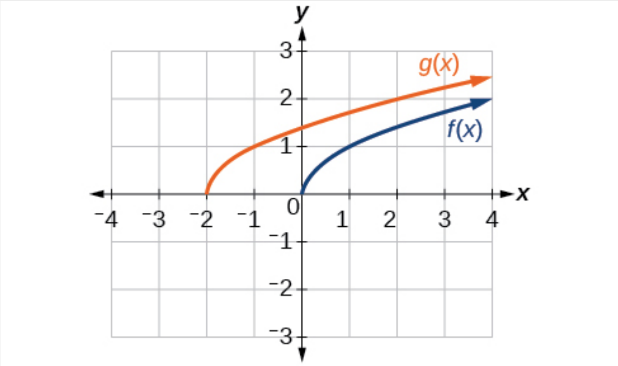
The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m+10)$.

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

G(m+10) can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Given the function $f(x) = x$, graph the original function f(x) and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

The graphs of f(x) and g(x) are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.



Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output (y-) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left.

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Graphing Combined Vertical and

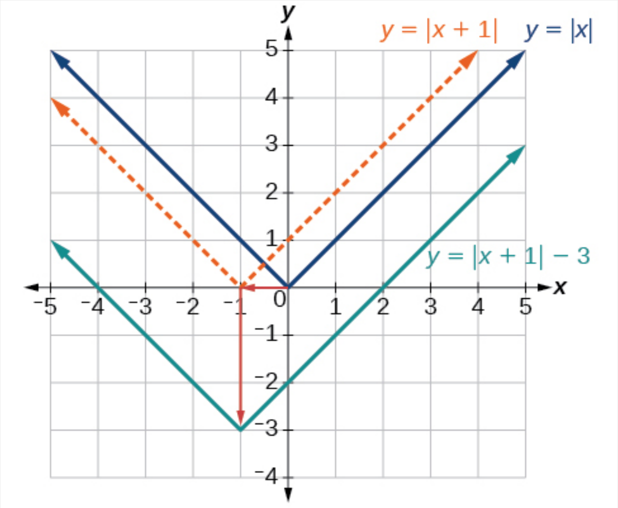
Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

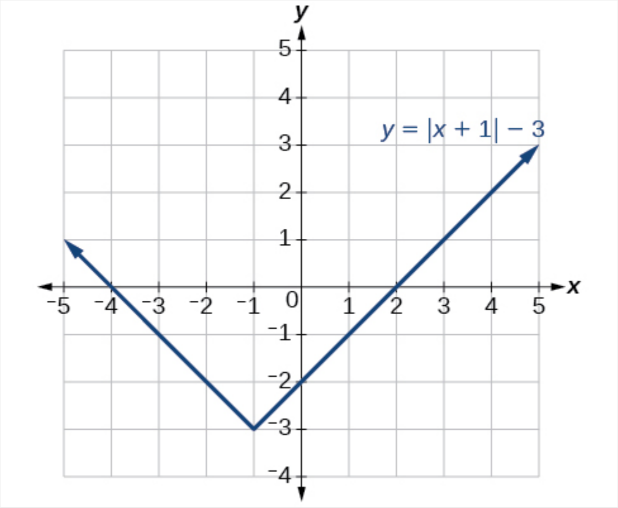
The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in [link].

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit: $(0,0)\to(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units: $(-1,0)\to(-1,-3)$

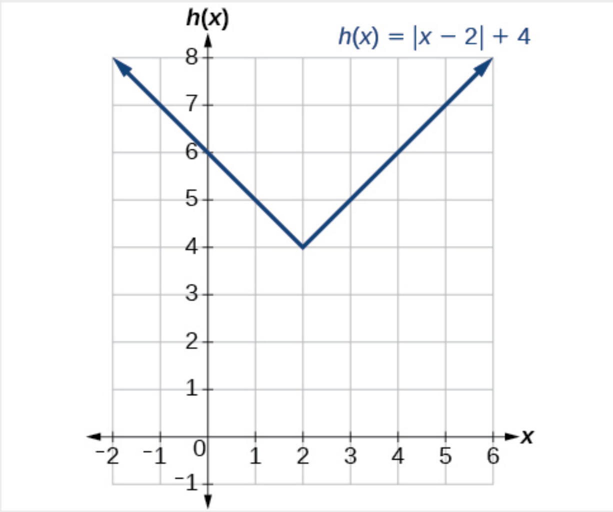


[link] shows the graph of h.



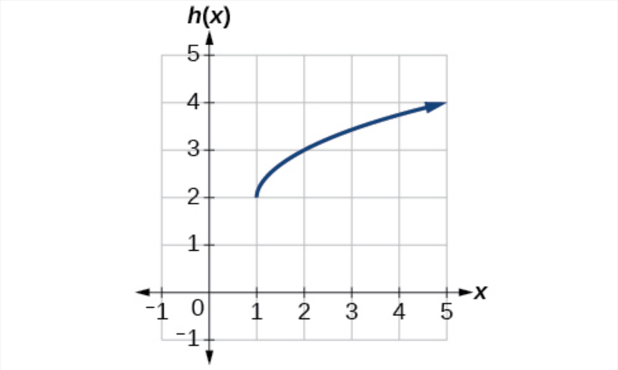
Given $f(x) = | x |$, sketch a graph of $h(x) = f(x$

$-2)+4.$



Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in [link], which is a transformation of the toolkit square root function.



The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as
$h(x)=f(x-1)+2$

Using the formula for the square root function, we can write
$h(x)=\sqrt{x-1}+2$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1,\infty)$ and range $[2,\infty)$.

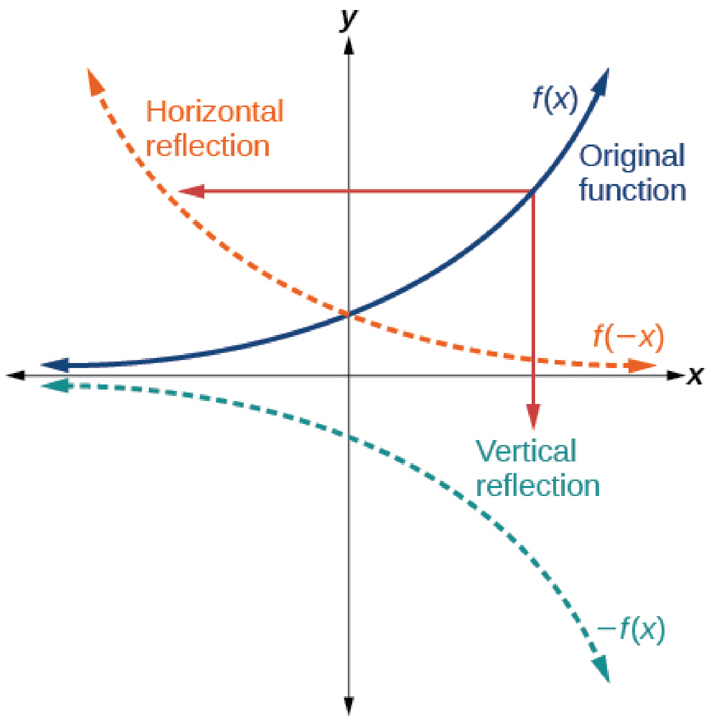
Write a formula for a transformation of the toolkit reciprocal function $f(x)=\frac{1}{x}$ that

shifts the function’s graph one unit to the right and one unit up.
g(x)= 1 x-1 +1

Vertical and horizontal reflections of a function.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the *x*- or *y*-axis. A **vertical reflection** reflects a graph vertically across the *x*-axis, while a **horizontal reflection** reflects a graph horizontally across the *y*-axis. The reflections are shown in [link].



Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the *x*-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the *y*-axis.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$

is a **horizontal reflection** of the function f(x), sometimes called a reflection about the *y*-axis.

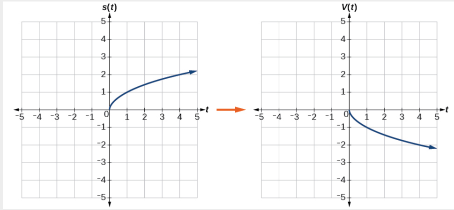
Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = t$ (a) vertically and (b) horizontally.

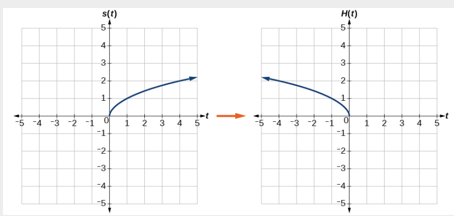
1. Reflecting the graph vertically means that each output value will be reflected over the horizontal *t*-axis as shown in [link]. Vertical reflection of the square root function



Because each output value is the opposite of the original output value, we can write $V(t) = -s(t)$ or $V(t) = -\ t$

Notice that this is an outside change, or vertical shift, that affects the output s(t) values, so the negative sign belongs outside of the function.

2. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in [link].
   Horizontal reflection of the square root function



Because each input value is the opposite of the original input value, we can write $H(t) = s(-t)$ or $H(t) =\ -t$

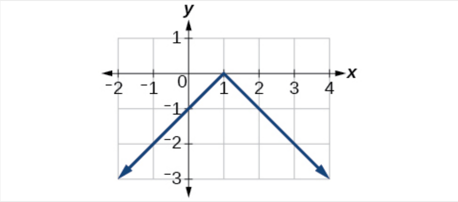
Notice that this is an inside change or horizontal change that affects the input

values, so the negative sign is on the inside of the function.

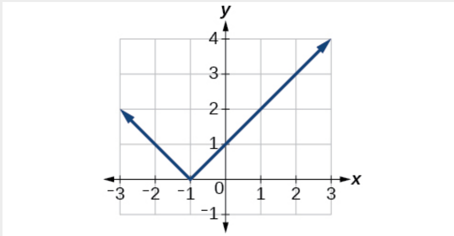
Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0,\infty)$ and range $[0,\infty)$, the vertical reflection gives the V(t) function the range $(-\infty,0]$ and the horizontal reflection gives the H(t) function the domain $(-\infty,0]$.

Reflect the graph of $f(x)=|x-1|$ (a) vertically and (b) horizontally.

1.



2.



Reflecting a Tabular Function Horizontally and Vertically

A function f(x) is given as [link]. Create a table for the functions below.

1. $g(x) = -f(x)$
2. $h(x) = f(-x)$

x	2	4	6	8
f(x)	1	3	7	11

1. For g(x), the negative sign outside the function indicates a vertical reflection, so the *x*-values stay the same and each

output value will be the opposite of the original output value. See [link].

x	2	4	6	8
g(x)	–1	–3	–7	–11

2. For h(x), the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the h(x) values stay the same as the f(x) values. See [link].

x	−2	−4	−6	−8
h(x)	1	3	7	11

A function f(x) is given as [link]. Create a table for the functions below.

1. $g(x) = -f(x)$
2. $h(x) = f(-x)$

x	−2	0	2	4
f(x)	5	10	15	20

1. $g(x) = -f(x)$

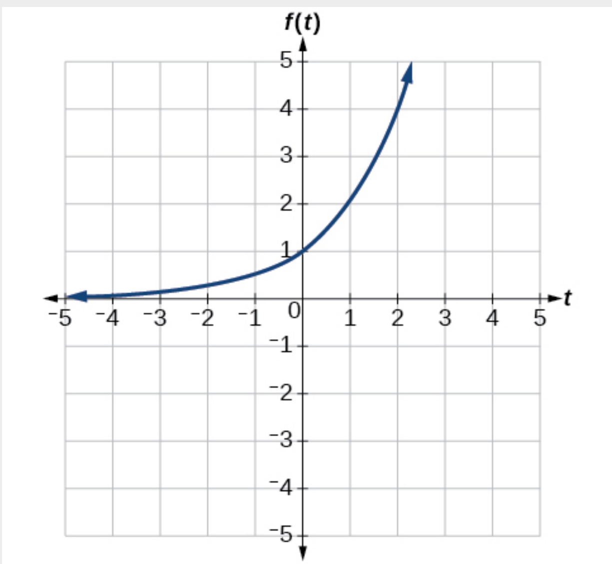
x	2	0	2	4
g(x)	−5	−10	−15	−20

2. $h(x) = f(-x)$

x	2	0	2	4
h(x)	15	10	5	unknown

Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -\ 2\ -t\ +1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = \ 2\ t$ shown in [link]. Sketch a graph of k(t).



This equation combines three transformations

into one equation.

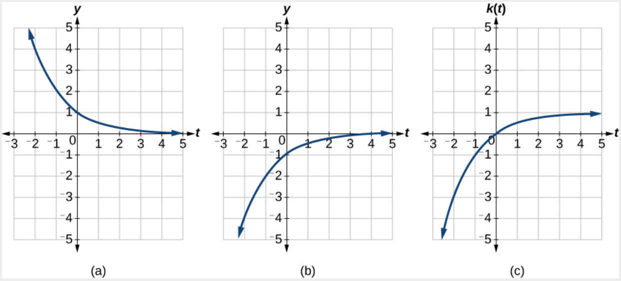
- A horizontal reflection: $f(-t) = 2 - t$
- A vertical reflection: $-f(-t) = -\ 2 - t$
- A vertical shift: $-f(-t) + 1 = -\ 2 - t\ + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (–1, 2).
2. Then, we apply a vertical reflection: (0, -1) (-1, –2).
3. Finally, we apply a vertical shift: (0, 0) (-1, -1).

This means that the original points, (0,1) and (1,2) become (0,0) and (-1,-1) after we apply the transformations.

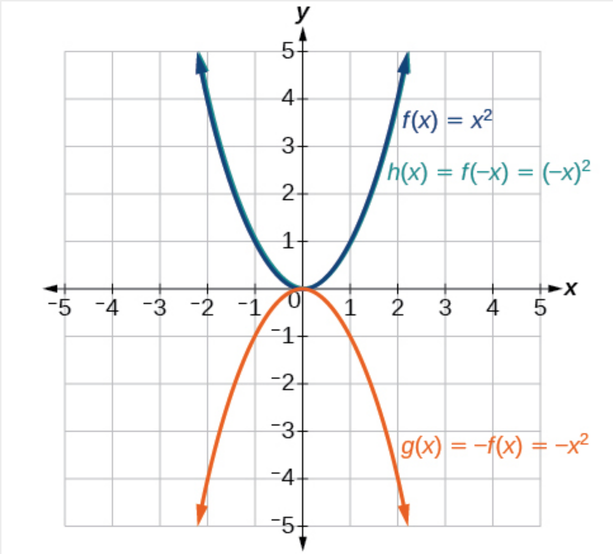
In [link], the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.



Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0,1)$.

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.



Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

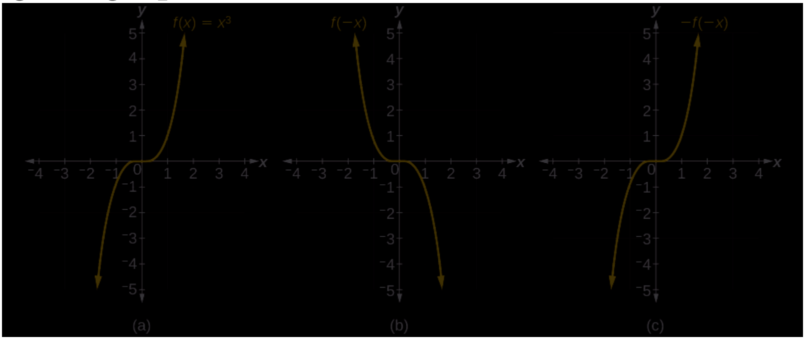
(a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the *y*-axis. Functions whose graphs are symmetric

about the *y*-axis are called **even functions.**

If the graphs of $f(x)= x 3$ or $f(x)= 1 x$ were reflected over *both* axes, the result would be the original graph, as shown in [link].



We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x)= 2 x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x)=0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$$f(x)=f(-x)$$

The graph of an even function is symmetric about the y- axis.

A function is called an **odd function** if for every input x
$f(x) = -f(-x)$
The graph of an odd function is symmetric about the origin.

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's

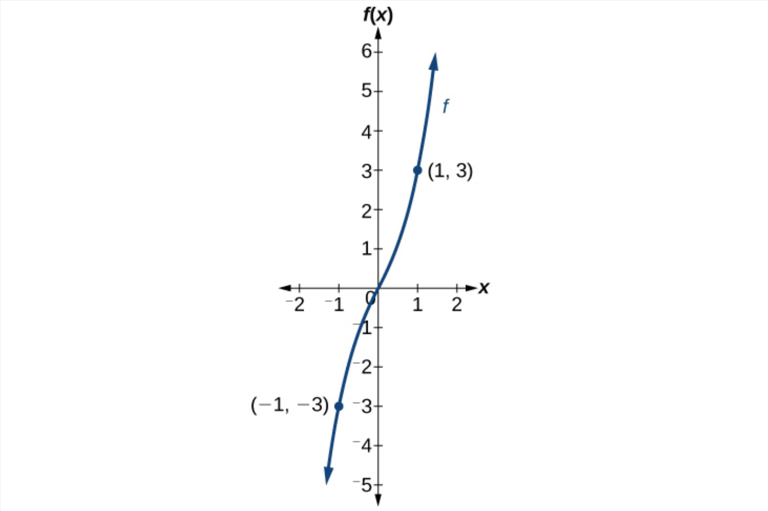
begin with the rule for even functions.
$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.
$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in [link]. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.



Is the function $f(s)=s^4+3s^2+7$ even, odd, or neither?
even

Vertical stretch and compression

Graphing Functions Using Stretches and Compressions

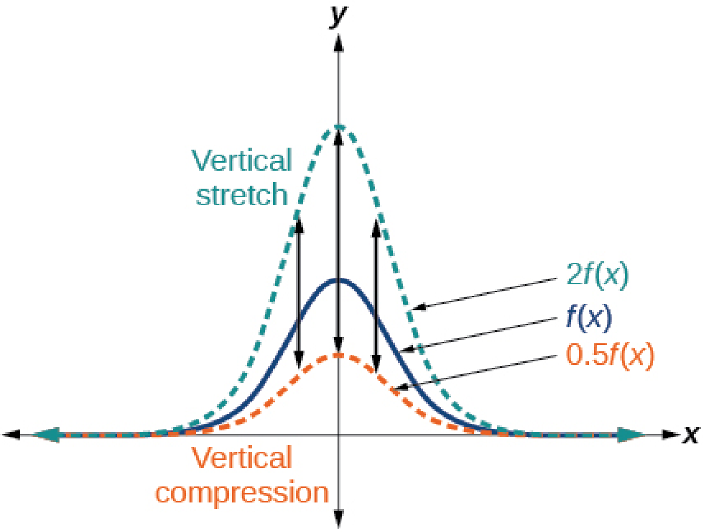
Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or

compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. [link] shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.



Vertical Stretches and Compressions

Given a function f(x), a new function g(x) = af(x), where a is a constant, is a **vertical stretch** or **vertical compression** of the function f(x).

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical

reflection.

Given a function, graph its vertical stretch.

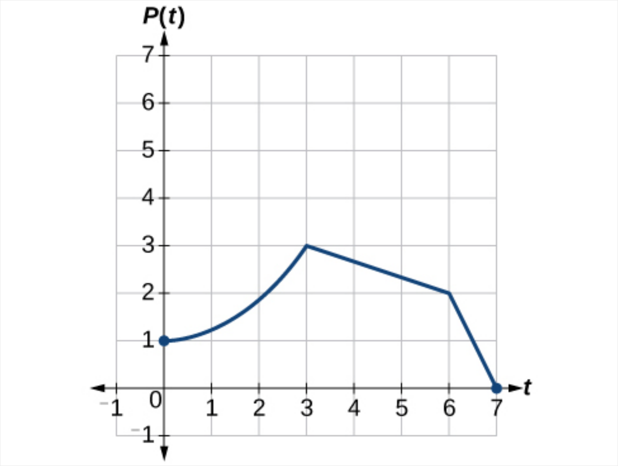
1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1$, the graph is stretched by a factor of a.

   If $0 < a < 1$, the graph is compressed by a factor of a.

   If $a < 0$, the graph is either stretched or compressed and also reflected about the *x*-axis.

Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in [link].

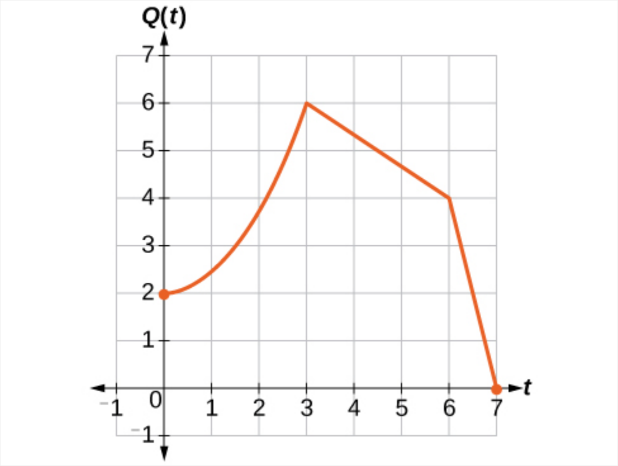


A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in [link].

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.
$(0,1) \rightarrow (0,2)$ $(3,3) \rightarrow (3,6)$ $(6,2) \rightarrow (6,4)$ $(7,0) \rightarrow (7,0)$



Symbolically, the relationship is written as
$Q(t) = 2P(t)$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Finding a Vertical Compression of a Tabular Function

A function f is given as [link]. Create a table for the function $g(x) = 1\ 2\ f(x)$.

x	2	4	6	8
f(x)	1	3	7	11

The formula $g(x) = 1\ 2\ f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = 1\ 2\ f(4) = 1\ 2\ (3) = 3\ 2$$

We do the same for the other values to produce [link].

x	2		4		6		8	
g(x)	1 2		3 2		7 2		11 2	

Analysis

The result is that the function g(x) has been compressed vertically by 1 2 . Each output value is divided in half, so the graph is half the original height.

A function f is given as [link]. Create a table for the function g(x)= 3 4 f(x).

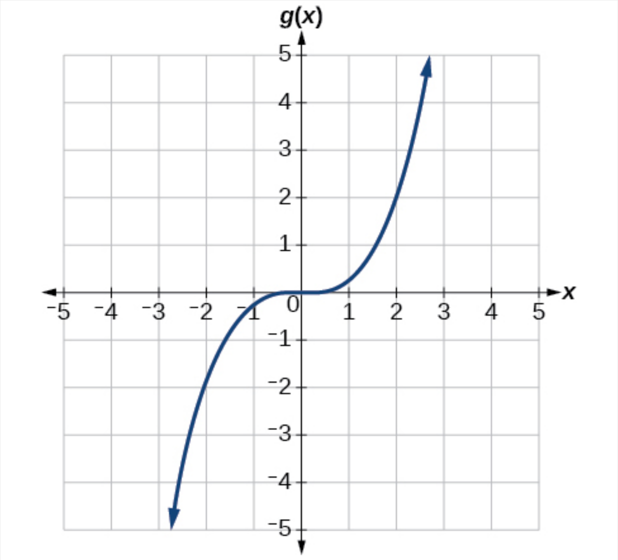
x	2		4		6		8	
f(x)	12		16		20		0	

x	2		4		6		8	

g(x)	9		12		15		0		

Recognizing a Vertical Stretch

The graph in [link] is a transformation of the toolkit function $f(x)= x^3$. Relate this new function g(x) to f(x), and then find a formula for g(x).



When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2)=2$. With the basic cubic

function at the same input, f(2)= 2 3 =8. Based on that, it appears that the outputs of g are 1 4 the outputs of the function f because g(2)= 1 4 f(2). From this we can fairly safely conclude that g(x)= 1 4 f(x).

We can write a formula for g by using the definition of the function f.
g(x)= 1 4 f(x)= 1 4 x 3

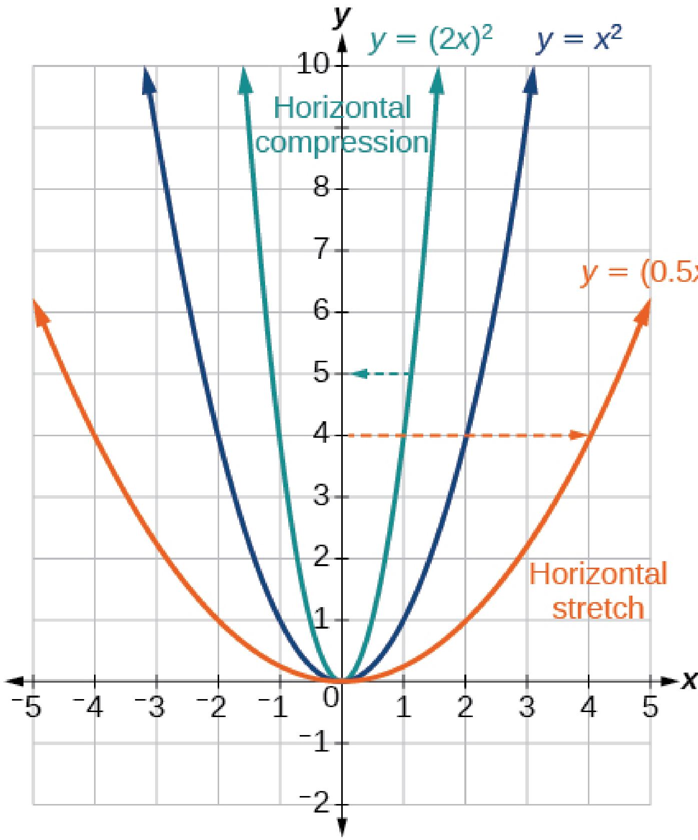
Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

g(x)=3x-2

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is

between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.



Given a function y = f(x), the form y = f(bx) results in a horizontal stretch or compression. Consider the function y = x 2 . Observe [link]. The graph of y = (0.5x) 2 is a horizontal stretch of the graph of the function y = x 2 by a factor of 2. The graph of y = (2x) 2 is a horizontal compression of the graph of the function y = x 2 by a factor of 2.

Horizontal Stretches and Compressions

Given a function f(x), a new function g(x) = f(bx), where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function f(x).

- If b > 1, then the graph will be compressed by 1 b .
- If 0 < b < 1, then the graph will be stretched by 1 b .
- If b < 0, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set g(x) = f(bx) where b > 1 for a compression or 0 < b < 1 for a stretch.

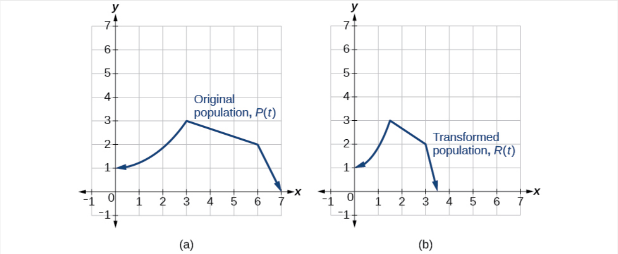
Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new

population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Symbolically, we could write
$R(1)=P(2)$, $R(2)=P(4)$, and in general,
$R(t)=P(2t)$.

See [link] for a graphical comparison of the original population and the compressed population.
(a) Original population graph (b) Compressed population graph



Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Finding a Horizontal Stretch for a Tabular

Function

A function $f(x)$ is given as [link]. Create a table for the function $g(x)=f\left(\frac{1}{2}x\right)$.

x	2	4	6	8
f(x)	1	3	7	11

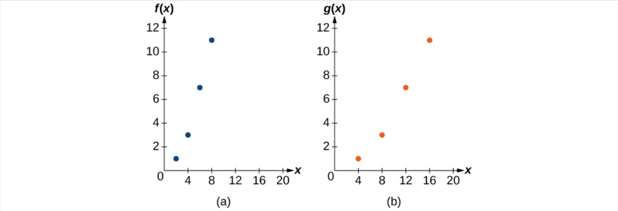
The formula $g(x)=f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2)=f\left(\frac{1}{2}\cdot 2\right)=f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4)=f\left(\frac{1}{2}\cdot 4\right)=f(2)=1$$

We do the same for the other values to produce [link].

x	4	8	12	16
g(x)	1	3	7	11

[link] shows the graphs of both of these sets of points.

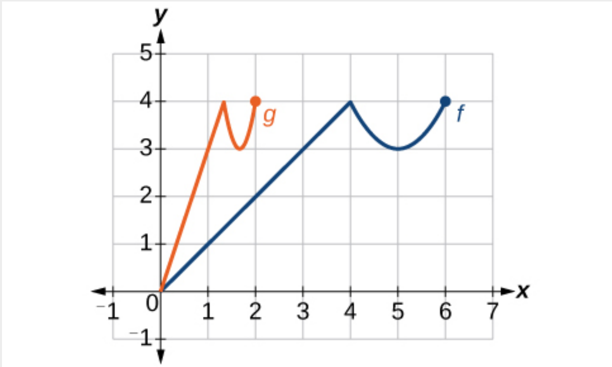


Analysis

Because each input value has been doubled, the result is that the function g(x) has been stretched horizontally by a factor of 2.

Recognizing a Horizontal Compression on a Graph

Relate the function g(x) to f(x) in [link].



The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6,4)$ and $g(x)$ ends at $(2,4)$, we can see that the x- values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right)=2$. We might also notice that $g(2)=f(6)$ and $g(1)=f(3)$. Either way, we can describe this relationship as $g(x)=f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

$g(x)=f(\ 1\ 3\ x\ )$ so using the square root function we get $g(x)=\ 1\ 3\ x$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by

2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f\left(b\left(x+\frac{p}{b}\right)\right)$$

Let's work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations
When combining vertical transformations written in the form af(x)+k, first vertically stretch by a and then vertically shift by k.
When combining horizontal transformations written in the form f(bx-h), first horizontally shift by h and then horizontally stretch by 1 b .
When combining horizontal transformations written in the form f(b(x-h)), first horizontally stretch by 1 b and then horizontally shift by h.
Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Finding a Triple Transformation of a Tabular Function

Given [link] for the function f(x), create a table of values for the function g(x)=2f(3x)+1.

x	6	12	18	24
f(x)	10	14	15	17

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by 1 3 , which means we multiply each x- value by 1 3 . See [link].

x	2	4	6	8
f(3x)	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See [link].

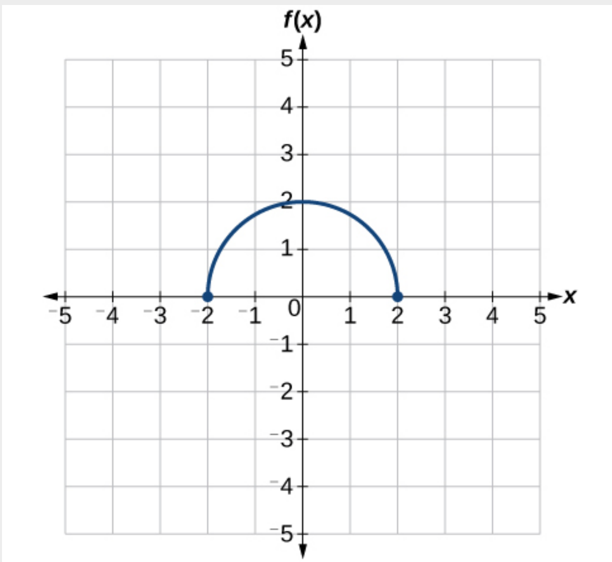
x	2	4	6	8
2f(3x)	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See [link].

x	2	4	6	8
$g(x)=2f(3x)+1$		29	31	35

Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in [link] to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

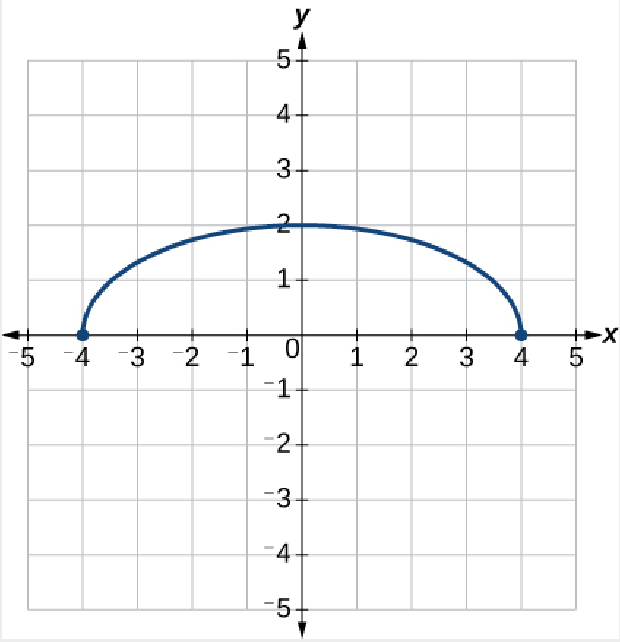


To simplify, let's start by factoring out the inside of the function.

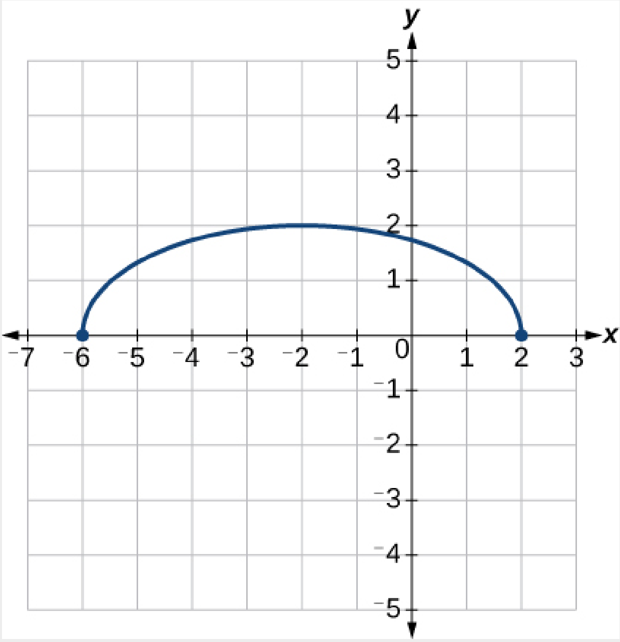
$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$

By factoring the inside, we can first

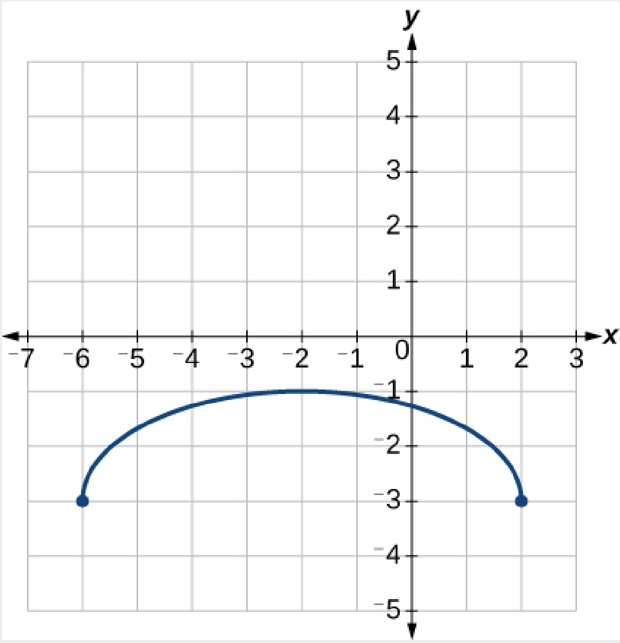
horizontally stretch by 2, as indicated by the 1 2 on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0). See [link].



Next, we horizontally shift left by 2 units, as indicated by $x+2$. See [link].



Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See [link].



Access this online resource for additional instruction and practice with transformation of functions.

- Function Transformations

Key Equations

Vertical shift	$g(x)=f(x)+k$ (up for $k>0$)
Horizontal shift	$g(x)=f(x-h)$ (right for $h>0$)
Vertical reflection	$g(x)=-f(x)$
Horizontal reflection	$g(x)=f(-x)$
Vertical stretch	$g(x)=af(x)$ ($a>1$)
Vertical compression	$g(x)=af(x)$ $(0<a<1)$
Horizontal stretch	$g(x)=f(bx)$ $(0<b<1)$
Horizontal compression	$g(x)=f(bx)$ ($b>1$)

Key Concepts

- A function can be shifted vertically by adding a constant to the output. See [link] and [link].
- A function can be shifted horizontally by adding a constant to the input. See [link], [link], and [link].
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See [link].
- Vertical and horizontal shifts are often combined. See [link] and [link].
- A vertical reflection reflects a graph about the x- axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y- axis. A graph can be reflected horizontally by multiplying the input by –1.

- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See [link].
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See [link].
- A function presented as an equation can be reflected by applying transformations one at a time. See [link].
- Even functions are symmetric about the y- axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither. See [link].
- A function can be compressed or stretched vertically by multiplying the output by a constant. See [link], [link], and [link].
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See [link], [link], and [link].
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a

horizontal transformation in any order. See [link] and [link].

Section Exercises

Verbal

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a

vertical compression?

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the *x*-axis from a reflection with respect to the *y*-axis?

How can you determine whether a function is odd or even from the formula of the function?

For a function f, substitute $(-x)$ for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x)=f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x)=-f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic

Write a formula for the function obtained when the graph of $f(x) = x$ is shifted up 1 unit and to the left 2 units.

Write a formula for the function obtained when the graph of $f(x) = | x |$ is shifted down 3 units and to the right 1 unit.

$g(x) = |x-1| - 3$

Write a formula for the function obtained when the graph of $f(x) = 1 x$ is shifted down 4 units and to the right 3 units.

Write a formula for the function obtained when the graph of $f(x) = 1 x 2$ is shifted up 2 units and to the left 4 units.

$g(x) = 1 (x+4) 2 + 2$

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

$y = f(x - 49)$

$y = f(x + 43)$

The graph of $f(x + 43)$ is a horizontal shift to the left 43 units of the graph of f.

$y = f(x + 3)$

$y = f(x - 4)$

The graph of f(x-4) is a horizontal shift to the right 4 units of the graph of f.

$y = f(x) + 5$

$y = f(x) + 8$

The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f.

$y = f(x) - 2$

$y = f(x) - 7$

The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of f.

$y=f(x-2)+3$

$y=f(x+4)-1$

The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

$f(x)=4(x+1)^2-5$

$g(x)=5(x+3)^2-2$

---

decreasing on $(-\infty,-3)$ and increasing on $(-3,\infty)$

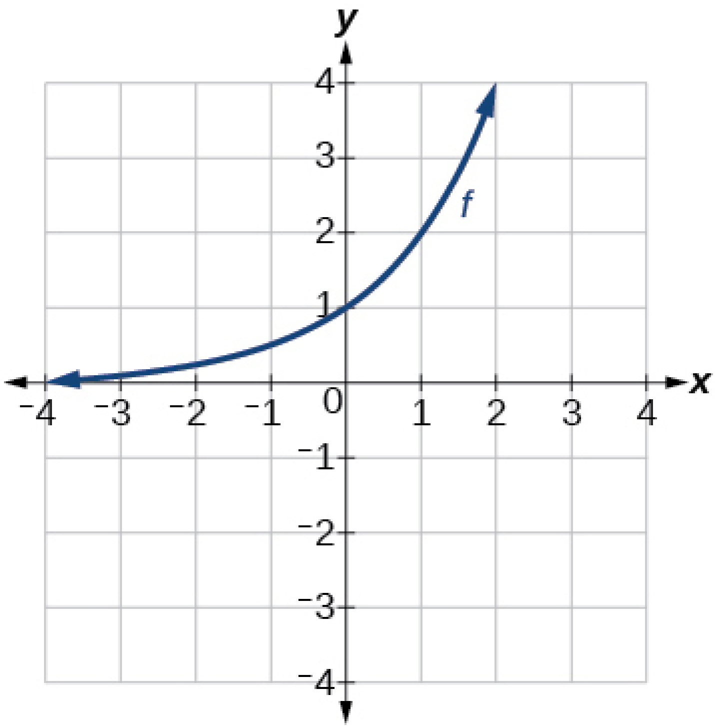
$a(x)=\sqrt{-x+4}$

$k(x)=-3\sqrt{x}-1$

decreasing on $(0,\infty)$

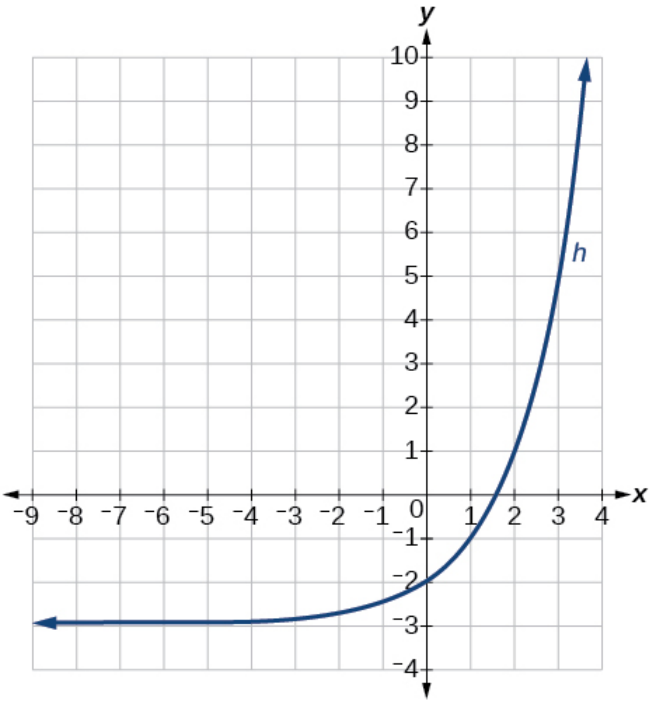
Graphical

For the following exercises, use the graph of $f(x)=2^x$ shown in [link] to sketch a graph of each transformation of $f(x)$.



$g(x)=2^x+1$

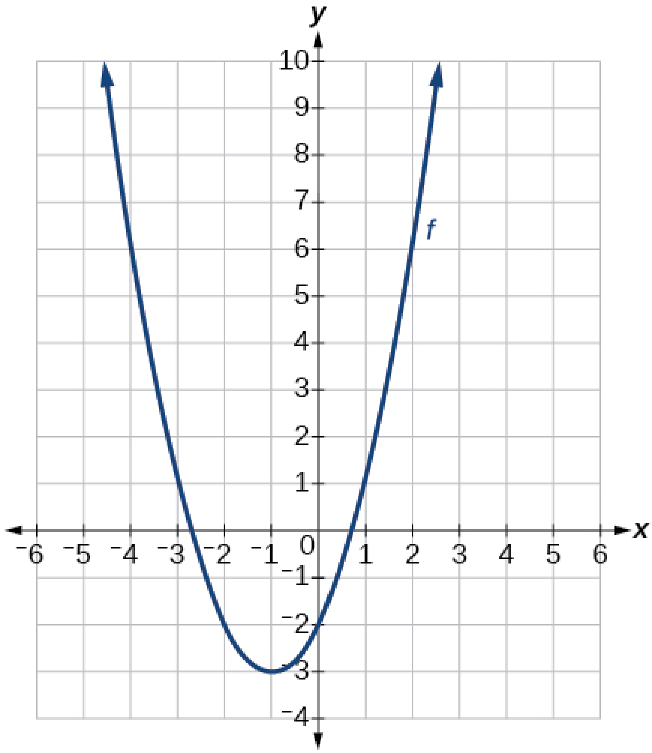
$h(x)=2^x-3$



$w(x) = 2\ x - 1$

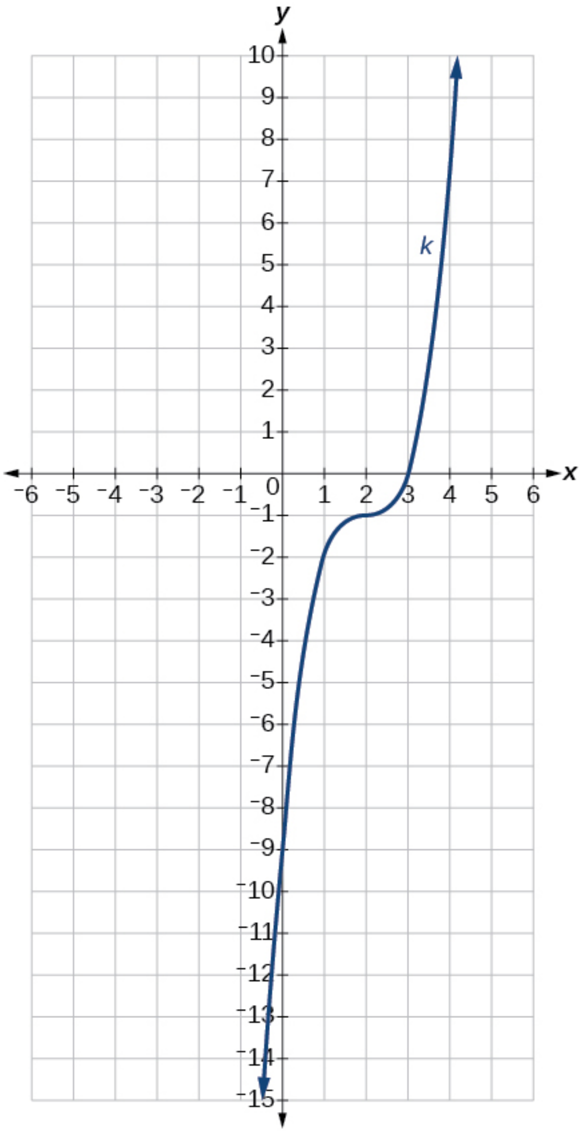
For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

$f(t) = (t+1)\ 2\ -3$



$$h(x) = |x - 1| + 4$$

$$k(x) = (x - 2)^3 - 1$$



$m(t) = 3 + \; t + 2$

Numeric

Tabular representations for the functions f,g, and h are given below. Write g(x) and h(x) as transformations of f(x).

x	−2	−1	0	1	2
f(x)	−2	−1	−3	1	2

x	−1	0	1	2	3
g(x)	−2	−1	−3	1	2

x	−2	−1	0	1	2
h(x)	−1	0	−2	2	3

$g(x) = f(x-1), h(x) = f(x) + 1$

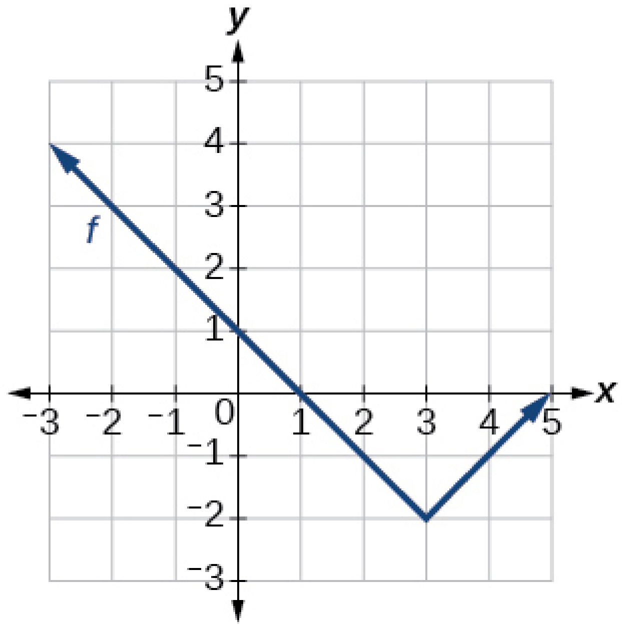
Tabular representations for the functions f,g, and h are given below. Write g(x) and h(x) as transformations of f(x).

x	−2	−1	0	1	2
f(x)	−1	−3	4	2	1

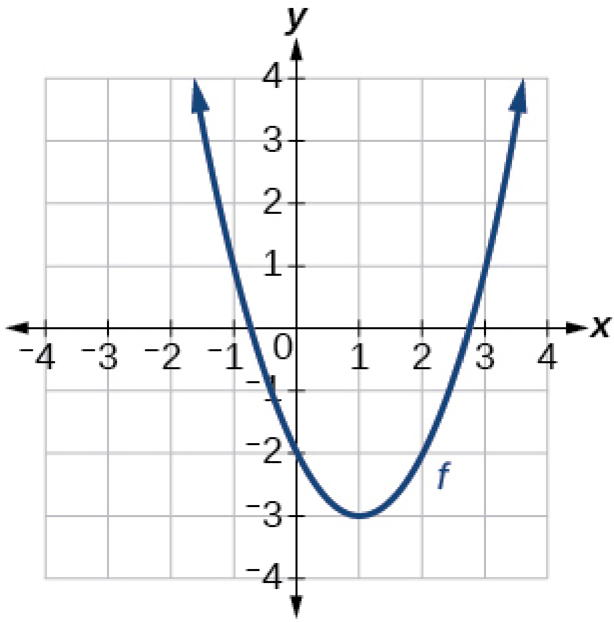
x	−3	−2	−1	0	1
g(x)	−1	−3	4	2	1

x	−2	−1	0	1	2
h(x)	−2	−4	3	1	0

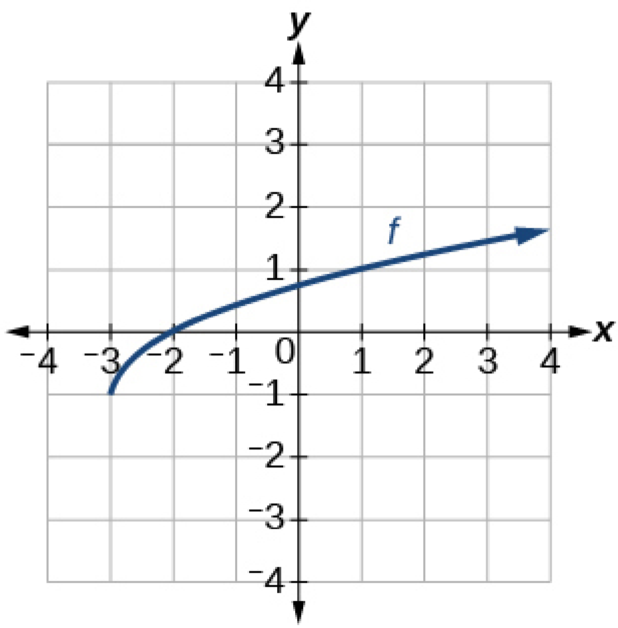
For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.



$f(x) = |x\text{-}3| - 2$

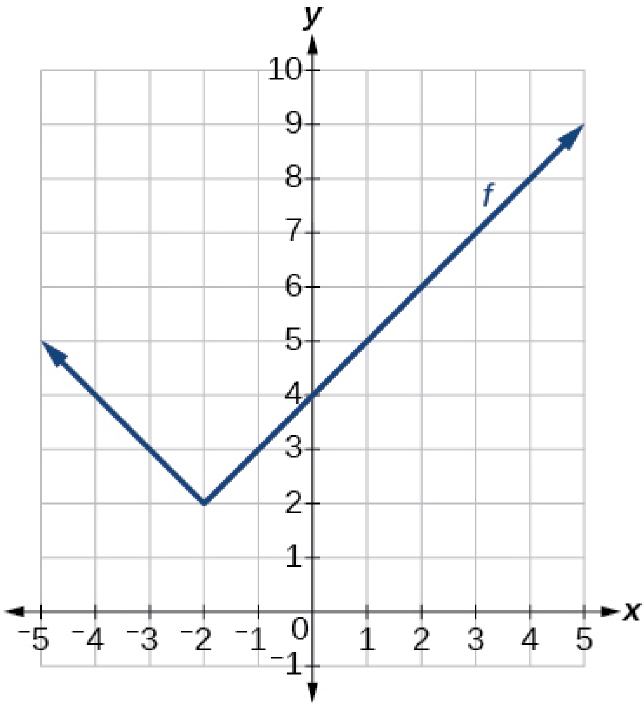


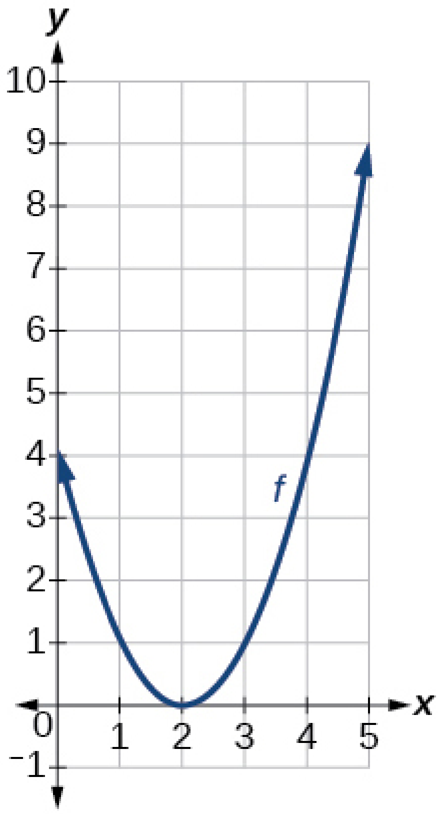
y
x
f



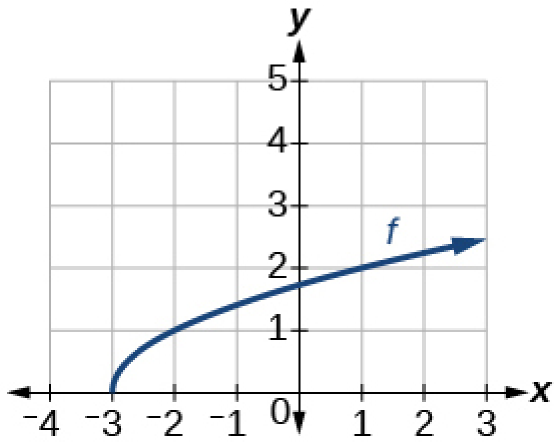
y
x
f

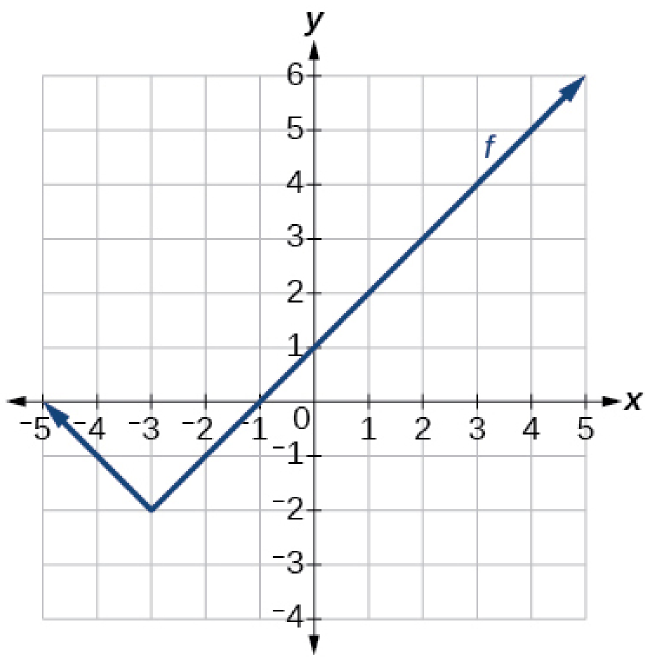
$f(x) = x + 3 \ -1$



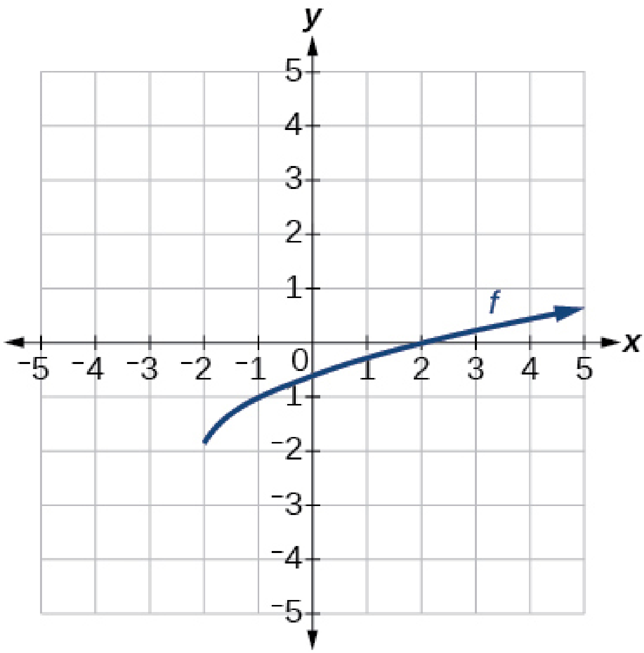


f(x) = (x-2) 2

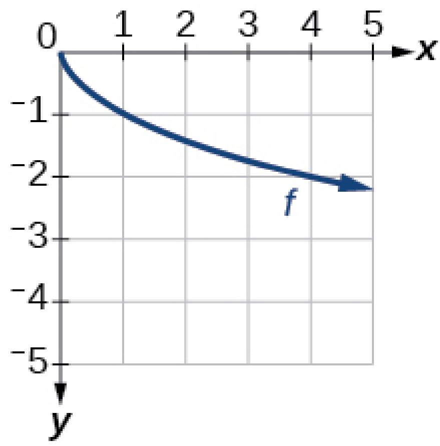




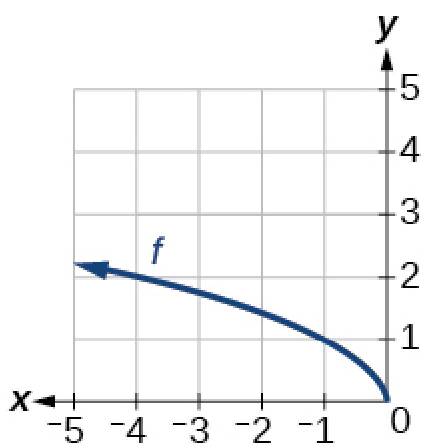
$f(x) = |x+3| - 2$



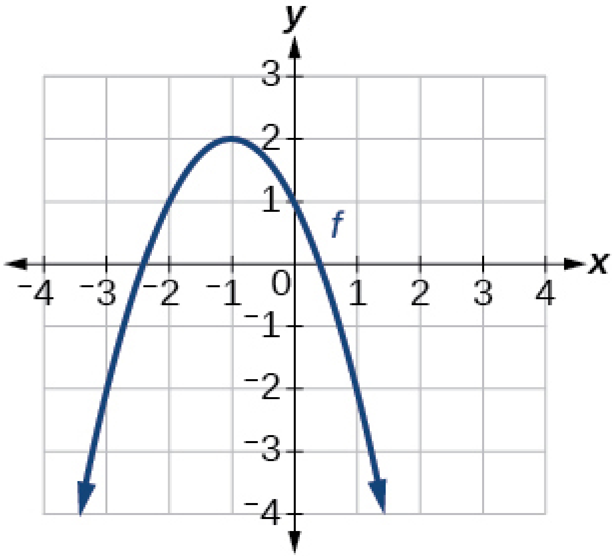
For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.



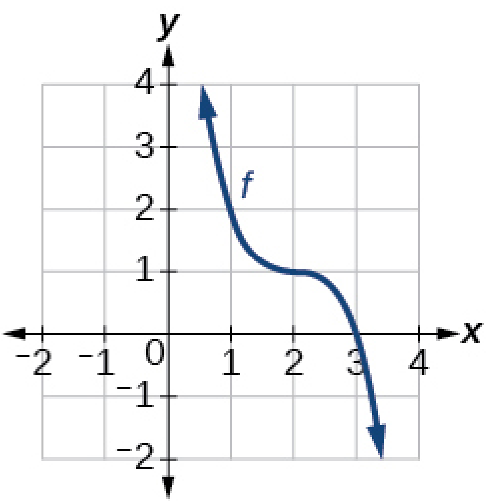
$f(x) = -x$

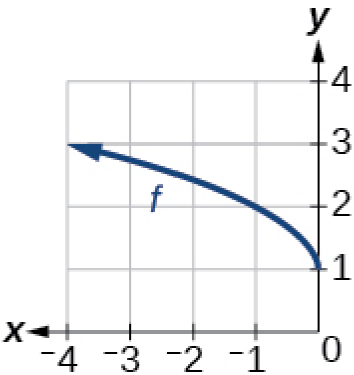


For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

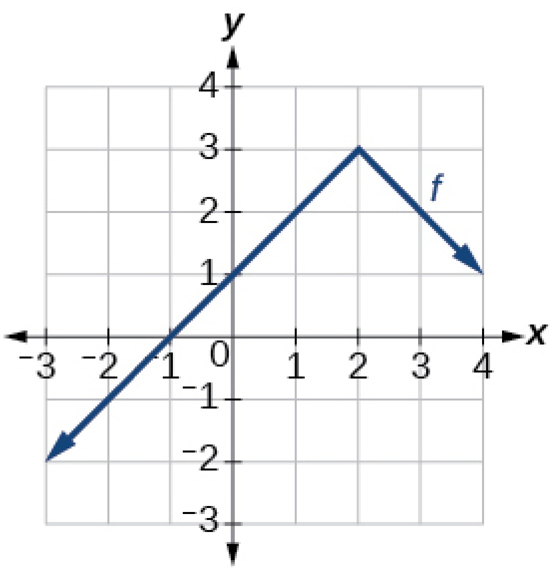


$f(x) = -(x+1)^2 + 2$





$f(x) = \ -x \ +1$



For the following exercises, determine whether the function is odd, even, or neither.

$f(x) = 3\ x\ 4$

---

even

$g(x) = \ x$

$h(x) = \ 1\ x\ + 3x$

---

odd

$f(x) = \ (x - 2)\ 2$

$g(x) = 2\ x\ 4$

---

even

$h(x) = 2x - \ x\ 3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

$g(x) = -f(x)$

The graph of g is a vertical reflection (across the x -axis) of the graph of f.

$g(x) = f(-x)$

$g(x) = 4f(x)$

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

$g(x) = 6f(x)$

$g(x) = f(5x)$

The graph of g is a horizontal compression by a factor of 1 5 of the graph of f.

$g(x) = f(2x)$

$g(x) = f(1 3 x)$

The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

$g(x)=f(\frac{1}{5}x)$

$g(x)=3f(-x)$

The graph of g is a horizontal reflection across the y -axis and a vertical stretch by a factor of 3 of the graph of f.

$g(x)=-f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

The graph of $f(x)=|x|$ is reflected over the y -axis and horizontally compressed by a factor of $\frac{1}{4}$.

$g(x)=|-4x|$

The graph of $f(x)= x$ is reflected over the x -axis and horizontally stretched by a factor of 2.

The graph of f(x)= 1 x 2 is vertically compressed by a factor of 1 3 , then shifted to the left 2 units and down 3 units.

g(x)= 1 3 (x+2) 2 −3

The graph of f(x)= 1 x is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

The graph of f(x)= x 2 is vertically compressed by a factor of 1 2 , then shifted to the right 5 units and up 1 unit.

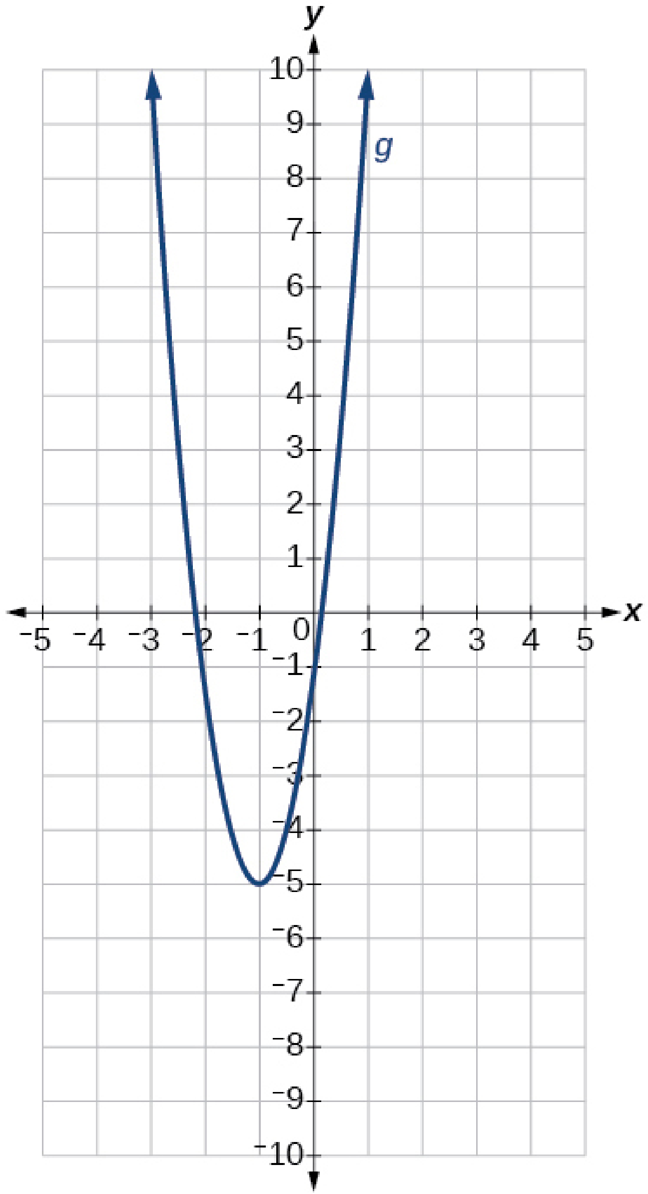
g(x)= 1 2 (x-5) 2 +1

The graph of f(x)= x 2 is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

g(x)=4 (x+1) 2 −5

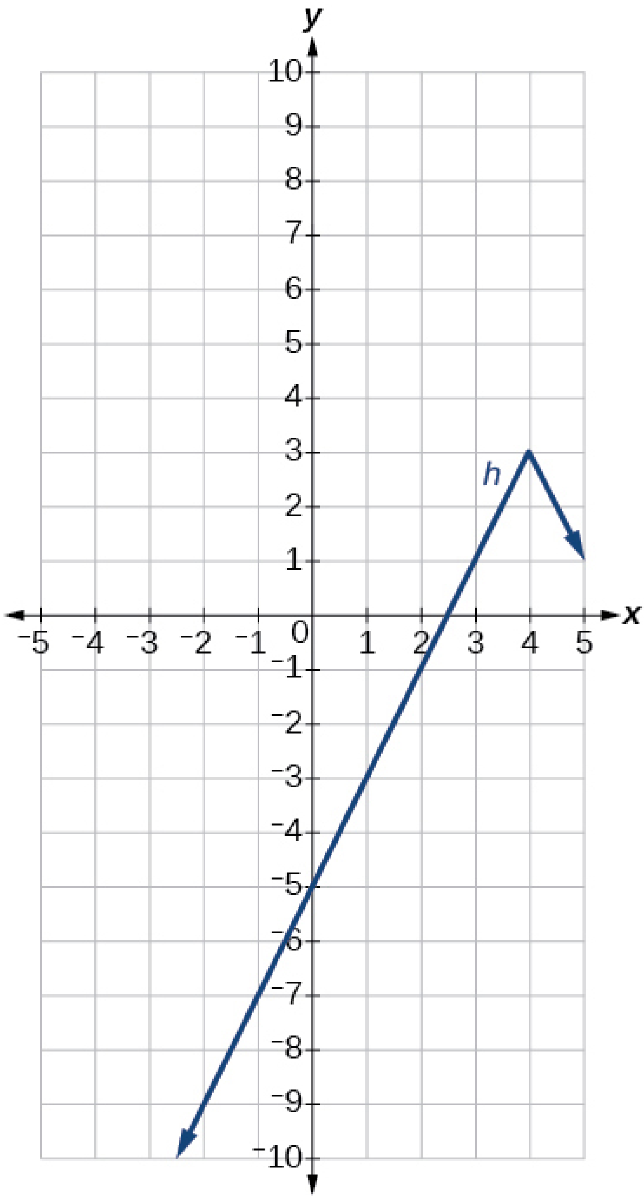
The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



$$g(x) = 5\ (x+3)\ 2\ -2$$

$$h(x) = -2|x-4|+3$$

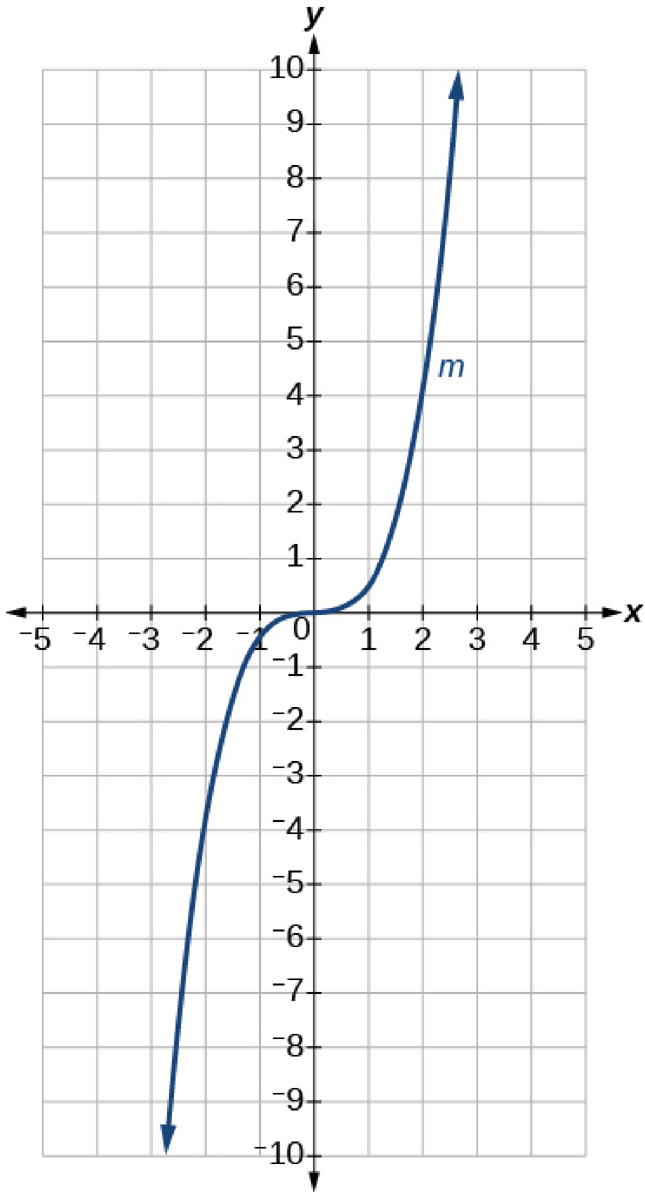
The graph of $f(x)=|x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



$k(x) = -3x - 1$

$m(x) = 1\ 2\ x\ 3$

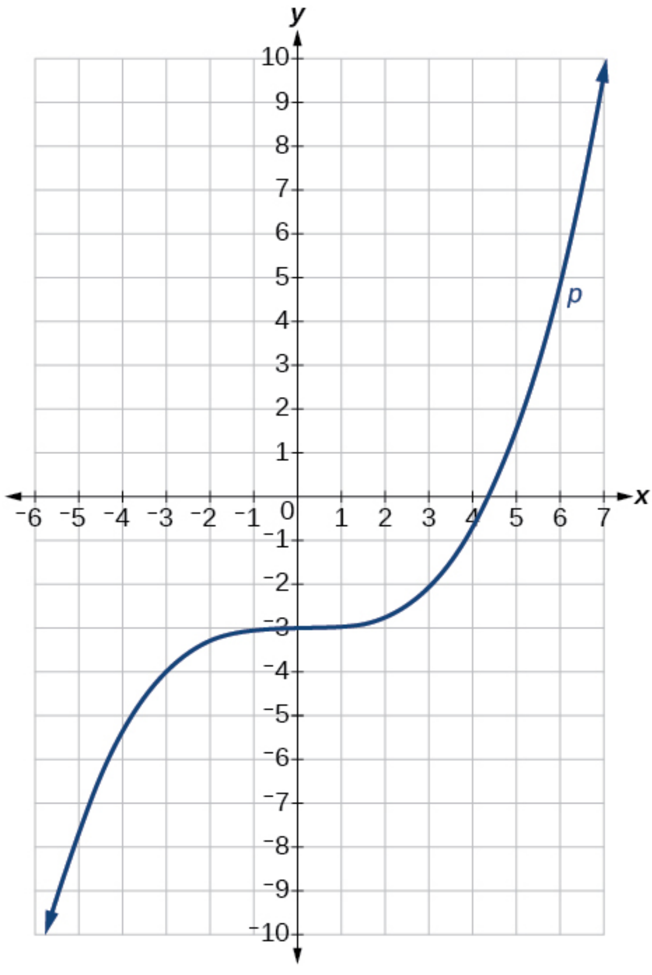
The graph of the function $f(x) = x\ 3$ is compressed vertically by a factor of $1\ 2$.



$n(x) = 1\ 3\ |x - 2|$

$p(x)=\left(\frac{1}{3}x\right)^3-3$

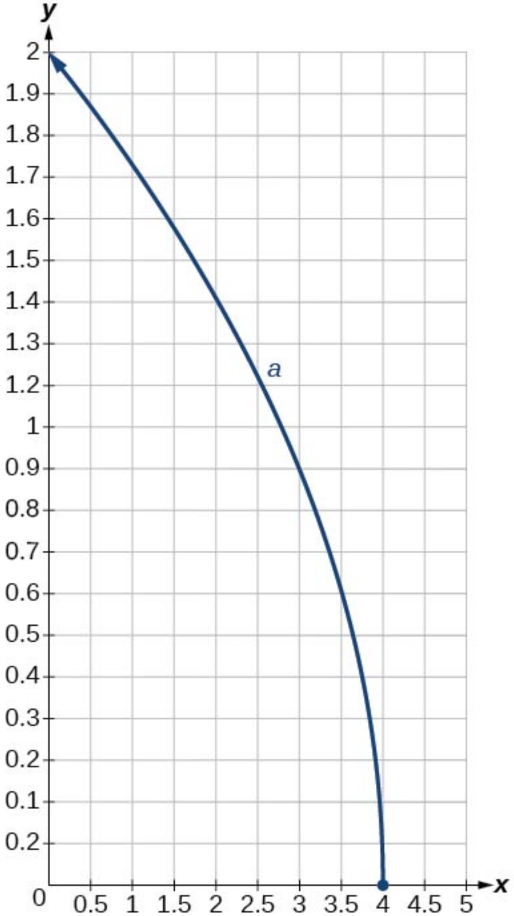
The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



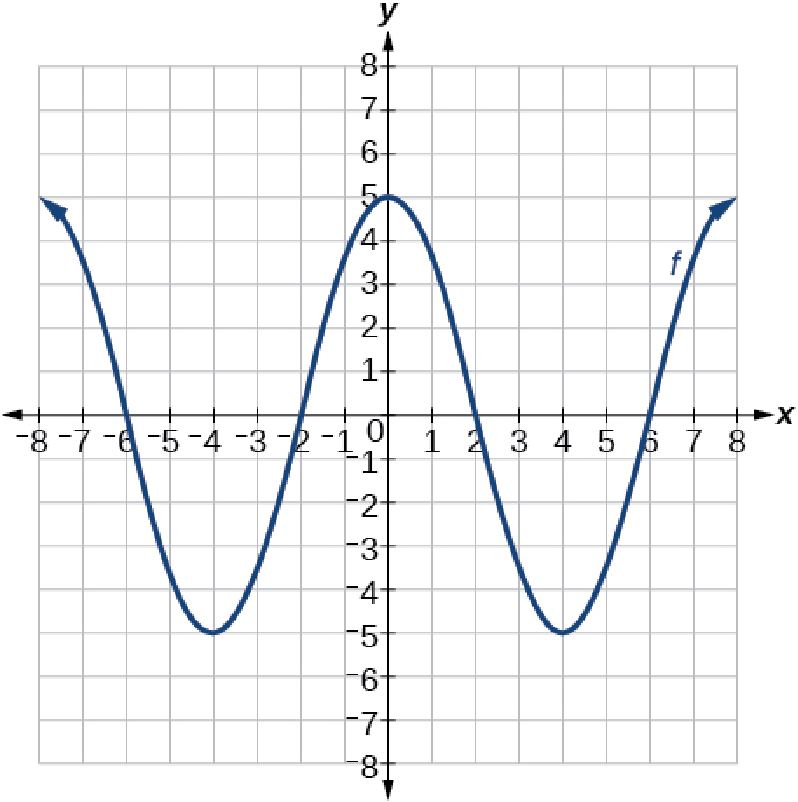
$q(x) = (14x)3 + 1$

$a(x) = -x + 4$

The graph of $f(x) = x$ is shifted right 4 units and then reflected across the vertical line $x = 4$.

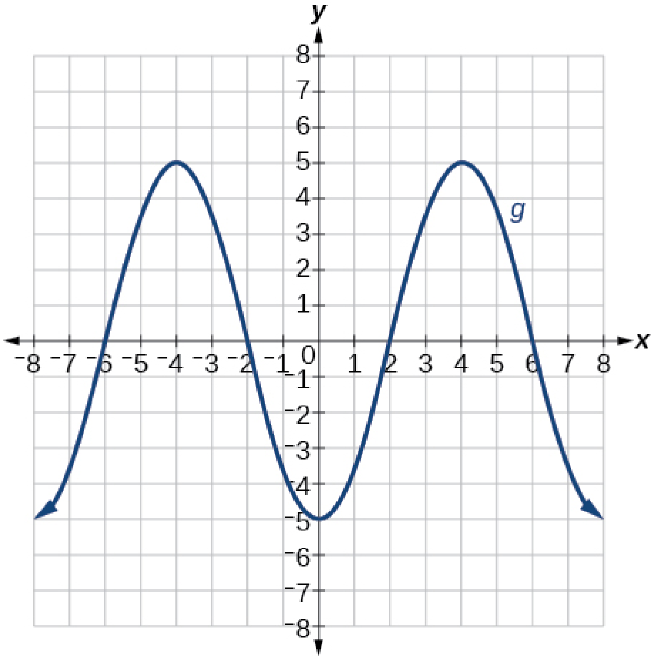


For the following exercises, use the graph in [link] to sketch the given transformations.



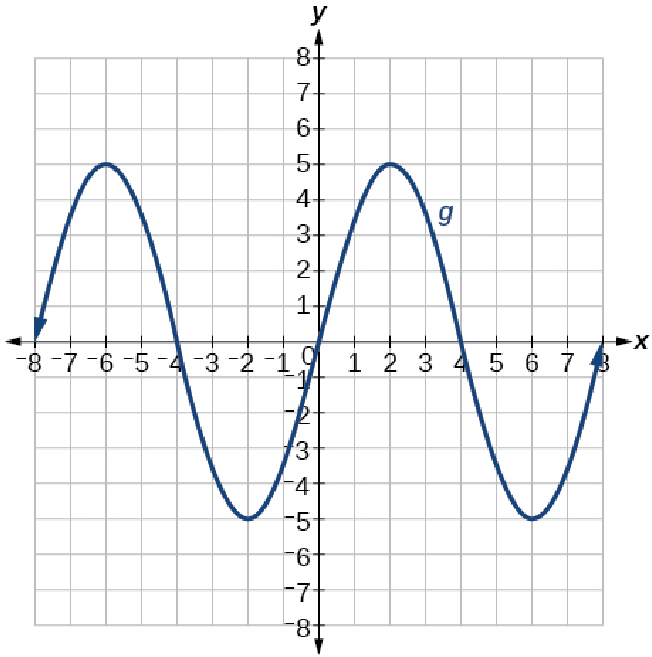
$g(x) = f(x) - 2$

$g(x) = -f(x)$



$g(x) = f(x+1)$

$g(x) = f(x-2)$



Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y- axis

horizontal compression

a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$

horizontal reflection

a transformation that reflects a function's

graph across the y-axis by multiplying the input by -1

horizontal shift
: a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
: a transformation that stretches a function's graph horizontally by multiplying the input by a constant $0<b<1$

odd function
: a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x)=-f(-x)$, and is symmetric about the origin

vertical compression
: a function transformation that compresses the function's graph vertically by multiplying the output by a constant $0<a<1$

vertical reflection
: a transformation that reflects a function's graph across the x-axis by multiplying the output by -1

vertical shift
: a transformation that shifts a function's graph up or down by adding a positive or negative

constant to the output

vertical stretch

a transformation that stretches a function’s graph vertically by multiplying the output by a constant $a > 1$

Absolute Value Functions
In this section you will:

- Graph an absolute value function.
- Solve an absolute value equation.

Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all

directions. As such, it is useful to consider distance as an absolute value function. In this section, we will continue our investigation of absolute value functions.

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign. Knowing this, we can use absolute value functions to solve some kinds of real-world problems.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$f(x) = |x| = \{ x \text{ if } x \geq 0 \;\; -x \text{ if } x < 0$

Using Absolute Value to Determine Resistance

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%$, $\pm 5\%$, or $\pm 10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm 5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

We can find that 5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

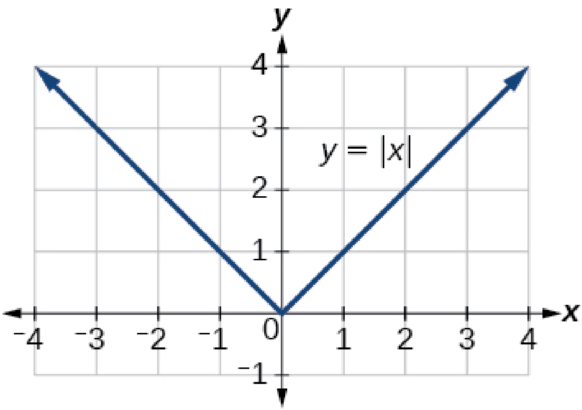
Students who score within 20 points of 80 will pass a test. Write this as a distance from 80

using absolute value notation.
using the variable p for passing, $\mid p-80 \mid \leq 20$

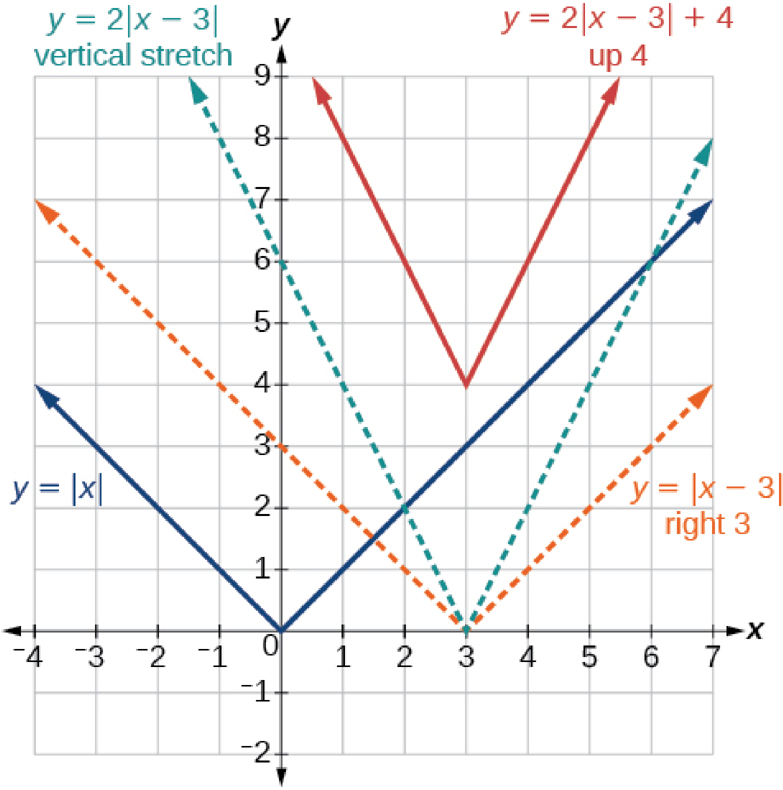
(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in [link].

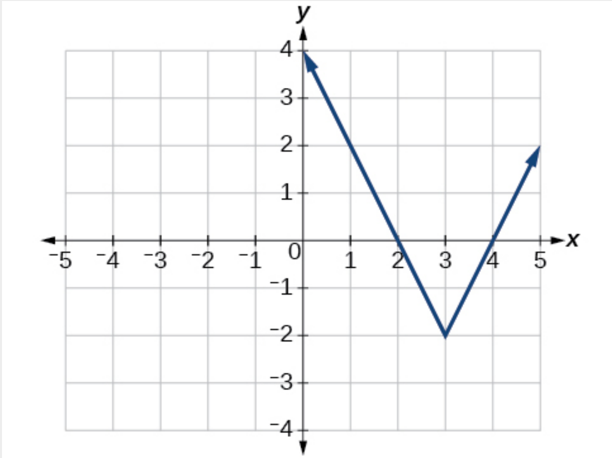


[link] shows the graph of $y=2|x-3|+4$. The graph of $y=|x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3,4)$ for this transformed function.

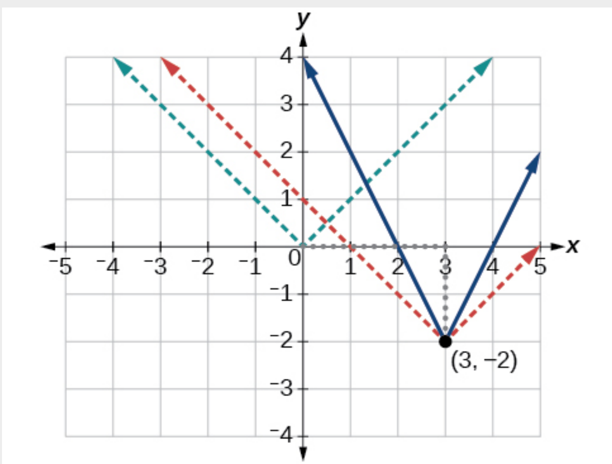


Writing an Equation for an Absolute Value Function Given a Graph

Write an equation for the function graphed in [link].

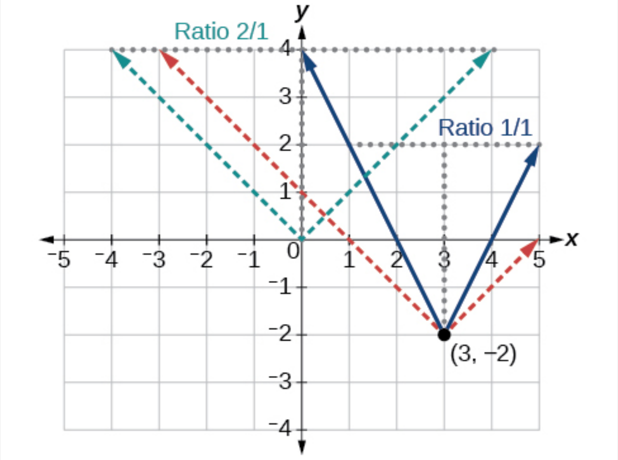


The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See [link].



We also notice that the graph appears vertically stretched, because the width of the

final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in [link].



From this information we can write the equation
$f(x) = 2|x-3|-2,$
treating the stretch as a vertical stretch,or $f(x) = |2(x-3)|-2,$
treating the stretch as a horizontal compression.

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x *and* $f(x)$.

$f(x)=a|x-3|-2$

Now substituting in the point $(1,\ 2)$

$2\ =\ a|1-3|-2\ 4\ =\ 2a\ a\ =\ 2$

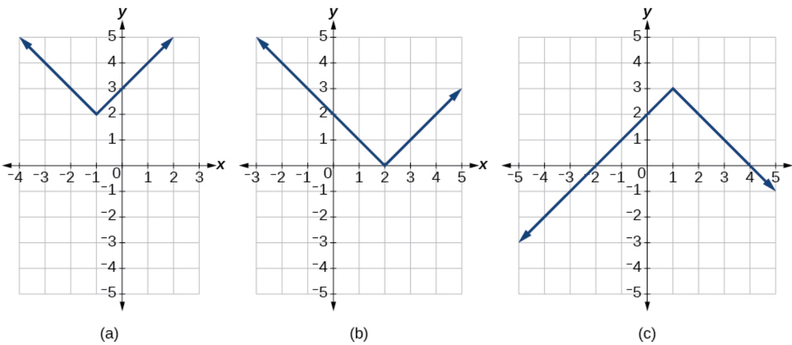
Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

$f(x)=-|\ x+2\ |+3$

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see [link]).



Solving an Absolute Value Equation

In Other Type of Equations, we touched on the concepts of absolute value equations. Now that we understand a little more about their graphs, we can take another look at these types of equations. Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve

independently.

$2x-6 = 8$ or $2x-6 = -8$ $2x = 14$ $2x = -2$ $x = 7$ $x = -1$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An absolute value equation is an equation in which the unknown variable appears in absolute value bars. For example,
$|x|=4,\ |2x-1|=3,$or $|5x+2|-4=9$

> Solutions to Absolute Value Equations
> For real numbers A and B , an equation of the form $|A|=B$, with $B\geq 0$, will have solutions when $A=B$ or $A=-B$. If $B<0$, the equation $|A|=B$ has no solution.

> **Given the formula for an absolute value function, find the horizontal intercepts of its graph.**
>
> 1. Isolate the absolute value term.
> 2. Use $|A|=B$ to write $A=B$ or $-A=B$, assuming $B>0$.

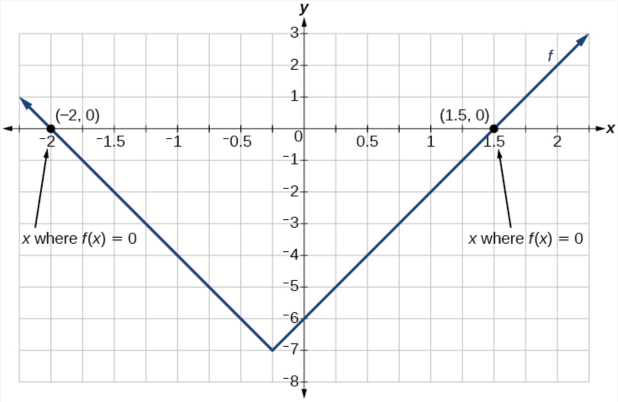
3. Solve for x.

Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

$0 = |4x+1| - 7$ Substitute 0 for f(x). $7 = |4x+1|$
Isolate the absolute value on one side of the equation.
$7 = 4x+1$ or $-7 = 4x+1$
Break into two separate equations and solve. $6 = 4x$ $-8 = 4x$ $x = 6\ 4 = 1.5$ $x = -8\ 4 = -2$

The function outputs 0 when $x = 3\ 2$ or $x = -2$. See [link].



For the function $f(x) = |\, 2x - 1 \,| - 3$, find the values of x such that $f(x) = 0$.

$x = -1$ or $x = 2$

Should we always expect two answers when solving $|\, A \,| = B$?

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |\, 3x - 5 \,| = 1$.

Access these online resources for additional instruction and practice with absolute value.

- Graphing Absolute Value Functions
- Graphing Absolute Value Functions 2

Key Concepts

- Applied problems, such as ranges of possible values, can also be solved using the absolute value function. See [link].
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction. See [link].
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable. See [link].

Section Exercises

Verbal

How do you solve an absolute value equation?

Isolate the absolute value term so that the equation is of the form $|A|=B.$ Form one equation by setting the expression inside the absolute value symbol, $A,$ equal to the expression on the other side of the equation, $B.$

Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

How can you tell whether an absolute value function has two *x*-intercepts without graphing the function?

When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

The graph of the absolute value function does not cross the x-axis, so the graph is either completely above or completely below the x-axis.

How can you use the graph of an absolute value function to determine the *x*-values for which the function values are negative?

Algebraic

Describe all numbers x that are at a distance of

4 from the number 8. Express this set of numbers using absolute value notation.

Describe all numbers x that are at a distance of 12 from the number -4. Express this set of numbers using absolute value notation.

$|x+4|=12$

Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this set of numbers using absolute value notation.

Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this set of numbers using absolute value notation.

$|f(x)-8|<0.03$

For the following exercises, find the *x*- and *y*-intercepts of the graphs of each function.

$f(x)=4|x-3|+4$

$f(x) = -3|x-2|-1$

$(0,-7)$; no x-intercepts

$f(x) = -2|x+1|+6$

$f(x) = -5|x+2|+15$

$(0,5),(1,0),(-5,0)$

$f(x) = 2|x-1|-6$

$(0,-4),(4,0),(-2,0)$

$f(x) = |-2x+1|-13$

$(0,-12),(-6,0),(7,0)$

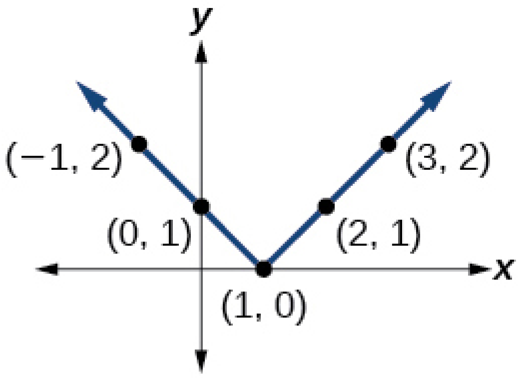
$f(x) = -|x-9|+16$

$(0,7),(25,0),(-7,0)$

Graphical

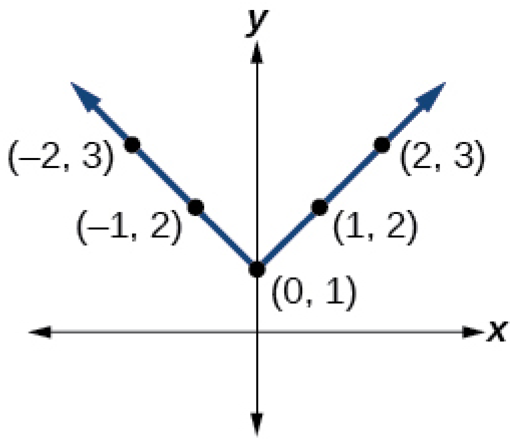
For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

$y = |x - 1|$



$y = |x + 1|$

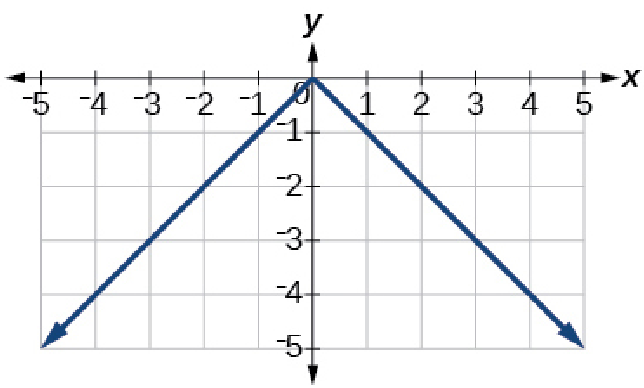
$y = |x| + 1$



For the following exercises, graph the given functions by hand.

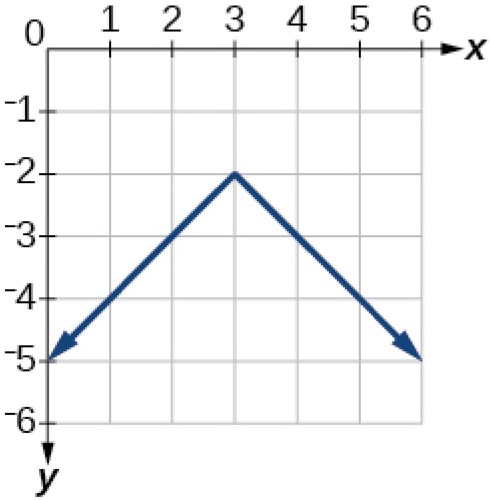
$y = |x| - 2$

$y = -|x|$



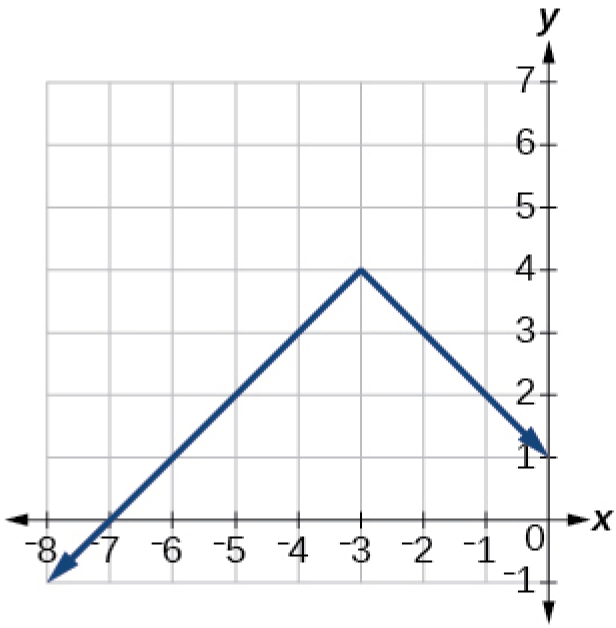
$$y = -|x| - 2$$

$$y = -|x - 3| - 2$$



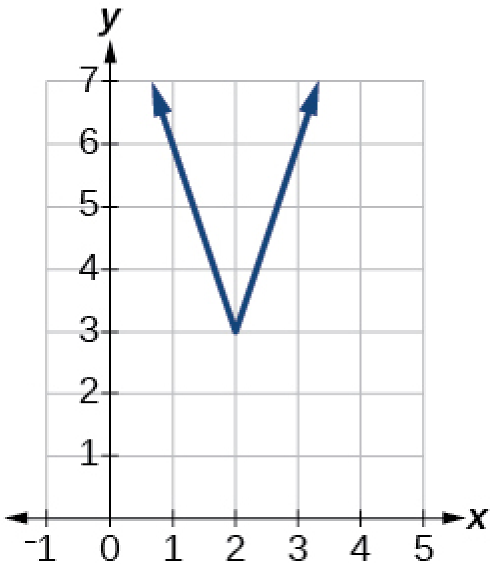
$$f(x) = -|x - 1| - 2$$

$$f(x) = -|x + 3| + 4$$



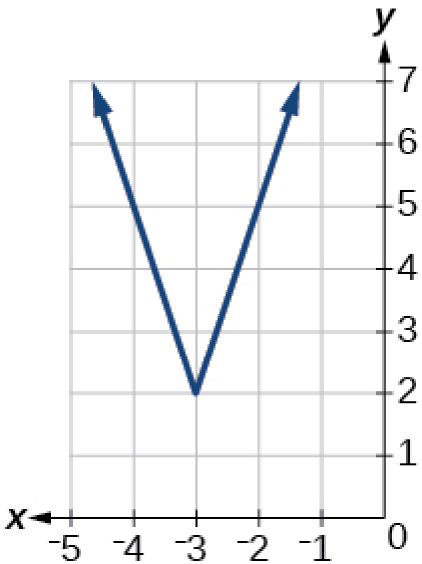
$f(x) = 2|x + 3| + 1$

$f(x) = 3|x - 2| + 3$



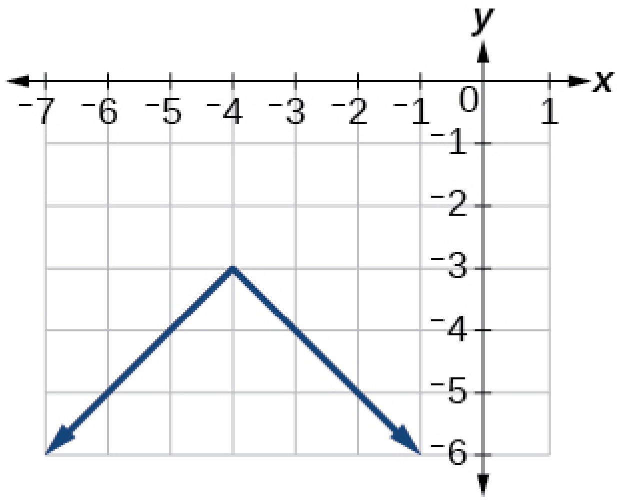
$$f(x) = |\ 2x - 4\ | - 3$$

$$f(\ x\ ) = |\ 3x + 9\ | + 2$$



$f(x) = -|x-1| - 3$

$f(x) = -|x+4| - 3$



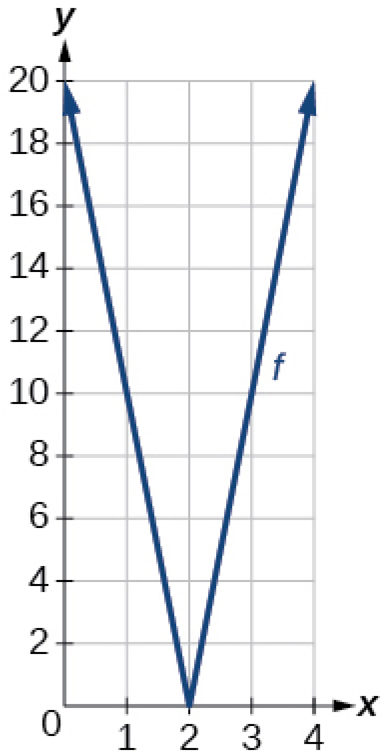
$f(x)= 1\ 2\ |\ x+4\ |-3$

Technology

Use a graphing utility to graph $f(x)=10|x-2|$ on the viewing window $[\ 0,4\ ]$. Identify the corresponding range. Show the graph.

---

range: $[\ 0,20\ ]$

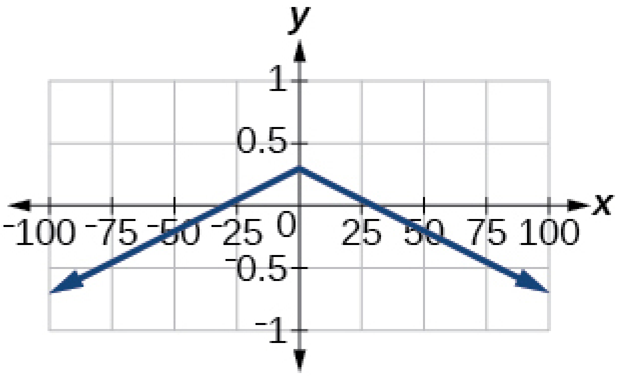


Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using a graphing utility. Specify the viewing window.

$f(x) = -0.1|0.1(0.2 - x)| + 0.3$

x- intercepts:



$f(x)=4\times\ 10\ 9\ |\ x-(5\times\ 10\ 9\ )\ |+2\times\ 10\ 9$

Extensions

For the following exercises, solve the inequality.

If possible, find all values of a such that there are no x- intercepts for $f(x)=2|\ x+1\ |+a$.

If possible, find all values of a such that there are no y -intercepts for $f(x)=2|\ x+1\ |+a$.

There is no solution for a that will keep the function from having a y -intercept. The

absolute value function always crosses the y-intercept when $x = 0$.

Real-World Applications

Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

$| p - 0.08 | \le 0.015$

Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

A machinist must produce a bearing that is within 0.01 inches of the correct diameter of

5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

$$|x-5.0|\leq 0.01$$

The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

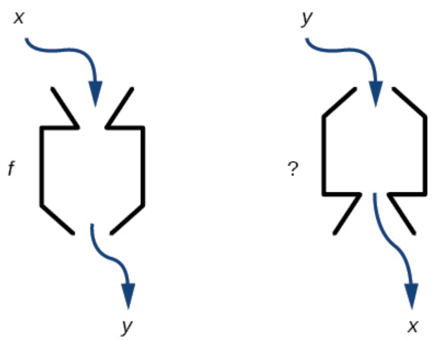
Inverse Functions

In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. [link] provides a visual representation of this question. In this section, we will consider the reverse nature of functions.
Can a function "machine" operate in reverse?



Verifying That Two Functions Are Inverse Functions

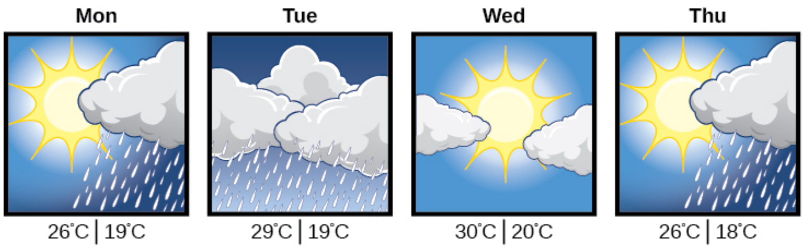
Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24°C.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from [link] for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.



At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \qquad 26 \cdot \frac{9}{5} = F - 32 \qquad F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1. In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}(f(x))=f^{-1}(y)=x$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x))=x$ or $f(g(x))=x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y=4x$ and $y=\frac{1}{4}x$ are inverse functions.

$$(f^{-1}\circ f)(x)=f^{-1}(4x)=\frac{1}{4}(4x)=x$$

and

$$(f\circ f^{-1})(x)=f\left(\frac{1}{4}x\right)=4\left(\frac{1}{4}x\right)=x$$

A few coordinate pairs from the graph of the function $y=4x$ are $(-2,\ -8)$, $(0,\ 0)$, and $(2,\ 8)$. A few coordinate pairs from the graph of the function $y=\frac{1}{4}x$ are $(-8,\ -2)$, $(0,\ 0)$, and $(8,\ 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Inverse Function

For any one-to-one function $f(x)=y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y)=x$. This can also be written as $f^{-1}(f(x))=x$ for all x in the domain of f. It also follows that $f(f^{-1}(x))=x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read " f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x)\neq\frac{1}{f(x)}$$

and not all functions have inverses.

Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2)=4$

and $f(5)=12$, what are the corresponding input and output values for the inverse function?

The inverse function reverses the input and output quantities, so if
$f(2)=4$, then $f^{-1}(4)=2$; $f(5)=12$, then $f^{-1}(12)=5$.

Alternatively, if we want to name the inverse function g, then $g(4)=2$ and $g(12)=5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See [link].

$(x,f(x))$	$(x,g(x))$
$(2,4)$	$(4,2)$
$(5,12)$	$(12,5)$

Given that $h^{-1}(6)=2$, what are the

corresponding input and output values of the original function h?

$h(2)=6$

Given two functions f(x) and g(x), test whether the functions are inverses of each other.

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If both statements are true, then $g= f^{-1}$ and $f= g^{-1}$. If either statement is false, then both are false, and $g\neq f^{-1}$ and $f\neq g^{-1}$.

Testing Inverse Relationships Algebraically

If $f(x)= 1\ x+2$ and $g(x)= 1\ x -2$, is $g= f^{-1}$?

$g(f(x))= 1\ (1\ x+2) -2 =x+2-2 =x$

so

$g= f^{-1}$ and $f= g^{-1}$

This is enough to answer yes to the question, but we can also verify the other formula.

$f(g(x)) = \frac{1}{\frac{1}{x}-2+2} = \frac{1}{\frac{1}{x}} = x$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

No

Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

$f(g(x)) = \frac{x^3}{27} \neq x$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the

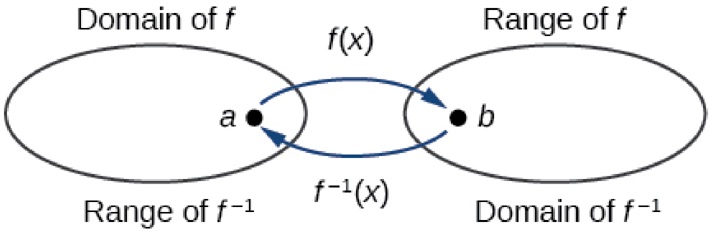
cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?
Yes

Domain and range of a function and its inverse

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in [link].



When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x)= x$ is $f^{-1}(x)= x^2$, because a square “undoes” a square root; but the square is only the inverse of the square root on the domain $[0,\infty)$, since that is the range of $f(x)= x$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x)= x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the “inverse” is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function

$f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $[1, \infty)$.
- The domain of f^{-1} = range of f = $[0, \infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in [link]. We restrict the domain in such a fashion that the function assumes all *y*-values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x)=c$	$f(x)=x$	$f(x)=x$	$f(x)=x$	$f(x)=1$

Reciprocal squared	Cube root	Square root	Absolute value
$f(x)=\frac{1}{x^2}$	$f(x)=\sqrt[3]{x}$	$f(x)=\sqrt{x}$	$f(x)=\lvert x\rvert$

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

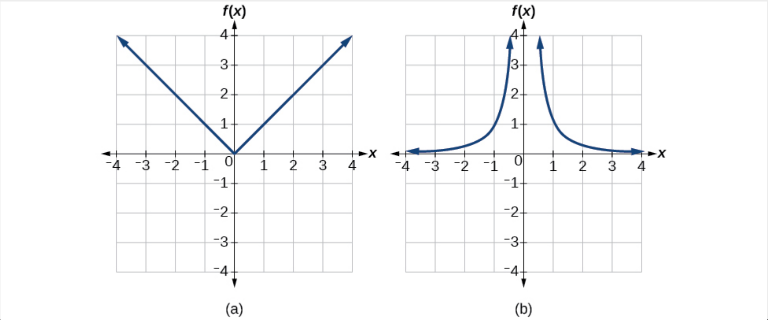
The absolute value function can be restricted to the domain $[0,\infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0,\infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in [link]. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

(a) Absolute value (b) Reciprocal squared



The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Interpreting the Inverse of a Tabular Function

A function f(t) is given in [link], showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
f(t	20	40	60	70

) (miles)

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70)=90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a)=b$, then $f^{-1}(b)=a$. By this definition, if we are given $f^{-1}(70)=a$, then we are looking for a value a so that $f(a)=70$. In this case, we are looking for a t so that $f(t)=70$, which is when $t=90$.

Using [link], find and interpret (a) $f(60)$, and (b) $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

1. $f(60)=50$. In 60 minutes, 50 miles are traveled.
2. $f^{-1}(60)=70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

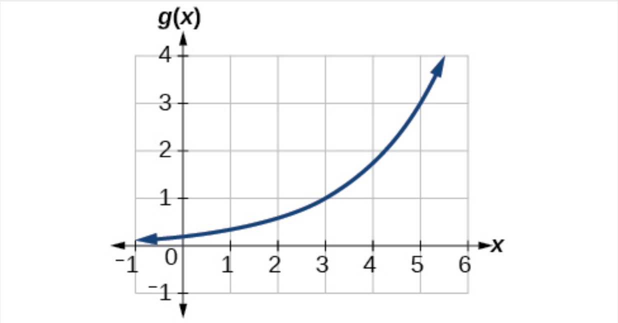
We saw in Functions and Function Notation that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the *y*-axis of the given graph.
2. Read the inverse function's output from the *x*-axis of the given graph.

Evaluating a Function and Its Inverse from a Graph at Specific Points

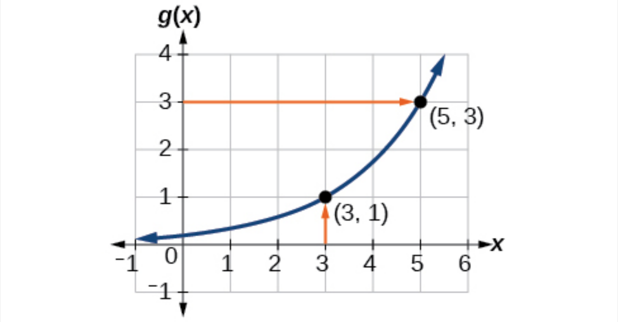
A function g(x) is given in [link]. Find g(3) and $g^{-1}(3)$.



To evaluate g(3), we find 3 on the *x*-axis and find the corresponding output value on the *y*-axis. The point $(3,1)$ tells us that $g(3)=1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of *x* for which

$g(x)=3$. By looking for the output value 3 on the vertical axis, we find the point $(5,3)$ on the graph, which means $g(5)=3$, so by definition, $g^{-1}(3)=5$. See [link].



Using the graph in [link], (a) find $g^{-1}(1)$, and (b) estimate $g^{-1}(4)$.

a. 3; b. 5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—

for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$C=\frac{5}{9}(F-32)$

$C=\frac{5}{9}(F-32)$ $C\cdot\frac{9}{5}=F-32$ $F=\frac{9}{5}C+32$

By solving in general, we have uncovered the inverse function. If

$C=h(F)=\frac{5}{9}(F-32),$

then

$F=h^{-1}(C)=\frac{9}{5}C+32.$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Solve for x in terms of y given $y=\frac{1}{3}(x-5)$

$x=3y+5$

Solving to Find an Inverse Function

Find the inverse of the function $f(x)=\frac{2}{x-3}+4$.

$y=\frac{2}{x-3}+4$ Set up an equation. $y-4=\frac{2}{x-3}$ Subtract 4 from both sides. $x-3=\frac{2}{y-4}$ Multiply both sides by $x-3$ and divide by $y-4$. $x=\frac{2}{y-4}+3$ Add 3 to both sides.

So $f^{-1}(y)=\frac{2}{y-4}+3$ or $f^{-1}(x)=\frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating [link].

x	1	2	5	$f^{-1}(y)$
f(x)	3	2	5	y

Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x)=2+ x-4$

.

$$y=2+ x-4 \quad (y-2)^2 = x-4 \quad x= (y-2)^2 +4$$

So $f^{-1}(x)= (x-2)^2 +4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$.

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself

must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

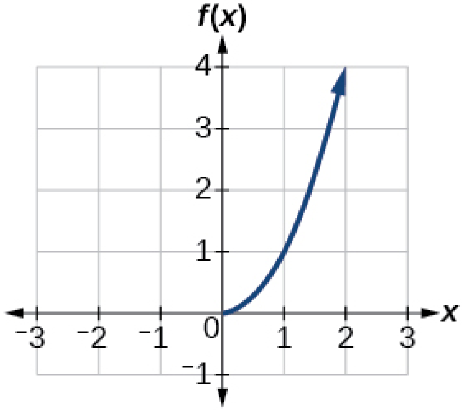
What is the inverse of the function $f(x)=2-\sqrt{x}$? State the domains of both the function and the inverse function.

$f^{-1}(x)=(2-x)^2$;domainoff:$[0,\infty)$;domainof f^{-1}:$(-\infty,2]$

Quadratic function with domain restricted to $[0,\infty)$. Square and square-root functions on the non-negative domain

Finding Inverse Functions and Their Graphs

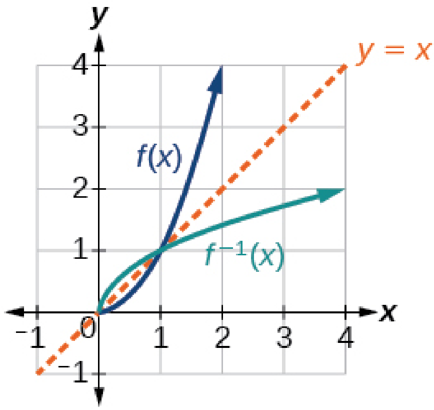
Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x)=x^2$ restricted to the domain $[0,\infty)$, on which this function is one-to-one, and graph it as in [link].



Restricting the domain to $[0,\infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x)=\sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x- axis for the input to both f and f^{-1}?

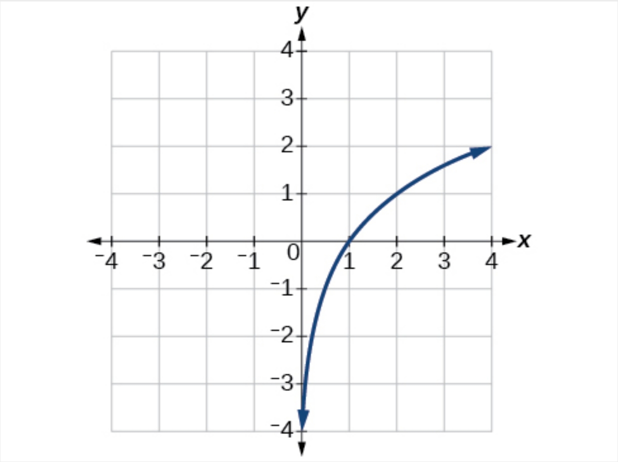
We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of f(x) reflected about the diagonal line $y=x$, which we will call the identity line, shown in [link].



This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Finding the Inverse of a Function Using Reflection about the Identity Line

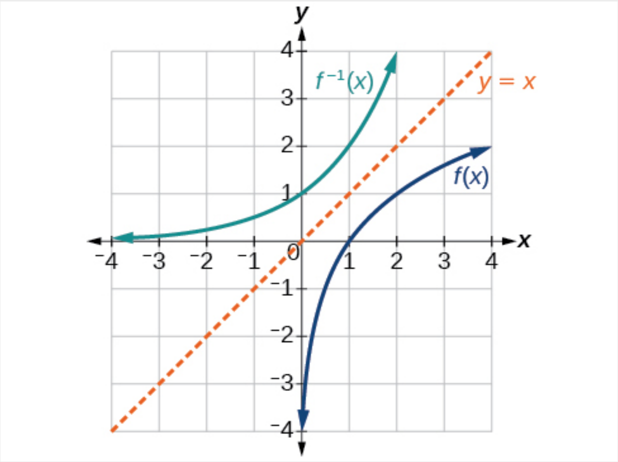
Given the graph of f(x) in [link], sketch a graph of $f^{-1}(x)$.



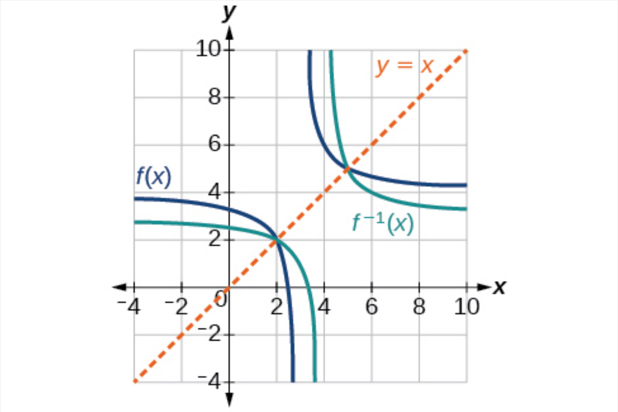
This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1,0)$ reflects to $(0,1)$ and the point $(4,2)$ reflects to $(2,4)$. Sketching the inverse on the same axes as the original graph gives [link].

The function and its inverse, showing reflection about the identity line



Draw graphs of the functions f and f −1 from [link].



Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Access these online resources for additional instruction and practice with inverse functions.

- Inverse Functions
- Inverse Function Values Using Graph
- Restricting the Domain and Finding the Inverse

Visit this website for additional practice questions from Learningpod.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$. See [link], [link], and

[link].
- Each of the toolkit functions has an inverse. See [link].
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See [link].
- The inverse of a function can be determined at specific points on its graph. See [link].
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y. See [link], [link], and [link].
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$. See [link].

Section Exercises

Verbal

Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y -values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y -values repeat and the function is one-to-one.

Why do we restrict the domain of the function $f(x) = x\ 2$ to find the function's inverse?

Can a function be its own inverse? Explain.

Yes. For example, $f(x) = 1\ x$ is its own inverse.

Are one-to-one functions either always increasing or always decreasing? Why or why not?

How do you find the inverse of a function algebraically?

Given a function $y=f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the

inverse, $y = f^{-1}(x)$.

Algebraic

Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

$f(x) = x + 3$

$f^{-1}(x) = x - 3$

$f(x) = x + 5$

$f(x) = 2 - x$

$f^{-1}(x) = 2 - x$

$f(x) = 3 - x$

$f(x) = x\ x + 2$

$f^{-1}(x) = -2x \; x-1$

$f(x) = 2x+3 \; 5x+4$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

$f(x) = (x+7)^2$

---

domain of $f(x): [-7, \infty)$; $f^{-1}(x) = x - 7$

$f(x) = (x-6)^2$

$f(x) = x^2 - 5$

---

domain of $f(x): [0, \infty)$; $f^{-1}(x) = x + 5$

Given $f(x) = x^2 + x$ and $g(x) = 2x \; 1-x$:

1. Find f(g(x)) and g(f(x)).
2. What does the answer tell us about the relationship between f(x) and g(x)?

---

a. $f(g(x))=x$ and $g(f(x))=x$. b. This tells us that f and g are inverse functions

For the following exercises, use function composition to verify that f(x) and g(x) are inverse functions.

$f(x)= x-1 3$ and $g(x)= x 3 +1$

$f(g(x))=x, g(f(x))=x$

$f(x)= -3x+5$ and $g(x)= x-5 -3$

Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

$f(x)= x$

---

one-to-one

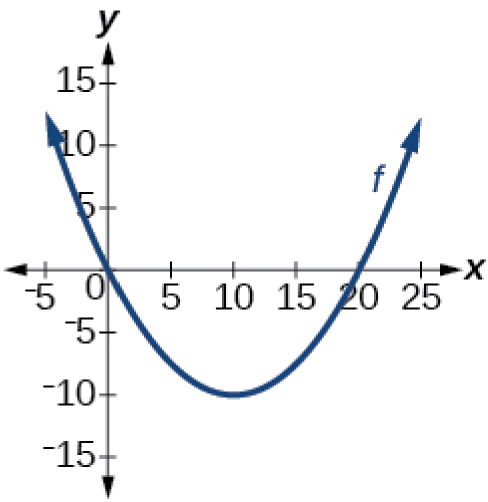
$f(x)= 3x+1 3$

$f(x)= -5x+1$

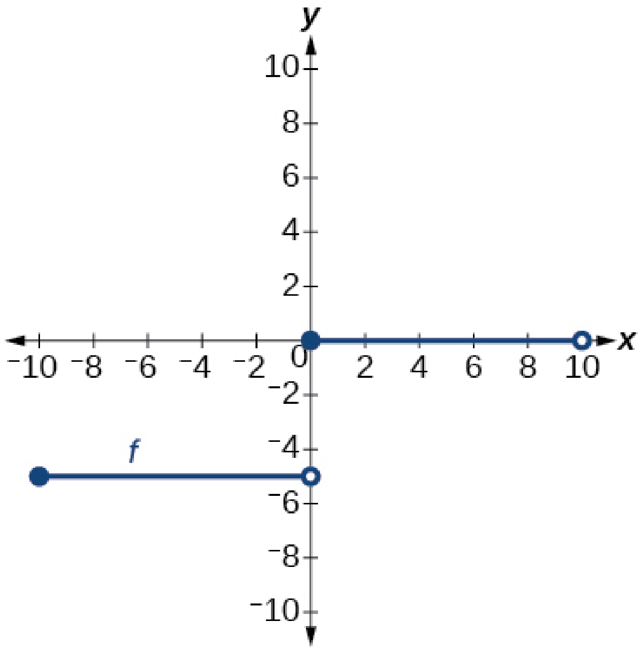
one-to-one

$f(x) = x^3 - 27$

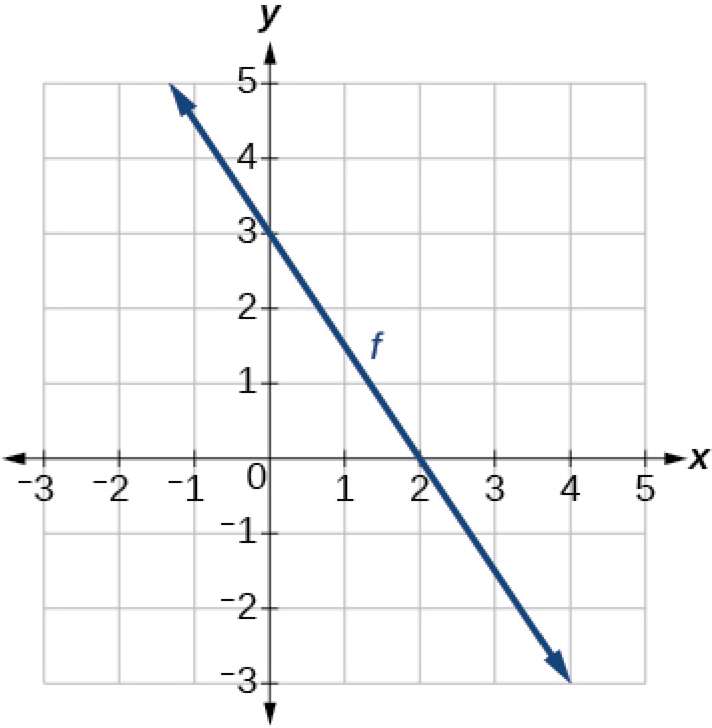
For the following exercises, determine whether the graph represents a one-to-one function.



not one-to-one



For the following exercises, use the graph of f shown in [link].



Find f(0).

3

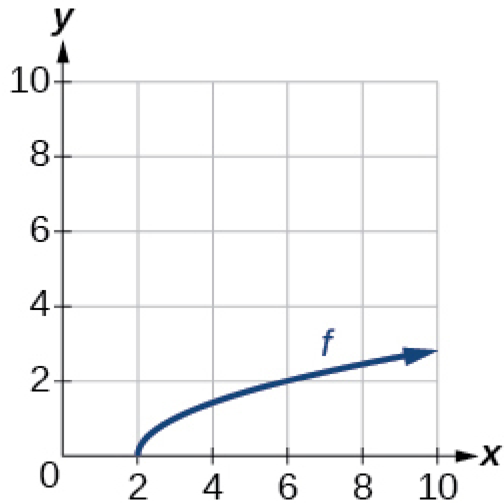
Solve f(x) = 0.

Find $f^{-1}(0)$.

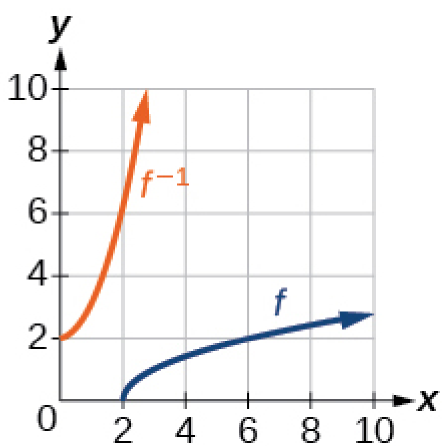
2

Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in [link].



Sketch the graph of f^{-1}.



Find f(6) and $f^{-1}(2)$.

If the complete graph of f is shown, find the domain of f.

$[2,10]$

If the complete graph of f is shown, find the range of f.

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

If $f(6)=7$, find $f^{-1}(7)$.

6

If $f(3)=2$, find $f^{-1}(2)$.

If $f^{-1}(-4)=-8$, find $f(-8)$.

-4

If $f^{-1}(-2)=-1$, find $f(-1)$.

For the following exercises, use the values listed in [link] to evaluate or solve.

x	f(x)
0	8
1	0
2	7
3	4
4	2
5	6
6	5
7	3
8	9
9	1

Find $f(1)$.

0

Solve $f(x)=3$.

Find $f^{-1}(0)$.

1

Solve $f^{-1}(x)=7$.

Use the tabular representation of f in [link] to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
f(x)	1	4	7	12	16

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

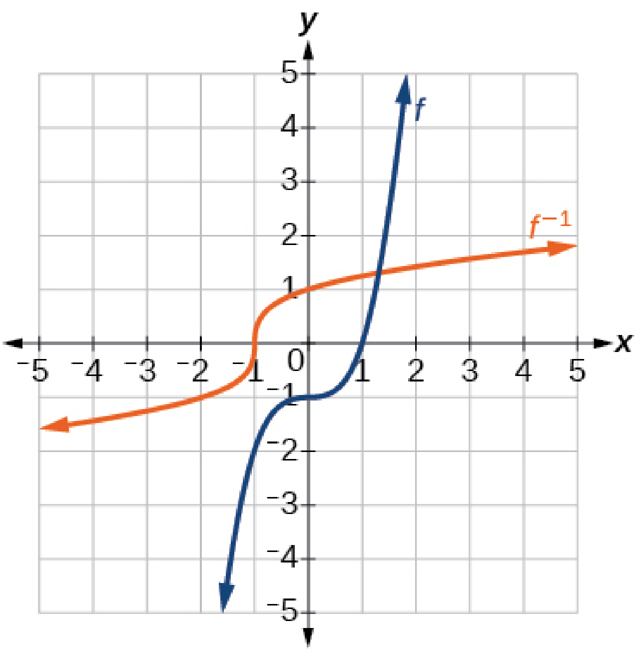
Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

$f(x) = \frac{3}{x-2}$

$f(x) = x^3 - 1$

$f^{-1}(x) = (1+x)^{1/3}$



Find the inverse function of $f(x)=\frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x)=\frac{9}{5}x+32$. Find the inverse function, if it exists, and explain its meaning.

$f^{-1}(x)=\frac{5}{9}(x-32)$. Given the Fahrenheit temperature, x, this formula allows you to calculate the Celsius temperature.

The circumference C of a circle is a function of its radius given by $C(r)=2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t)=50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call

this function t(d). Find t(180) and interpret its meaning.

---

t(d)= d 50 , t(180)= 180 50 . The time for the car to travel 180 miles is 3.6 hours.

Chapter Review Exercises

Functions and Function Notation

For the following exercises, determine whether the relation is a function.

{ (a,b),(c,d),(e,d) }

---

function

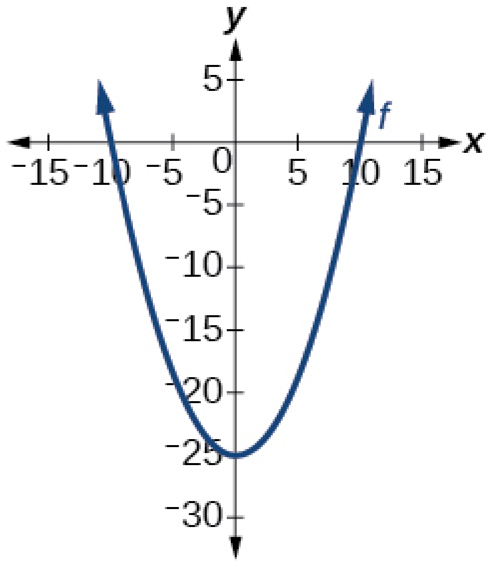
{ (5,2),(6,1),(6,2),(4,8) }

y 2 + 4 = x, for x the independent variable and y the dependent variable

---

not a function

Is the graph in [link] a function?



For the following exercises, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a+h)$.

$f(x) = -2x^2 + 3x$

$f(-3) = -27$; $f(2) = -2$; $f(-a) = -2a^2 - 3a$; $-f(a) = 2a^2 - 3a$; $f(a+h) = -2a^2 + 3a - 4ah + 3h - 2h^2$

$f(x) = 2|3x - 1|$

For the following exercises, determine whether the

functions are one-to-one.

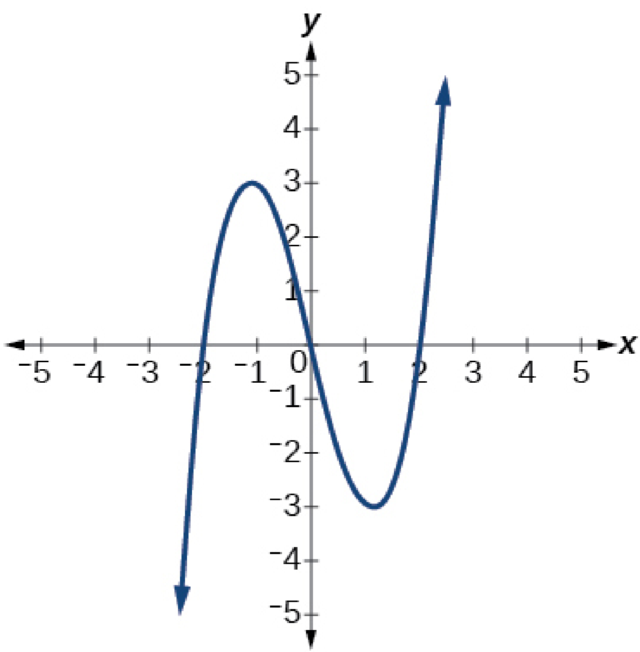
$f(x) = -3x + 5$

---

one-to-one

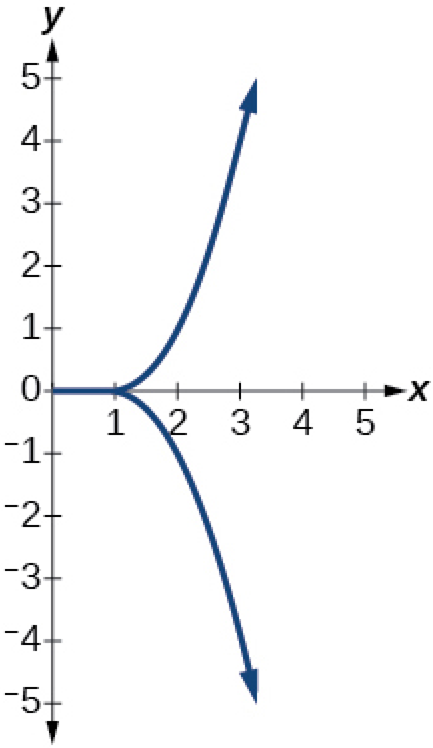
$f(x) = | x - 3 |$

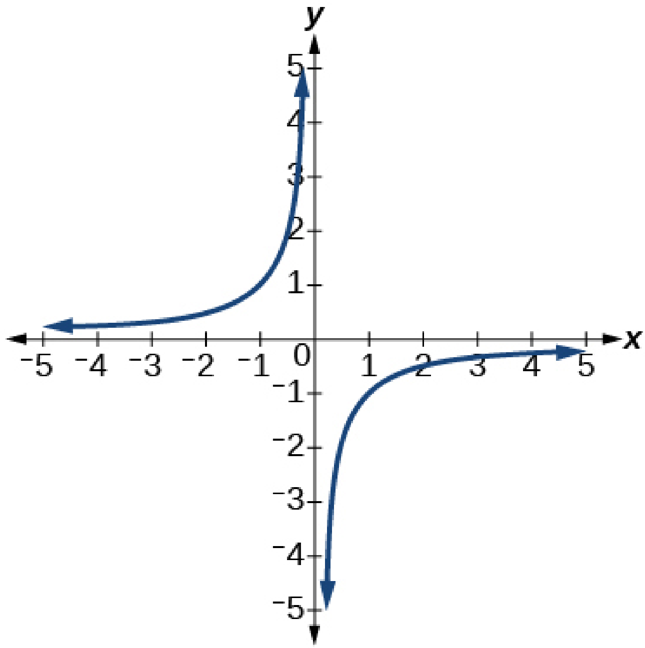
For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.



---

function



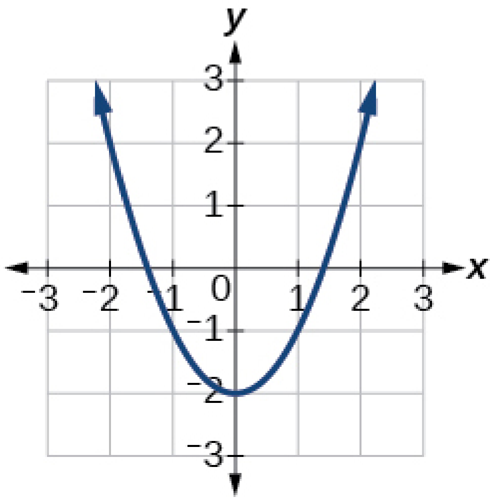


function

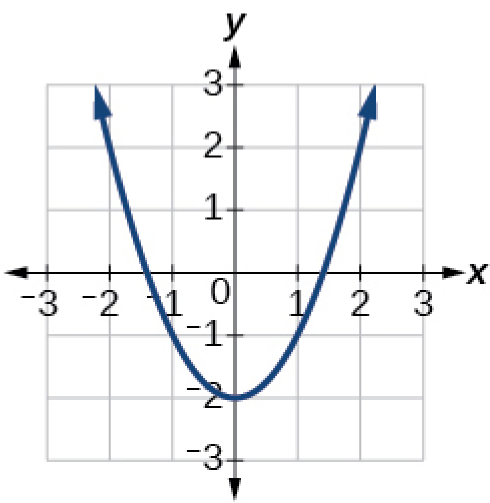
For the following exercises, graph the functions.

$f(x) = | x + 1 |$

$f(x) = x^2 - 2$



For the following exercises, use [link] to approximate the values.



f(2)

$f(-2)$

2

If $f(x) = -2$, then solve for x.

If $f(x) = 1$, then solve for x.

$x = -1.8$ or or $x = 1.8$

For the following exercises, use the function $h(t) = -16t^2 + 80t$ to find the values.

$\frac{h(2) - h(1)}{2 - 1}$

$\frac{h(a) - h(1)}{a - 1}$

$\frac{-64 + 80a - 16a^2}{-1 + a} = -16a + 64$

Domain and Range

For the following exercises, find the domain of each function, expressing answers using interval notation.

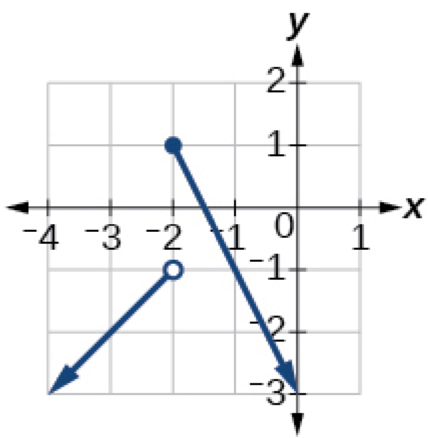
$f(x) = 2\ 3x + 2$

$f(x) = x - 3\ x\ 2\ - 4x - 12$

$(\ - \infty, -2\ ) \cup (\ -2, 6\ ) \cup (\ 6, \infty\ \ )$

$f(x) = x - 6\ x - 4$

Graph this piecewise function: $f(x) = \{\ x + 1 \quad x < -2\ \ -2x - 3 \quad x \geq -2$



Rates of Change and Behavior of Graphs

For the following exercises, find the average rate of

change of the functions from $x=1$ to $x=2$.

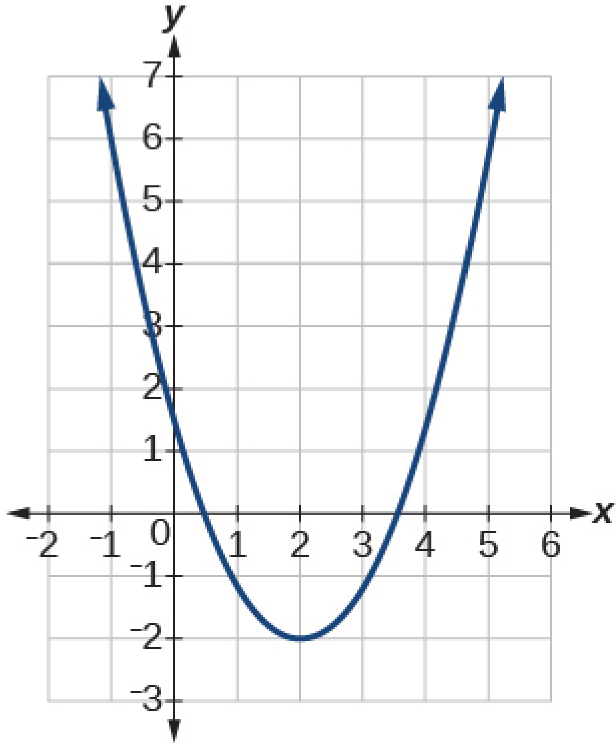
$f(x)=4x-3$

$f(x)=10x^2+x$

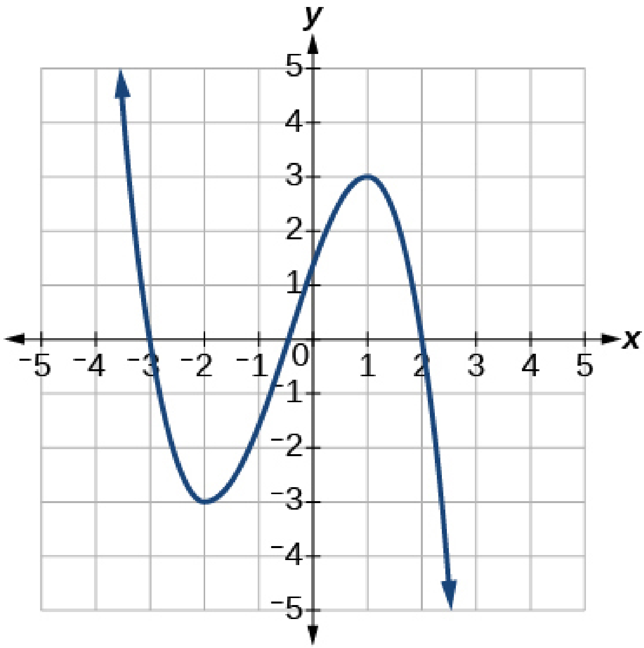
31

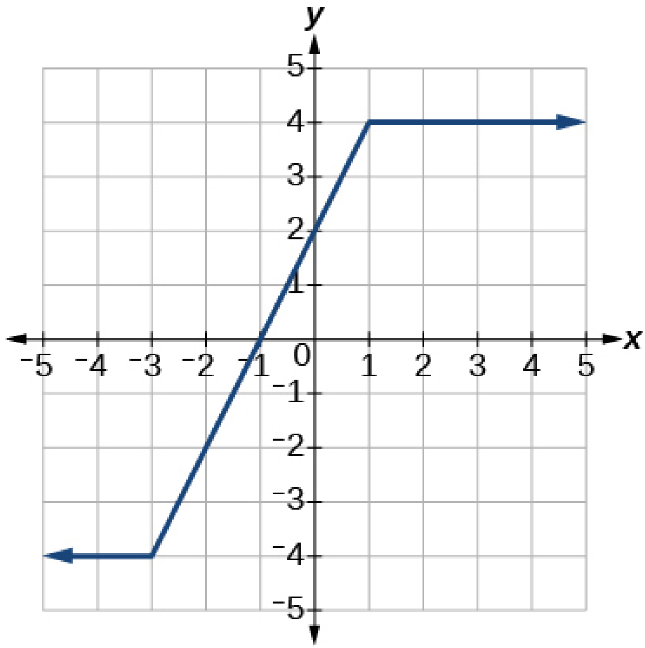
$f(x)=-2x^2$

For the following exercises, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.



increasing $(2, \infty)$; decreasing $(-\infty, 2)$


y
x
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
1
2
3
4
5



---

increasing $(-3,1)$; constant $(-\infty,-3)\cup(1,\infty)$

Find the local minimum of the function graphed in [link].

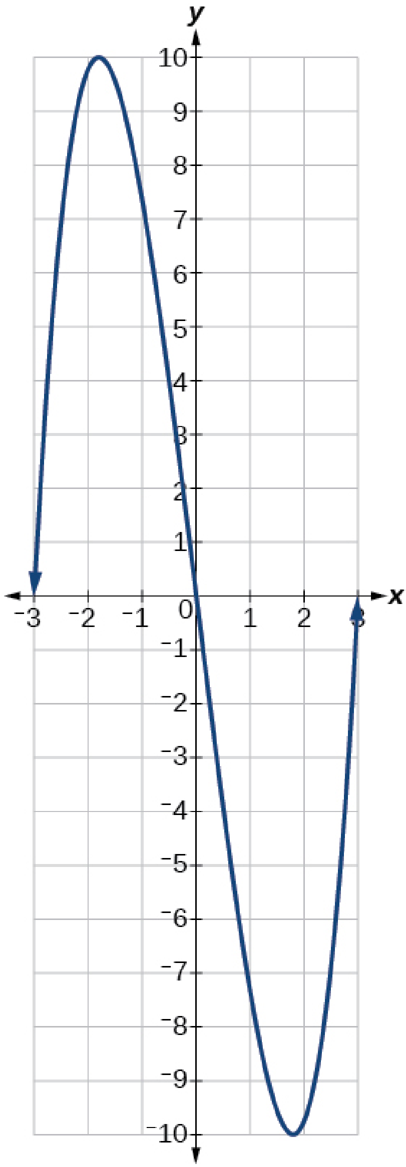
Find the local extrema for the function graphed in [link].

---

local minimum $(-2,-3)$; local maximum $(1,3)$

For the graph in [link], the domain of the function is $[-3,3]$. The range is $[-10,10]$. Find the absolute minimum of the function on this interval.

Find the absolute maximum of the function graphed in [link].



Absolute Maximum: 10

Composition of Functions

For the following exercises, find (f∘g)(x) and (g∘f)(x) for each pair of functions.

$f(x)=4-x, g(x)=-4x$

$f(x)=3x+2, g(x)=5-6x$

$(f\circ g)(x)=17-18x;(g\circ f)(x)=-7-18x$

$f(x)= x\ 2\ +2x, g(x)=5x+1$

$f(x)= x+2\ , g(x)= 1\ x$

$(f\circ g)(x)= 1\ x\ +2\ ;(g\circ f)(x)= 1\ x+2$

$f(x)= x+3\ 2\ , g(x)= 1-x$

For the following exercises, find (f∘g) and the domain for (f∘g)(x) for each pair of functions.

$f(x)= x+1\ x+4\ , g(x)= 1\ x$

$(f\circ g)(x)= 1+x\ 1+4x\ , x\neq 0, x\neq -\ 1\ 4$

$f(x)= 1\ x+3\ ,g(x)= 1\ x-9$

$f(x)= 1\ x\ ,g(x)= x$

$(\ f\circ g\ )(x)= 1\ x\ ,x>0$

$f(x)= 1\ x\ 2\ -1\ ,g(x)= x+1$

For the following exercises, express each function H as a composition of two functions f and g where $H(x)=(f\circ g)(x)$.

$H(x)= 2x-1\ 3x+4$

---

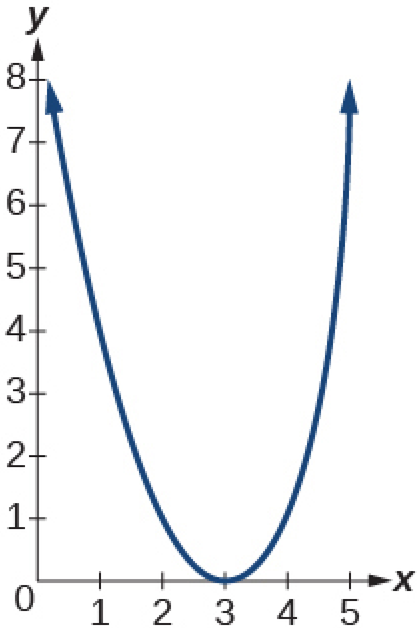
sample: $g(x)= 2x-1\ 3x+4\ ;f(x)= x$

$H(x)= 1\ (3\ x\ 2\ -4)\ -3$

Transformation of Functions

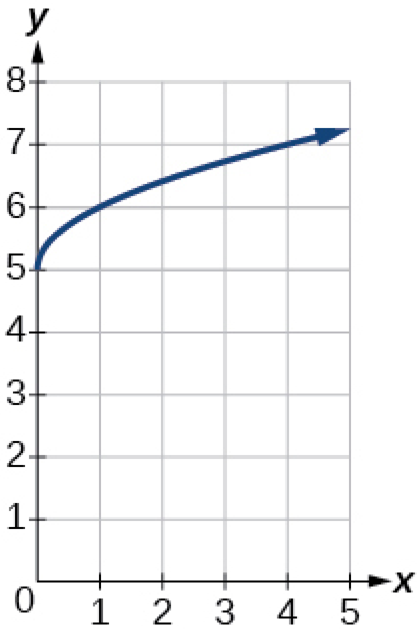
For the following exercises, sketch a graph of the given function.

$f(x)= (x-3)\ 2$



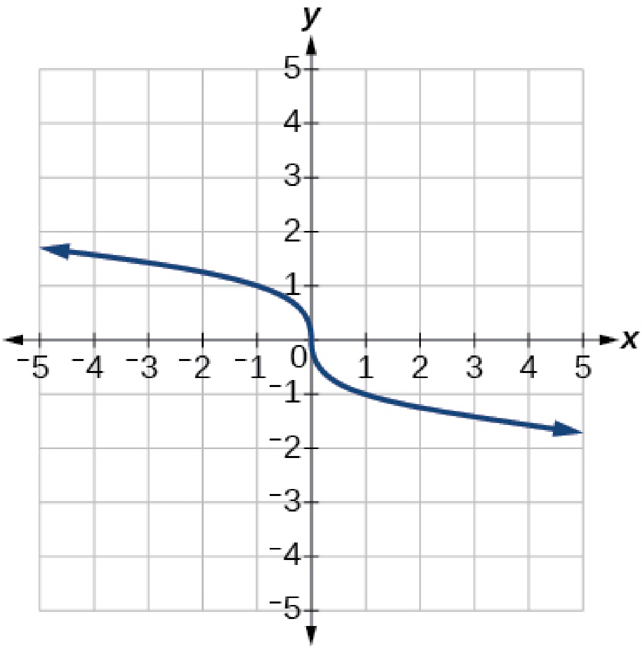
$f(x) = (x+4)\ 3$

$f(x) = x\ +5$



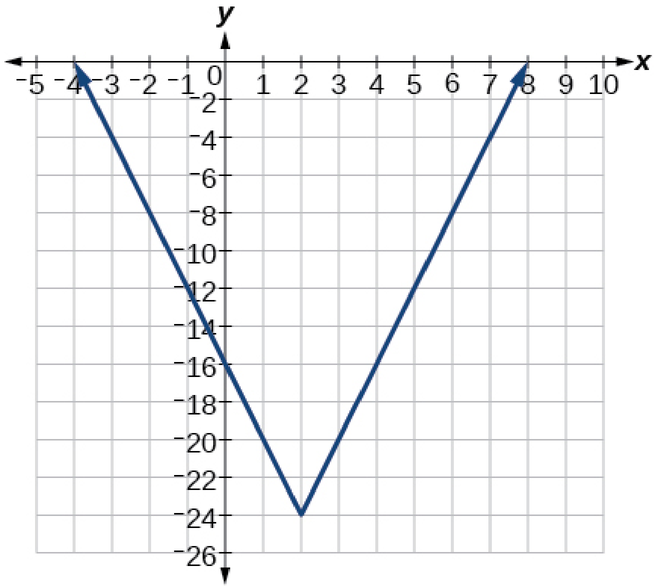
$f(x) = - \ x \ 3$

$f(x) = \ -x \ 3$



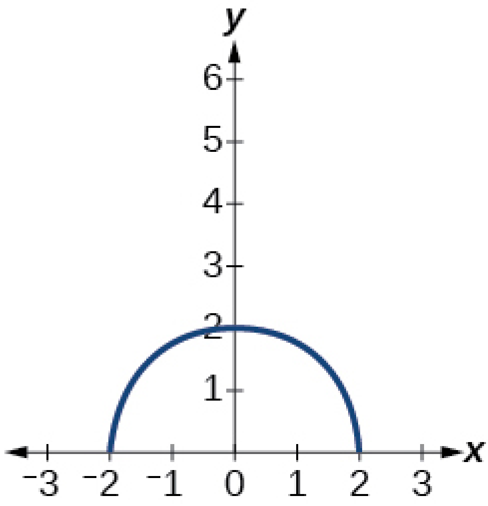
$f(x)=5\ \ -x\ \ -4$

$f(x)=4[\ |\ x-2\ |-6\ ]$

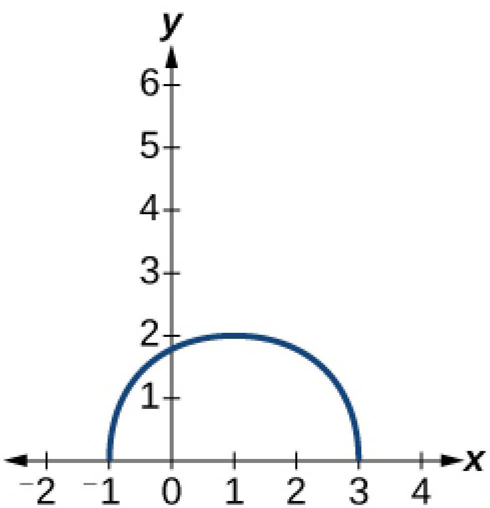


$$f(x) = -(x+2)^2 - 1$$

For the following exercises, sketch the graph of the function g if the graph of the function f is shown in [link].

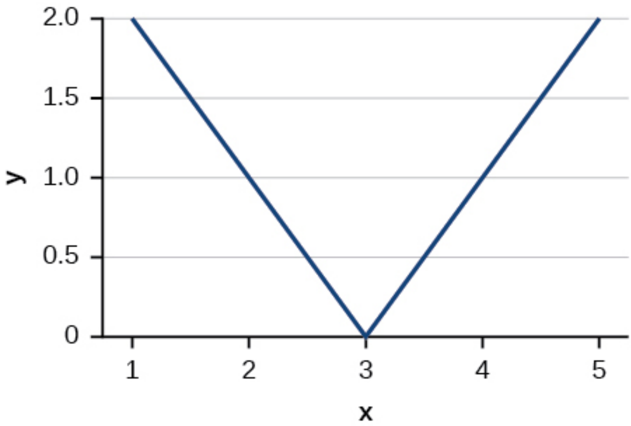


$g(x)=f(x-1)$

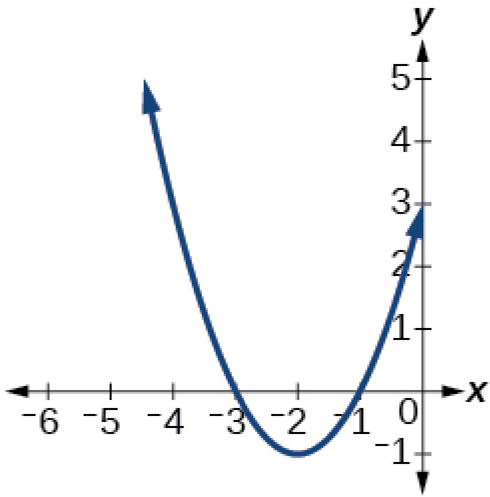


$g(x)=3f(x)$

For the following exercises, write the equation for the standard function represented by each of the graphs below.



$f(x) = |x - 3|$



For the following exercises, determine whether each function below is even, odd, or neither.

$f(x)=3\ x\ 4$

---

even

$g(x)=\ x$

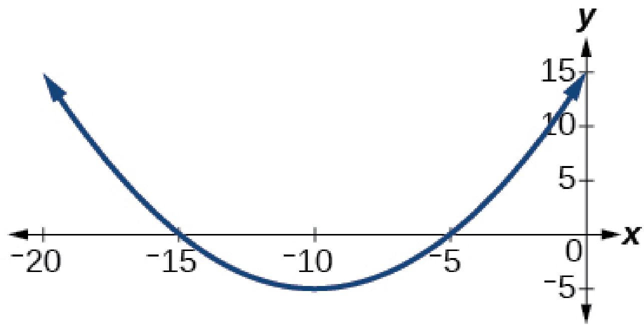
$h(x)=\ 1\ x\ +3x$

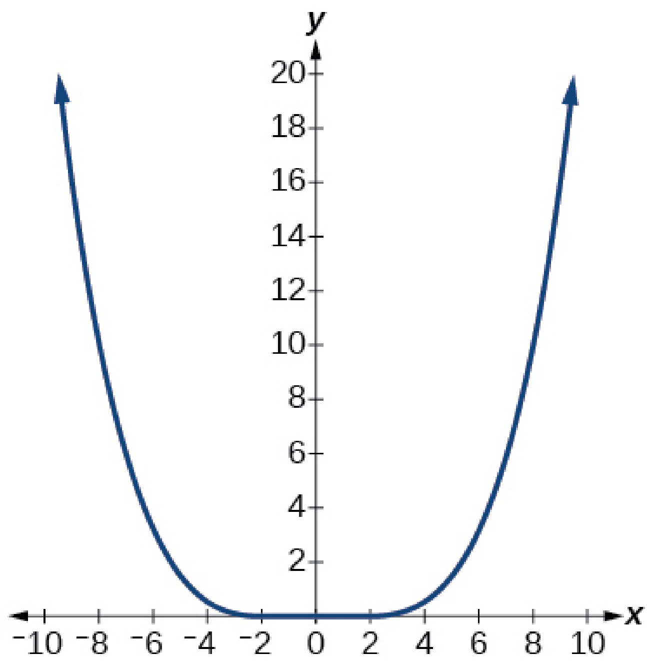
---

odd

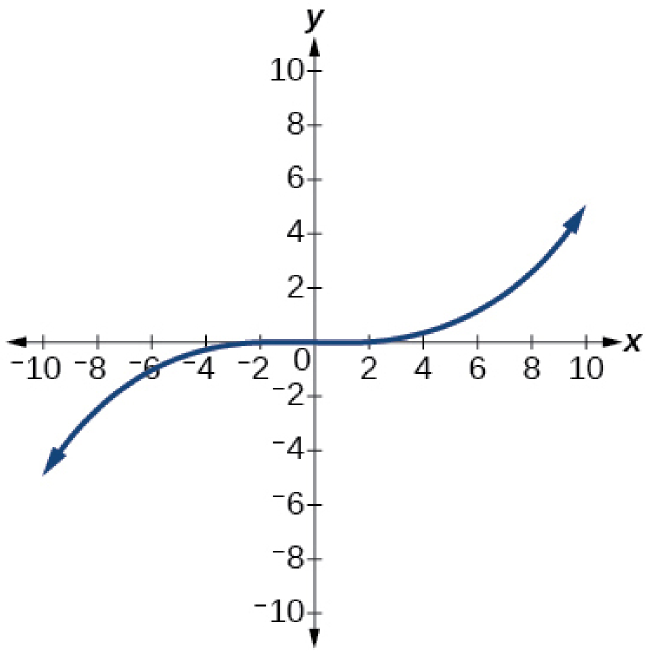
For the following exercises, analyze the graph and

determine whether the graphed function is even, odd, or neither.



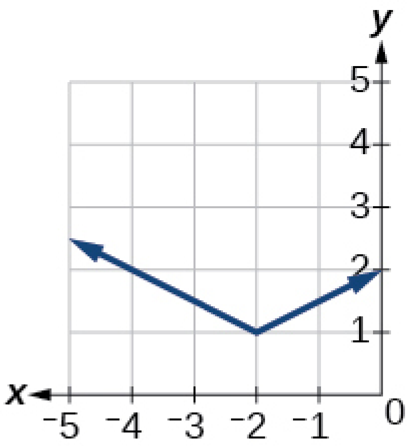


even

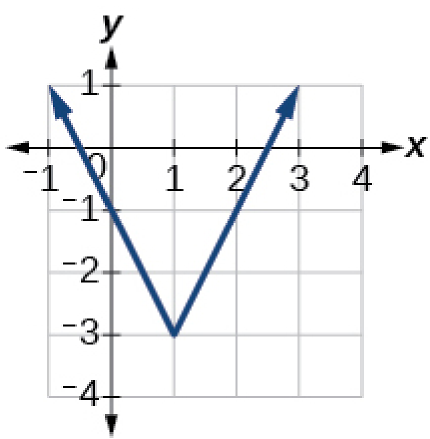


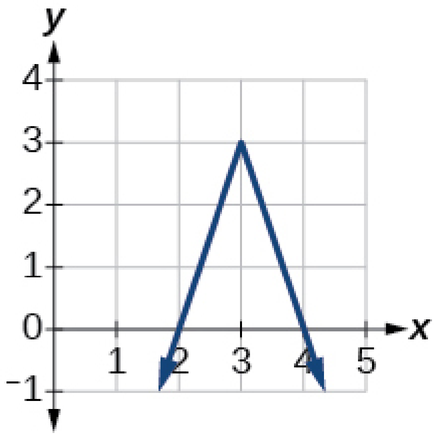
Absolute Value Functions

For the following exercises, write an equation for the transformation of $f(x) = |x|$.



$$f(x)=\frac{1}{2}|x+2|+1$$



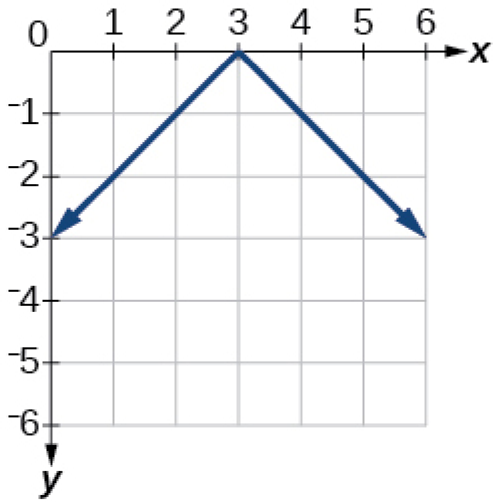


$f(x) = -3|x - 3| + 3$

For the following exercises, graph the absolute value function.

$f(x) = |x - 5|$

$f(x) = -|x - 3|$



$$f(x) = |\ 2x - 4\ |$$

For the following exercises, solve the absolute value equation.

$$|\ x + 4\ | = 18$$

$$x = -22, x = 14$$

$$|\ 1\ 3\ x + 5\ | = |\ 3\ 4\ x - 2\ |$$

For the following exercises, solve the inequality and express the solution using interval notation.

$|3x-2|<7$

$(-5\ 3,3)$

$|1\ 3\ x-2|\leq 7$

Inverse Functions

For the following exercises, find $f^{-1}(x)$ for each function.

$f(x)=9+10x$

$f^{-1}(x) = x\text{-}1$

$f(x)= x\ x+2$

For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

$f(x)= x^2+1$

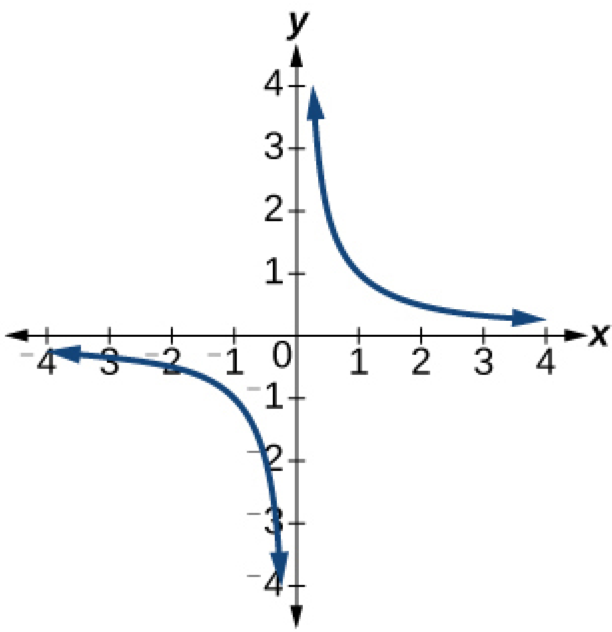
Given $f(x)= x^3-5$ and $g(x)= x+5\ 3$:

1. Find f(g(x)) and g(f(x)).
2. What does the answer tell us about the relationship between f(x) and g(x)?

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

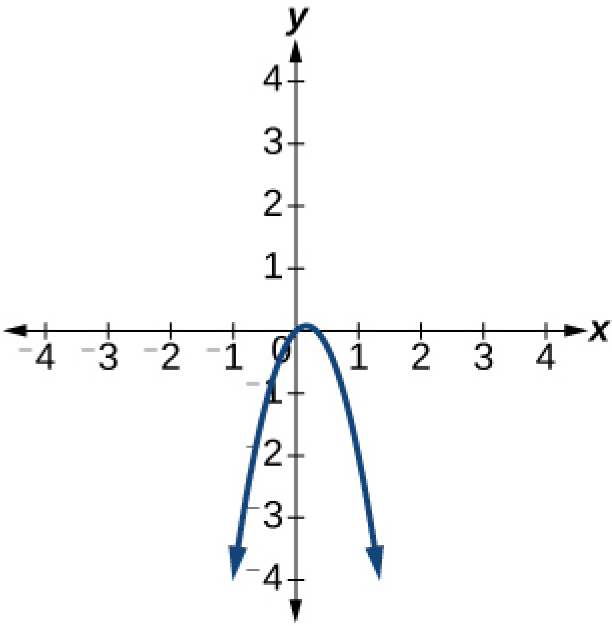
$f(x) = 1\ x$

The function is one-to-one.



$f(x)=-3x^2+x$

The function is not one-to-one.



If $f(5)=2$, find $f^{-1}(2)$.

5

If $f(1)=4$, find $f^{-1}(4)$.

Practice Test

For the following exercises, determine whether each of the following relations is a function.

$y = 2x + 8$

The relation is a function.

$\{ (2,1),(3,2),(-1,1),(0,-2) \}$

For the following exercises, evaluate the function $f(x) = -3x^2 + 2x$ at the given input.

$f(-2)$

-16

$f(a)$

Show that the function $f(x) = -2(x-1)^2 + 3$ is not one-to-one.

The graph is a parabola and the graph fails the horizontal line test.

Write the domain of the function $f(x)= 3-x$ in interval notation.

Given $f(x)=2 x 2 -5x$, find $f(a+1)-f(1)$.

$2 a 2 -a$

Graph the function $f(x)=\{ x+1 \quad \text{if } -2<x<3$
$-x \quad \text{if } x\geq 3$

Find the average rate of change of the function $f(x)=3-2 x 2 +x$ by finding $f(b)-f(a) b-a$.

$-2(a+b)+1$

For the following exercises, use the functions $f(x)=3-2 x 2 +x$ and $g(x)= x$ to find the composite functions.

$(g\circ f)(x)$

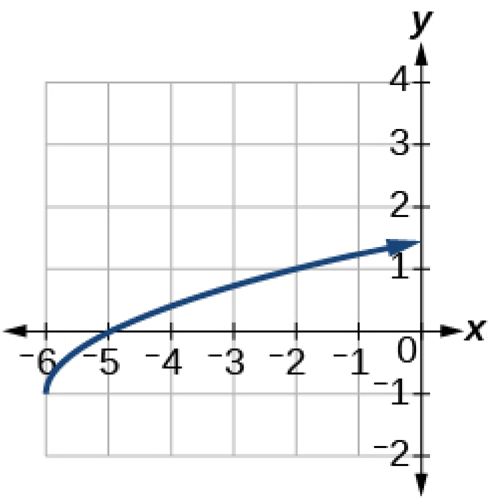
$(g\circ f)(1)$

2

Express $H(x) = 5\ x\ 2\ - 3x\ 3$ as a composition of two functions, f and g, where $(f \circ g)(x) = H(x)$.

For the following exercises, graph the functions by translating, stretching, and/or compressing a toolkit function.

$f(x) = x + 6\ - 1$



$f(x) = 1\ x + 2\ - 1$

For the following exercises, determine whether the functions are even, odd, or neither.

$f(x) = -5x^2 + 9x^6$

---

even

$f(x) = -5x^3 + 9x^5$

$f(x) = \frac{1}{x}$

---

odd

Graph the absolute value function $f(x) = -2|x - 1| + 3$.

Solve $|2x - 3| = 17$.

$x = -7$ and $x = 10$

Solve $-\left|\frac{1}{3}x - 3\right| \geq 17$. Express the solution in interval notation.

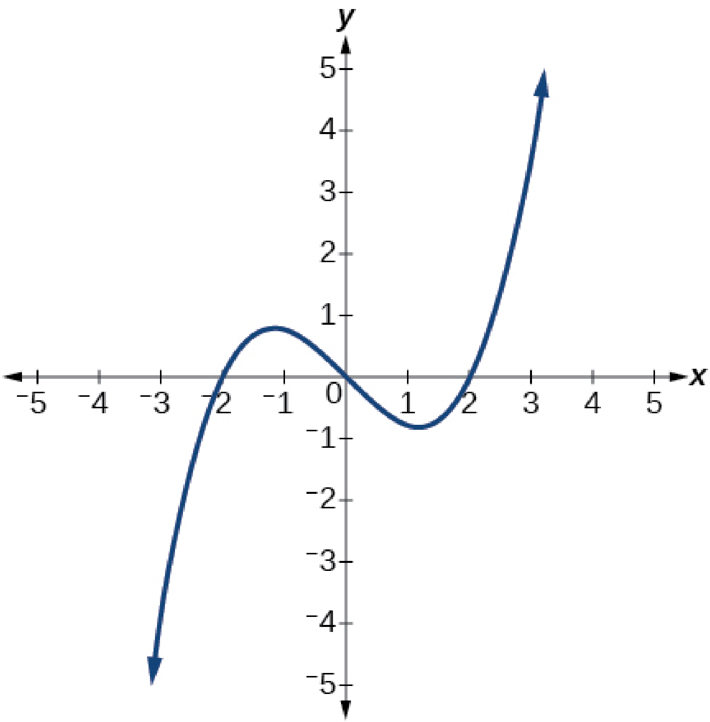
For the following exercises, find the inverse of the function.

$f(x) = 3x - 5$

$f^{-1}(x) = x + 5\ 3$

$f(x) = 4\ x + 7$

For the following exercises, use the graph of g shown in [link].



On what intervals is the function increasing?

$(-\infty, -1.1)$ and $(1.1, \infty)$

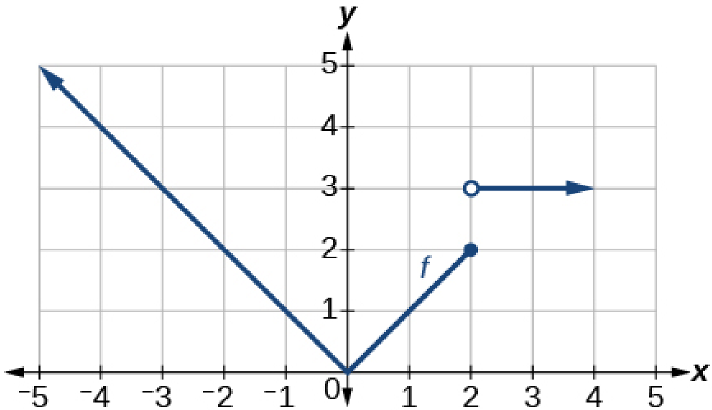
On what intervals is the function decreasing?

Approximate the local minimum of the function. Express the answer as an ordered pair.

$(1.1, -0.9)$

Approximate the local maximum of the function. Express the answer as an ordered pair.

For the following exercises, use the graph of the piecewise function shown in [link].



Find f(2).

$f(2) = 2$

Find $f(-2)$.

Write an equation for the piecewise function.

$f(x) = \{ \mid x \mid if x \leq 2 \ 3 if x > 2$

For the following exercises, use the values listed in [link].

x	F(x)
0	1
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

Find $F(6)$.

Solve the equation $F(x)=5$.

$x=2$

Is the graph increasing or decreasing on its domain?

Is the function represented by the graph one-to-one?

---

yes

Find $F^{-1}(15)$.

Given $f(x)=-2x+11$, find $f^{-1}(x)$.

$f^{-1}(x)=-\frac{x-11}{2}$

Glossary

inverse function
: for any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x))=x$ for all x in the domain of f; this also implies that $f(f^{-1}(x))=x$ for all x in the

domain of f^{-1}

Linear Functions

In this section you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over

time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train ([link]). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes[footnote].
http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Tabular representation of the function D showing selected input and output values The graph of $D(t)=83t+250$. Graphs of linear functions are lines because the rate of change is constant.

Representing Linear Functions

The function describing the train's motion is a linear function, which is defined as a function with a constant rate of change. This is a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each

method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the slope-

intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

Equation form $y = mx + b$ Function notation $f(x) = mx + b$

In the example of the train, we might use the notation $D(t)$ where the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$D(t) = 83t + 250$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in [link]. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

1 second 1 second 1 second

t	0	1	2	3
$D(t)$	250	333	416	499

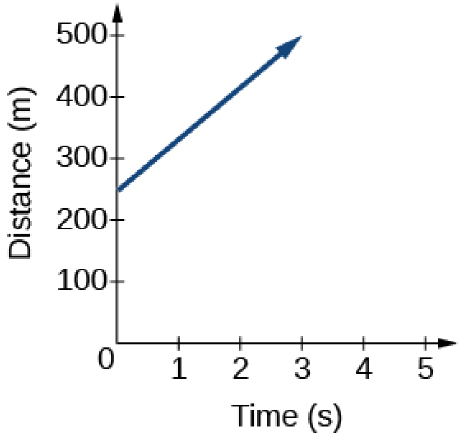
83 meters 83 meters 83 meters

Can the input in the previous example be any real number?
No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t)=83t+250$, to draw a graph as represented in [link]. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or *y*-intercept, of the line. We can see from the graph that the *y*-intercept in the train example we just saw is $(0,250)$ and represents the distance of the train from the station when it began moving at a constant speed.



Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x)=2x+1.$ Ask yourself what numbers can be input to the function. In other words, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line

$f(x)=mx+b$

where b is the initial or starting value of the function (when input, $x=0$), and m is the constant rate of change, or slope of the function. The *y*-intercept is at $(0,b)$.

Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in [link] depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d)=0.434d+14.696$. Restate this function in words.

(credit: Ilse Reijs and Jan-Noud Hutten)

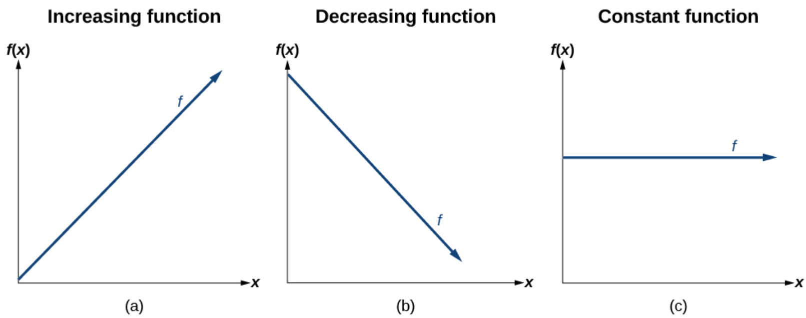
To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in [link]**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in [link]**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in [link]**(c)**.



Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x)=mx+b$ is an increasing function if $m>0$.
- $f(x)=mx+b$ is a decreasing function if $m<0$.
- $f(x)=mx+b$ is a constant function if $m=0$.

Deciding Whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day[footnote]. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the

function is increasing, decreasing, or constant.
http://
www.cbsnews.com/8301-501465_162-57400228-50146
teens-are-sending-60-texts-a-day-study-says/

1. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
2. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
3. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Analyze each function.

1. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
2. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes

sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
3. The cost function can be represented as $f(x)=50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Interpreting Slope as a Rate of Change

In the examples we have seen so far, the slope was provided to us. However, we often need to calculate the slope given input and output values. Recall that given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2 —which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m.

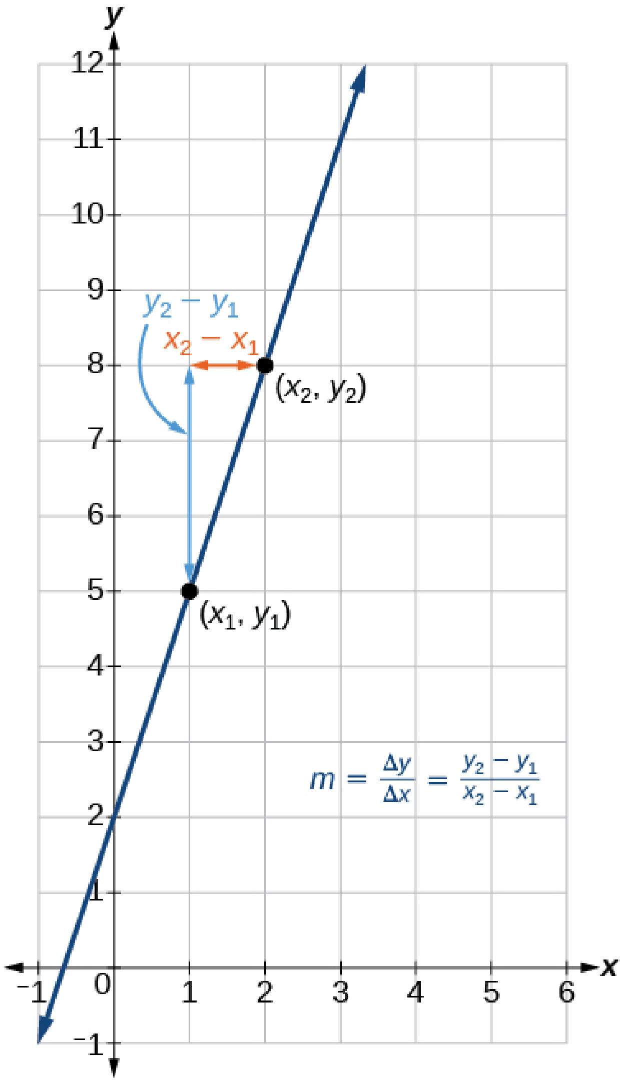
$m=$ change in output (rise) change in input (run) $=$

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that in function notation we can obtain two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

[link] indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness, or slant. The greater the absolute value of the slope, the steeper the slant is.



Are the units for slope always

units for the output units for the input ?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of

slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$$

We could also write the slope as $m=0.6$. The function is increasing because $m>0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used. Note that if we had reversed them, we would have obtained the same slope.

$$m = \frac{(-2)-(1)}{3-8} = \frac{-3}{-5} = \frac{3}{5}$$

If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

$m=\frac{4-3}{0-2}=\frac{1}{-2}=-\frac{1}{2}$; decreasing because $m<0$.

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27{,}800-23{,}400=4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4{,}400\text{ people}}{4\text{ years}}=1{,}100\ \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore

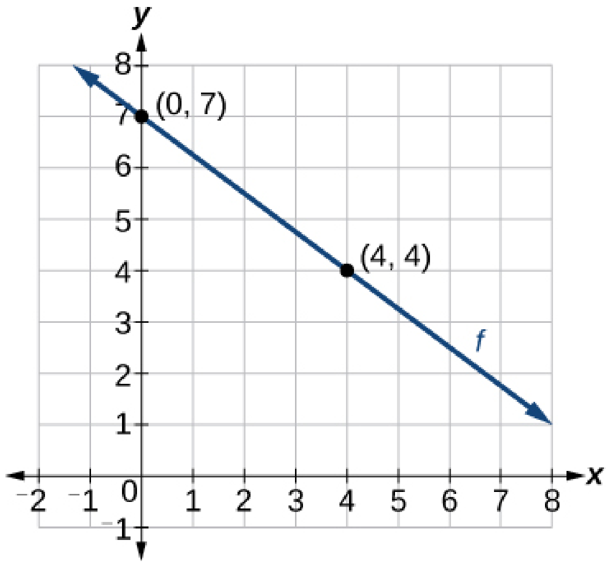
reasonable.

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

$m = \frac{1,868 - 1,442}{2,012 - 2,009} = \frac{426}{3} = 142$ people per year

Writing and Interpreting an Equation for a Linear Function

Recall from Equations and Inequalities that we wrote equations in both the slope-intercept form and the point-slope form. Now we can choose which method to use to write equations for linear functions based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in [link].



We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0,7)$ and $(4,4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-7}{4-0} = -\frac{3}{4}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

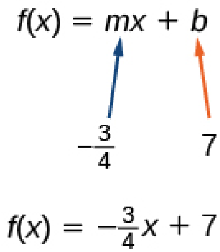
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we want to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the *y*-axis when the output value is 7. Therefore, $b = 7$. We now have

the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x)=-\frac{3}{4}x+7$, and the linear equation would be $y=-\frac{3}{4}x+7$.

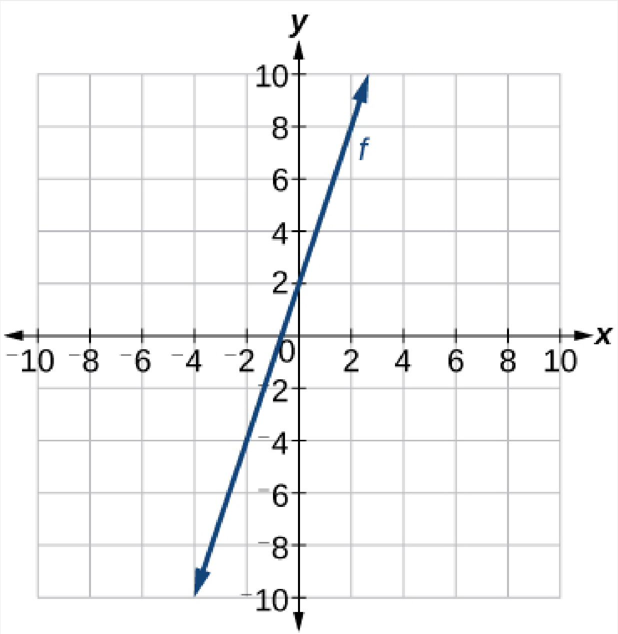
Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the *y*-axis to identify the *y*-intercept by visual inspection.
4. Substitute the slope and *y*-intercept into the slope-intercept form of a line equation.

Writing an Equation for a Linear Function

Write an equation for a linear function given a

graph of f shown in [link].



Identify two points on the line, such as $(0,2)$ and $(-2,-4)$. Use the points to calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-2}{-2-0} = \frac{-6}{-2} = 3$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

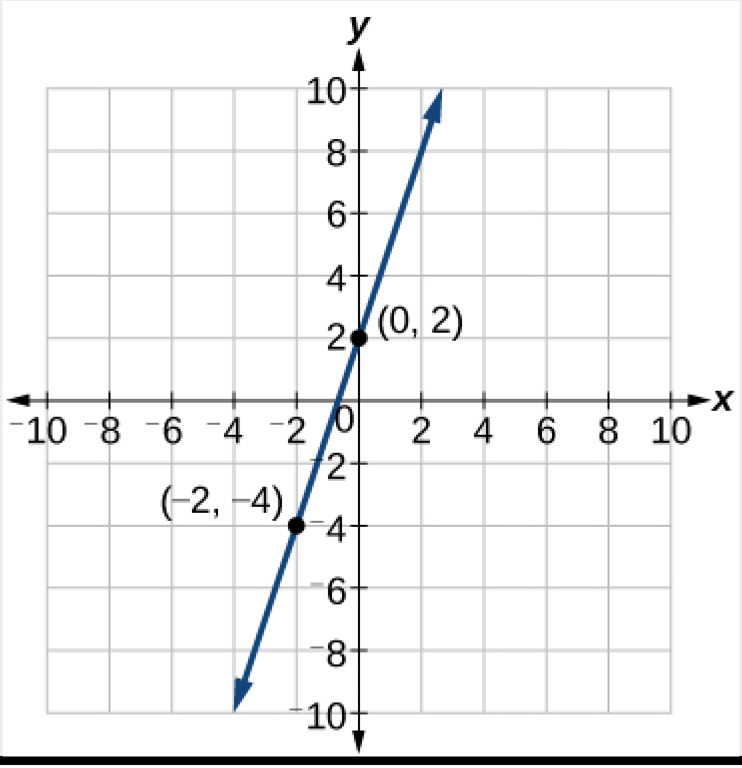
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y-(-4) &= 3(x-(-2)) \\ y+4 &= 3(x+2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$y+4 = 3(x+2) \quad y+4 = 3x+6 \quad y = 3x+2$$

Analysis

This makes sense because we can see from [link] that the line crosses the *y*-axis at the point $(0, 2)$, which is the *y*-intercept, so $b=2$.



Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of \$1,250 per month for the overhead, which includes his office rent. His production costs are \$37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

The fixed cost is present every month, \$1,250. The costs that can vary include the cost to produce each item, which is \$37.50. The variable cost, called the marginal cost, is represented by $37.5.$ The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x.$

Analysis

If Ben produces 100 items in a month, his monthly cost is found by substituting 100 for $x.$

$$C(100) = 1250 + 37.5(100) = 5000$$

So his monthly cost would be \$5,000.

Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2,$ and $f(8) = 1,$ find an equation for the function in

slope-intercept form.

We can write the given points using coordinates.

$f(3) = -2 \rightarrow (3,-2)$ $f(8) = 1 \rightarrow (8,1)$

We can then use the points to calculate the slope.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{8-3} = \frac{3}{5}$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$y - y_1 = m(x - x_1)$ $y-(-2) = \frac{3}{5}(x-3)$

We can use algebra to rewrite the equation in the slope-intercept form.

$y+2 = \frac{3}{5}(x-3)$ $y+2 = \frac{3}{5}x - \frac{9}{5}$ $y = \frac{3}{5}x - \frac{19}{5}$

If $f(x)$ is a linear function, with $f(2)=-11$, and $f(4) = -25$, write an equation for the function in slope-intercept form.

$y = -7x+3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

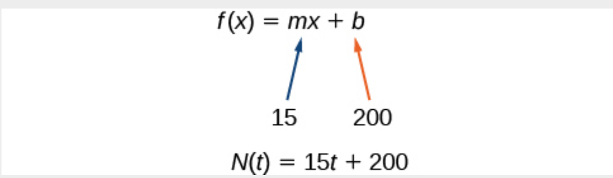
Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music

collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own at the end of one year?

The initial value for this function is 200 because he currently owns 200 songs, so $N(0)=200$, which means that $b=200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m=15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.



We can write the formula $N(t)=15t+200$.

With this formula, we can then predict how many songs Marcus will have at the end of one year (12 months). In other words, we can evaluate the function at $t=12$.

$$N(12) = 15(12)+200 = 180+200 = 380$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs: $(3,760)$ and $(5,920)$. We start by finding the rate of change.

$$m = \frac{920-760}{5-3} = \frac{\$160}{2 \text{ policies}} = \$80 \text{ per policy}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when

the number of policies increased by 2, so the rate of change is $80 per policy. Therefore, Ilya earns a commission of $80 for each policy sold during the week.

We can then solve for the initial value.
$I(n) = 80n+b$ $760 = 80(3)+b$ when $n=3, I(3)=760$ $760 - 80(3)=b$ $520 = b$

The value of b is the starting value for the function and represents Ilya's income when $n=0,$ or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.
$I(n)=80n+520$

Our final interpretation is that Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold.

Using Tabular Form to Write an Equation for a Linear Function

[link] relates the number of rats in a

population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

We can see from the table that the initial value for the number of rats is 1000, so $b=1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$P(w)=40w+1000$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2,1080)$ and $(6,1240)$

$m = 1240 - 1080\ 6 - 2 = 160\ 4 = 40$

Is the initial value always provided in a table of values like [link] ?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, *and solve for* b.

A new plant food was introduced to a young tree to test its effect on the height of the tree. [link] shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0		2		4		8		12	

***H*(*x*)**	12.5	13.5	14.5	16.5	18.5

$H(x) = 0.5x + 12.5$

Vertical stretches and compressions and reflections on the function $f(x) = x$This graph illustrates vertical shifts of the function $f(x) = x$.

Graphing Linear Functions

Now that we've seen and interpreted graphs of linear functions, let's take a look at how to create the graphs. There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third method is by using transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a

minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x)=2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1,2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2,4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Graphing by Plotting Points

Graph $f(x)= -\ 2\ 3\ x+5$ by plotting points.

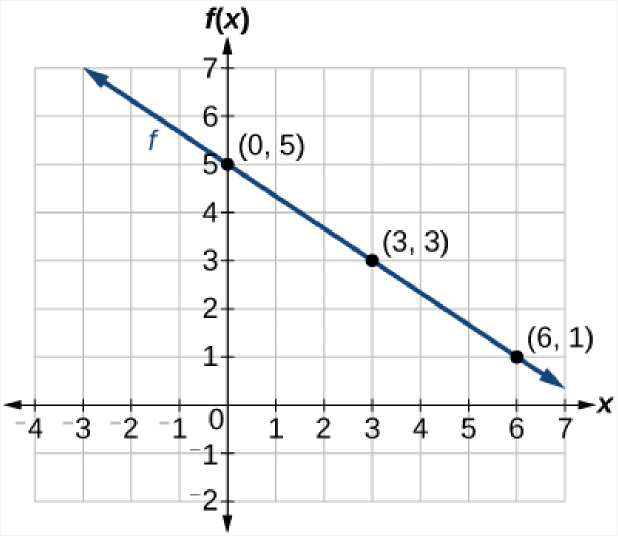
Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$x=0 \quad f(0)=-\frac{2}{3}(0)+5=5\Rightarrow(0,5)$$
$$x=3 \quad f(3)=-\frac{2}{3}(3)+5=3\Rightarrow(3,3)$$
$$x=6 \quad f(6)=-\frac{2}{3}(6)+5=1\Rightarrow(6,1)$$

Plot the coordinate pairs and draw a line through the points. [link] represents the graph of the function $f(x)=-\frac{2}{3}x+5$.

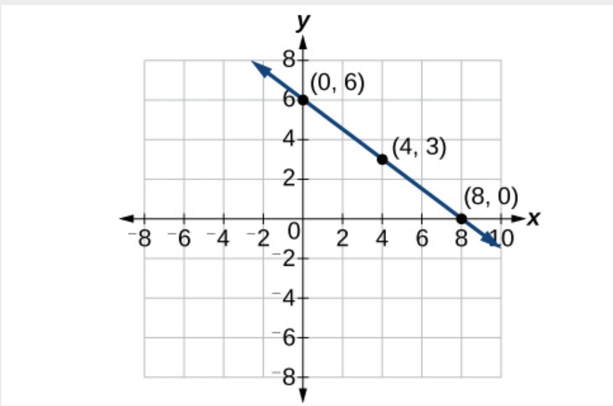
The graph of the linear function $f(x)=-\frac{2}{3}x+5$.



Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative, constant rate of change in the equation for the function.

Graph $f(x)=-\frac{3}{4}x+6$ by plotting points.



Graphing a Function Using *y*-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-

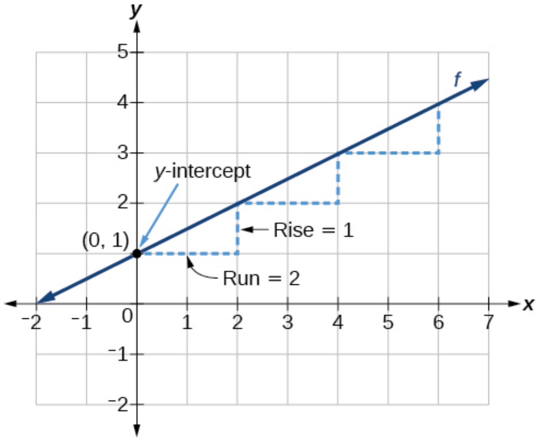
intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x=0$ in the equation.

The other characteristic of the linear function is its slope.

Let's consider the following function.

$f(x)=\frac{1}{2}x+1$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x=0$. The graph crosses the *y*-axis at $(0,1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0,1)$. We know that the slope is the change in the *y*-coordinate over the change in the *x*-coordinate. This is commonly referred to as rise over run, $m=\frac{\text{rise}}{\text{run}}$. From our example, we have $m=\frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0,1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in [link].



Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Do all linear functions have *y*-intercepts?

Yes. All linear functions cross the y-axis and therefore

have y-intercepts. (Note: *A vertical line is parallel to the y-axis does not have a y-intercept, but it is not a function.*)

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{rise}{run}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

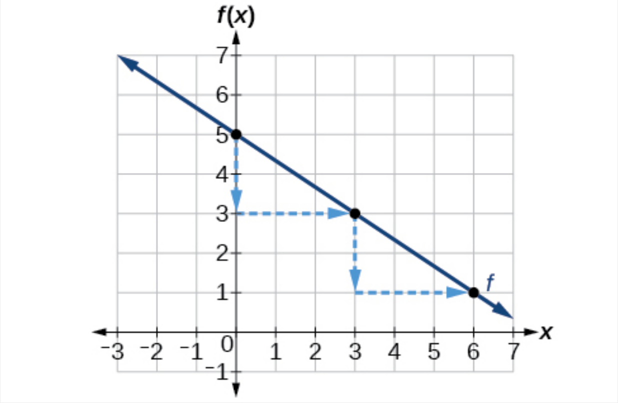
Graphing by Using the *y*-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

Evaluate the function at $x = 0$ to find the *y*-intercept. The output value when $x = 0$ is 5, so the graph will cross the *y*-axis at $(0,5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of –2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the *y*-intercept on the graph in [link]. From the initial value $(0,5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then drawing a line through the points.

Graph of $f(x) = -2/3x + 5$ and shows how to calculate the rise over run for the slope.



Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

Find a point on the graph we drew in [link] that has a negative x-value.

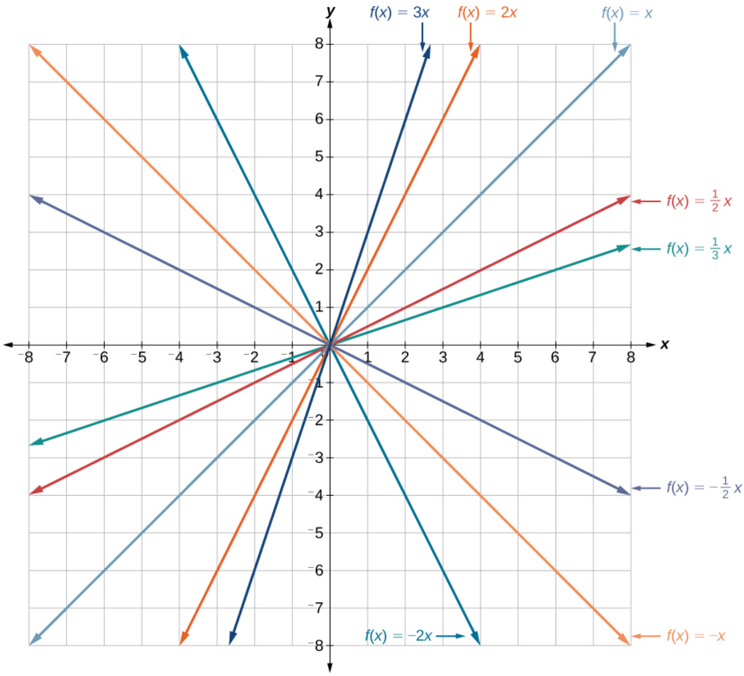
Possible answers include $(-3,7)$, $(-6,9)$, or $(-9,11)$.

Graphing a Function Using Transformations

Another option for graphing is to use a transformation of the identity function $f(x)=x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

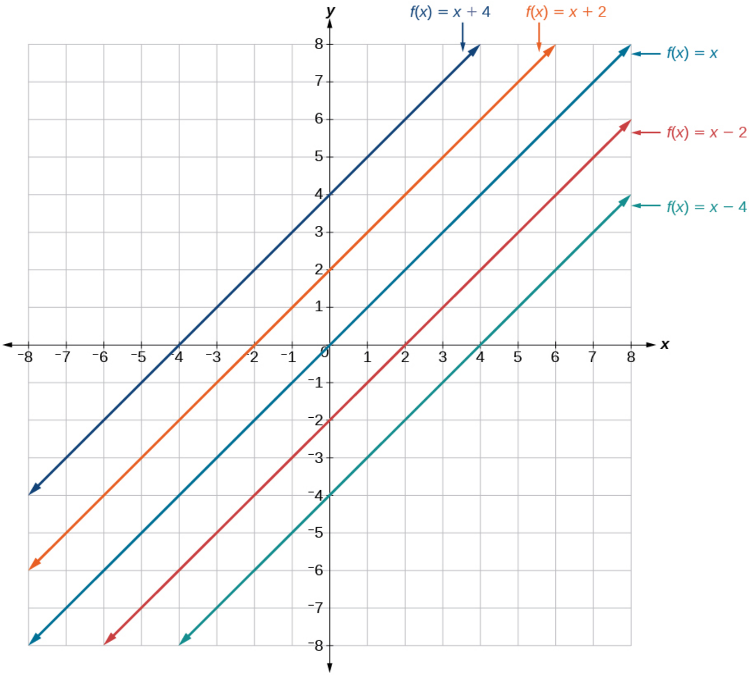
Vertical Stretch or Compression

In the equation $f(x)=mx$, the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in [link] that multiplying the equation of $f(x)=x$ by m stretches the graph of f by a factor of m units if $m>1$ and compresses the graph of f by a factor of m units if $0<m<1$. This means the larger the absolute value of m, the steeper the slope.



Vertical Shift

In $f(x)=mx+b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in [link] that adding a value of b to the equation of $f(x)=x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.



Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x)=mx+b$.

1. Graph $f(x)=x$.
2. Vertically stretch or compress the graph by a

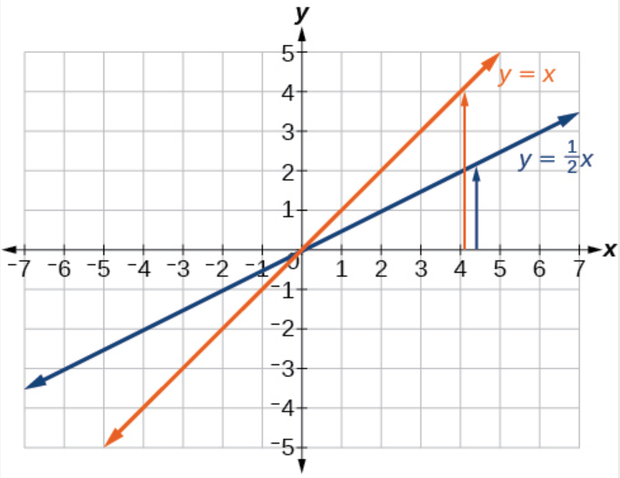
factor m.
3. Shift the graph up or down b units.

Graphing by Using Transformations

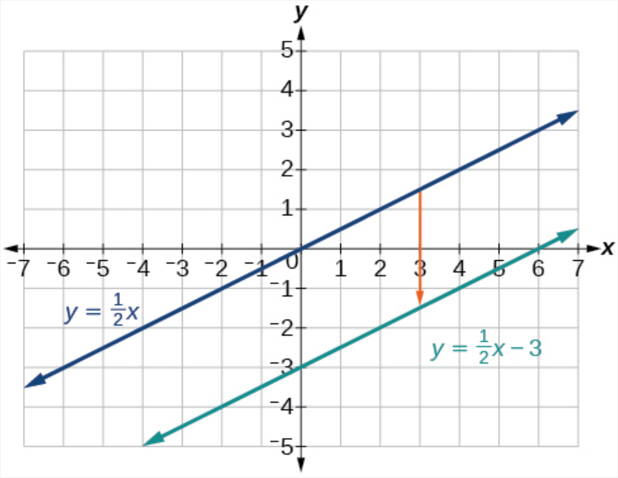
Graph $f(x)=\frac{1}{2}x-3$ using transformations.

The equation for the function shows that $m=\frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b=-3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in [link].

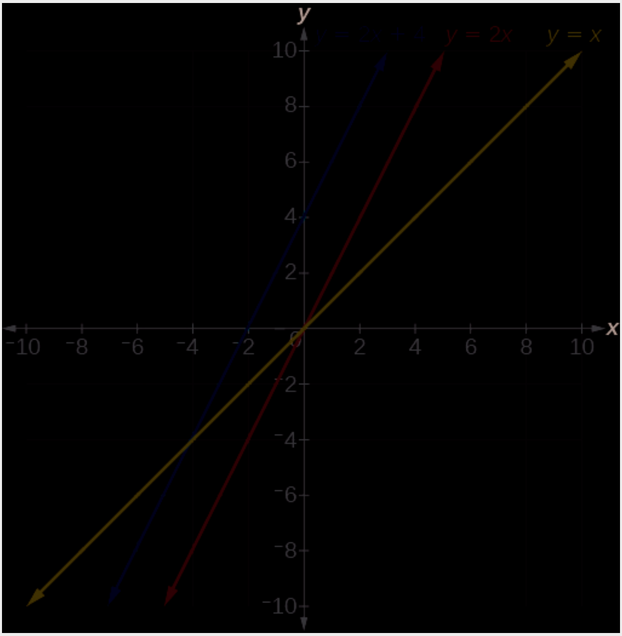
The function, $y=x$, compressed by a factor of $\frac{1}{2}$



Then show the vertical shift as in [link].
The function $y = \frac{1}{2}x$, shifted down 3 units



Graph $f(x) = 4 + 2x$ using transformations.



In [link] , could we have sketched the graph by reversing the order of the transformations?

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

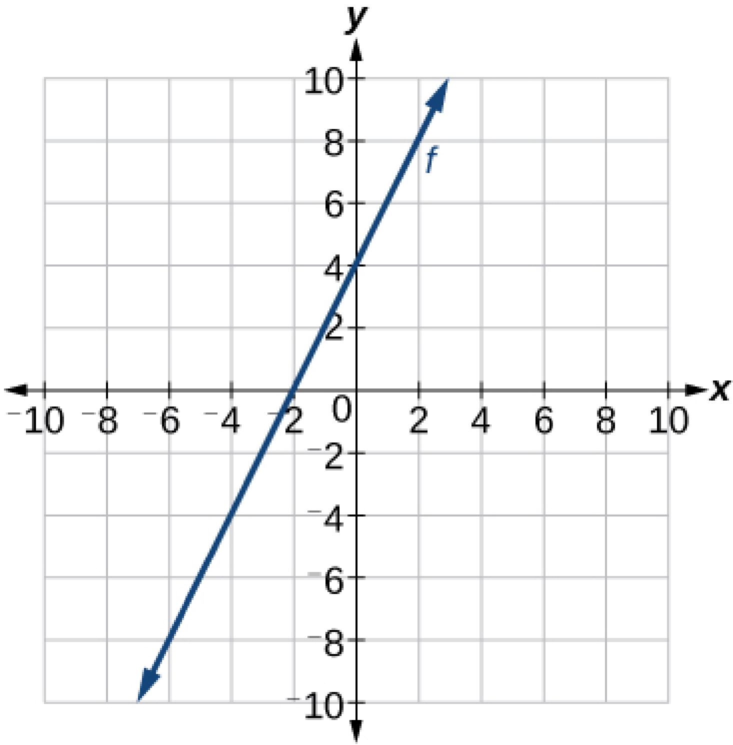
$f(2) = \frac{1}{2}(2) - 3 = 1 - 3 = -2$

A horizontal line representing the function $f(x) = 2$

Example of how a line has a vertical slope. 0 in the denominator of the slope. The vertical line, $x=2$, which does not represent a function

Writing the Equation for a Function from the Graph of a Line

Earlier, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at [link]. We can see right away that the graph crosses the *y*-axis at the point $(0,4)$ so this is the *y*-intercept.



Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point $(-2,0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m=\frac{\text{rise}}{\text{run}}=\frac{4}{2}=2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

$$y=2x+4$$

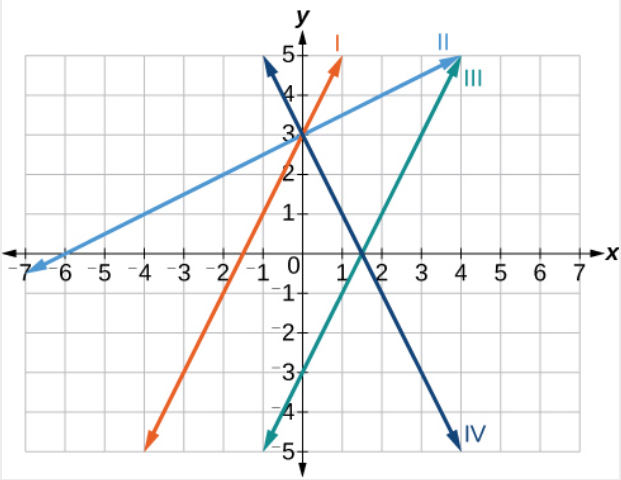
Given a graph of linear function, find the equation to describe the function.

1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in [link].

a. $f(x)=2x+3$ b. $g(x)=2x-3$ c. $h(x)=-2x+3$ d. $j(x)=\frac{1}{2}x+3$



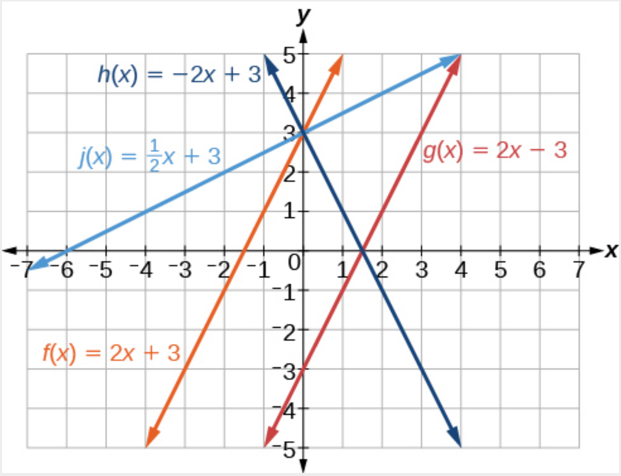
Analyze the information for each function.

1. This function has a slope of 2 and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different *y*-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0,3)$ so f must be represented by line I.
2. This function also has a slope of 2, but a *y*-intercept of $-3.$ It must pass through the point $(0,-3)$ and slant upward from left to right. It must be represented by line III.
3. This function has a slope of –2 and a *y*-

intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.

4. This function has a slope of $\frac{1}{2}$ and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0,3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in [link].



Finding the *x*-intercept of a Line

So far we have been finding the *y*-intercepts of a

function: the point at which the graph of the function crosses the *y*-axis. Recall that a function may also have an *x*-intercept, which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$f(x)=3x-6$

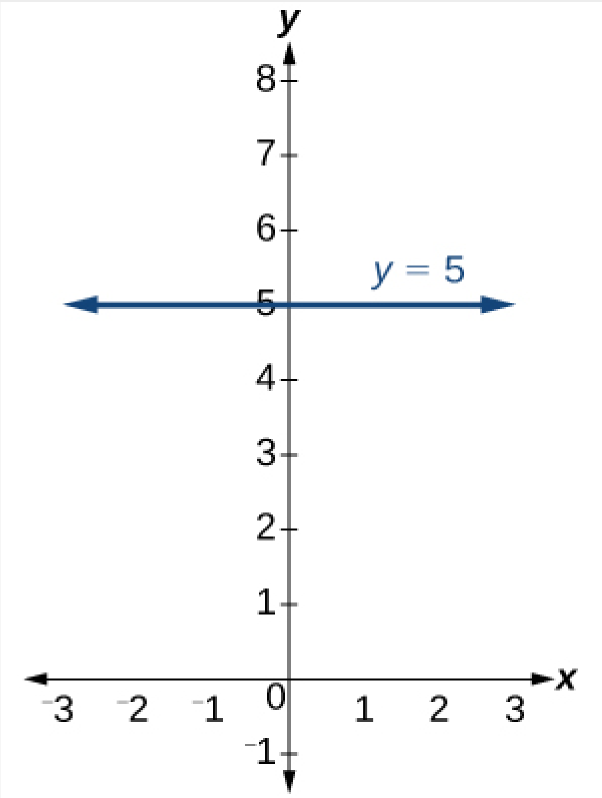
Set the function equal to 0 and solve for x.

$0 = 3x-6 \; 6 = 3x \; 2 = x \; x = 2$

The graph of the function crosses the *x*-axis at the point $(2,0)$.

Do all linear functions have *x*-intercepts?

No. However, linear functions of the form $y=c$, where c is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y=5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in **[link]**.



x-intercept
The *x*-intercept of the function is value of x when $f(x)=0$. It can be solved by the equation $0=mx+b$.

Finding an *x*-intercept

Find the *x*-intercept of $f(x)=\frac{1}{2}x-3$.

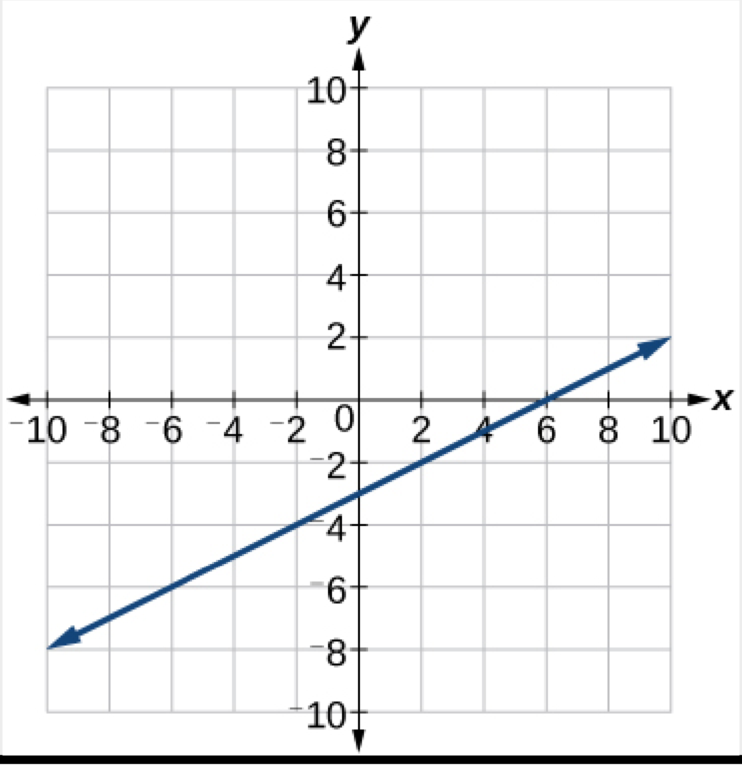
Set the function equal to zero to solve for x.

$$0 = \frac{1}{2}x - 3 \qquad 3 = \frac{1}{2}x \qquad 6 = x \qquad x = 6$$

The graph crosses the *x*-axis at the point $(6,0)$.

Analysis

A graph of the function is shown in [link]. We can see that the *x*-intercept is $(6,0)$ as we expected.

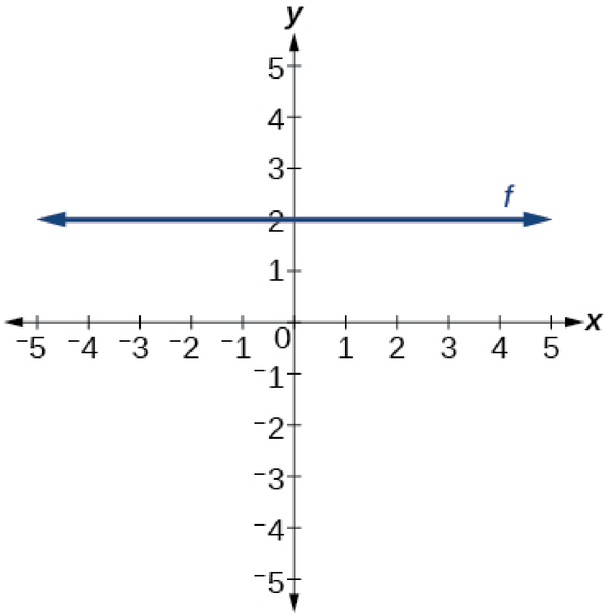


Find the *x*-intercept of $f(x)=\frac{1}{4}x-4.$

$(16, 0)$

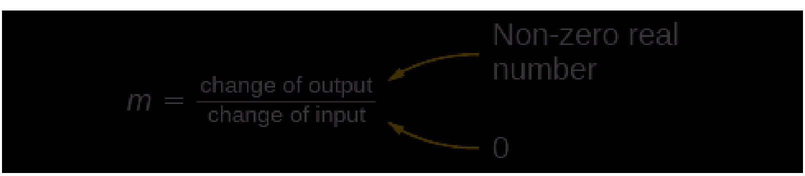
Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A horizontal line indicates a constant output, or *y*-value. In [link], we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m=0$ in the equation $f(x)=mx+b$, the equation simplifies to $f(x)=b.$ In other words, the value of the function is a constant. This graph represents the function $f(x)=2.$

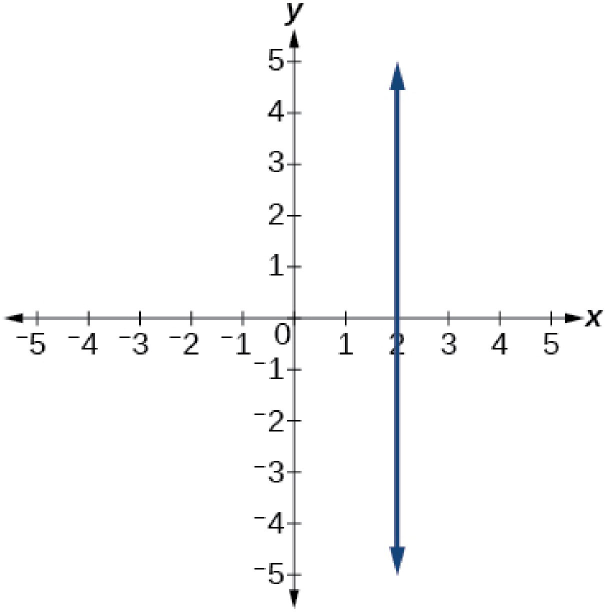


x	-4	-2	0	2	4
y	2	2	2	2	2

A vertical line indicates a constant input, or *x*-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.



A vertical line, such as the one in [link], has an *x*-intercept, but no *y*-intercept unless it's the line $x=0$. This graph represents the line $x=2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

Horizontal and Vertical Lines

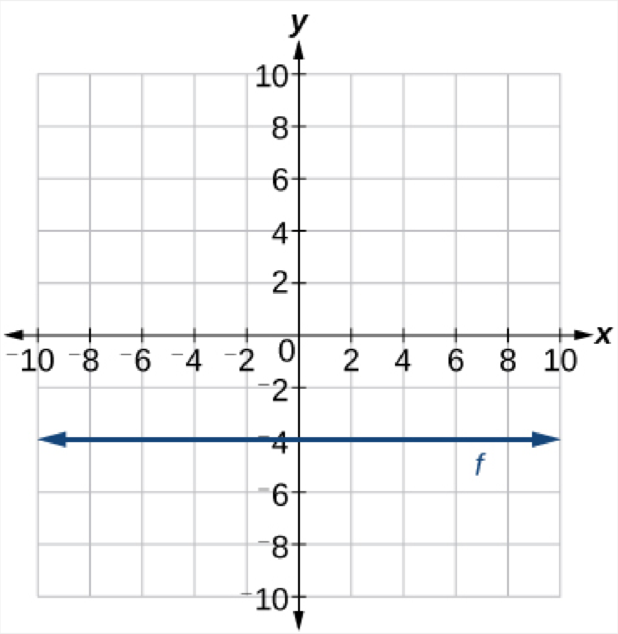
Lines can be horizontal or vertical.

A **horizontal line** is a line defined by an equation in the form $f(x)=b$.

A **vertical line** is a line defined by an equation in the form $x=a$.

Writing the Equation of a Horizontal Line

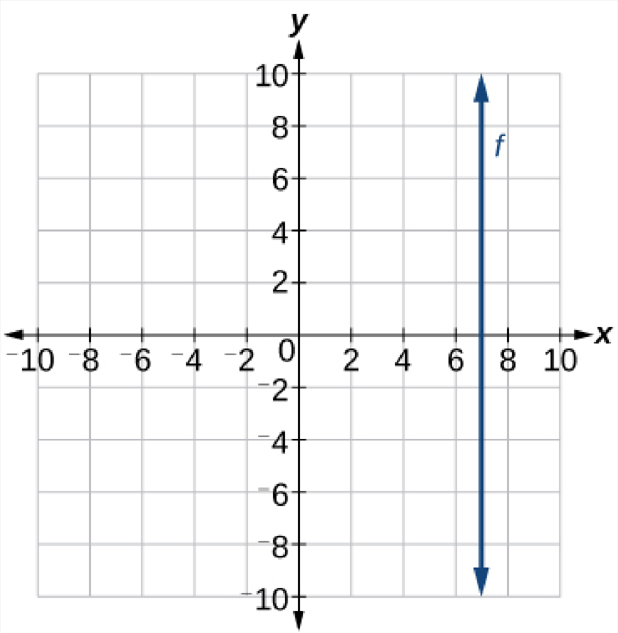
Write the equation of the line graphed in [link].



For any x-value, the y-value is -4, so the equation is $y=-4$.

Writing the Equation of a Vertical Line

Write the equation of the line graphed in [link].



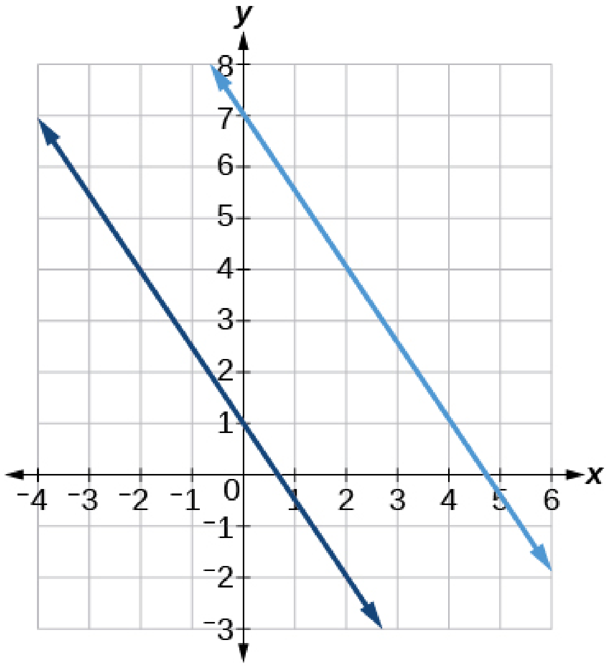
The constant *x*-value is 7, so the equation is $x=7$.

Parallel lines Perpendicular lines

Determining Whether Lines are Parallel or Perpendicular

The two lines in [link] are parallel lines: they will never intersect. They have exactly the same steepness, which means their slopes are identical.

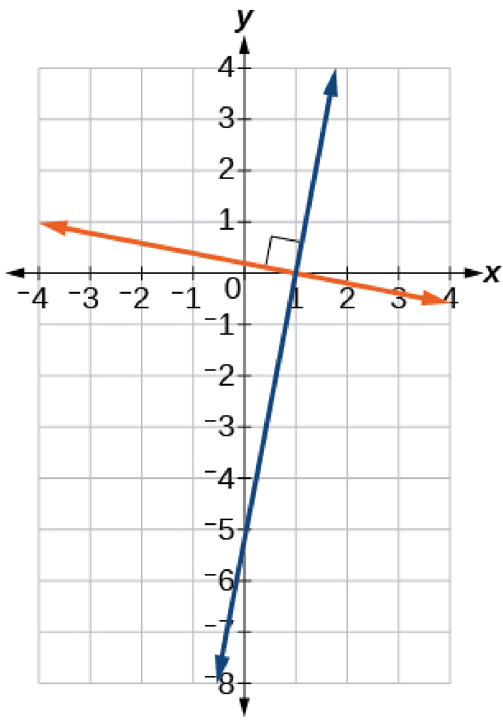
The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the other, they would become coincident.



We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the y-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{array}{l} f(x)=-2x+6 \\ f(x)=-2x-4 \end{array}\right\} \text{parallel} \qquad \left.\begin{array}{l} f(x)=3x+2 \\ f(x)=2x+2 \end{array}\right\} \text{not parallel}$$

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in [link] are perpendicular.



Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield -1.

$$m_1 m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$f(x) = \frac{1}{4}x + 2$ negative reciprocal of $\frac{1}{4}$ is -4
$f(x) = -4x + 3$ negative reciprocal of -4 is $\frac{1}{4}$

The product of the slopes is –1.

$-4\left(\frac{1}{4}\right) = -1$

Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$
are parallel if and only if $m_1 = m_2$

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect to form a right angle.

$f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$
are perpendicular if and only if
$m_1 m_2 = -1$, so $m_2 = -\frac{1}{m_1}$

Identifying Parallel and Perpendicular

Lines

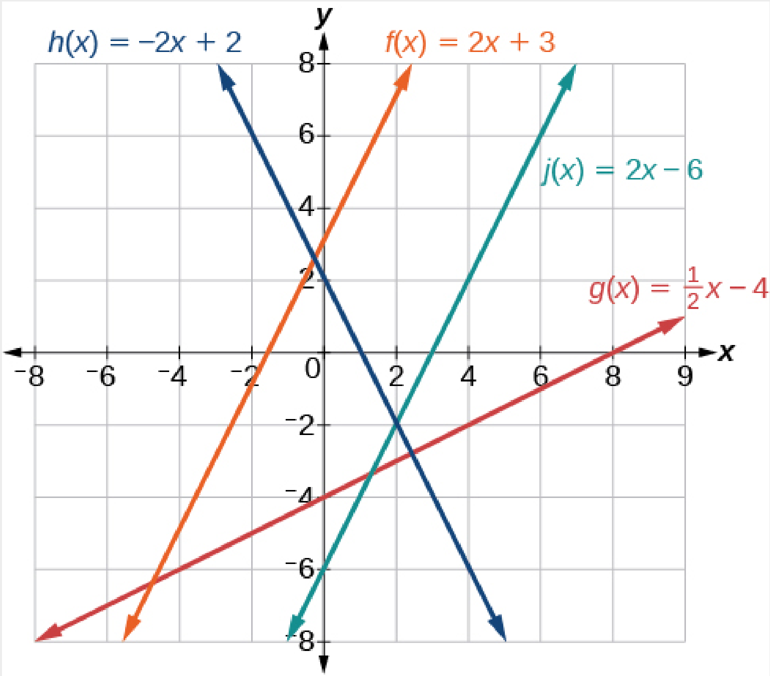
Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$f(x) = 2x+3$ $h(x) = -2x+2$ $g(x) = \frac{1}{2}x-4$ $j(x) = 2x-6$

Parallel lines have the same slope. Because the functions $f(x)=2x+3$ and $j(x)=2x-6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because -2 and $\frac{1}{2}$ are negative reciprocals, the functions $g(x)=\frac{1}{2}x-4$ and $h(x)=-2x+2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in [link].



The graph shows that the lines $f(x)=2x+3$ and $j(x)=2x–6$ are parallel, and the lines $g(x)=\frac{1}{2}x–4$ and $h(x)=-2x+2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the equation shown.
$f(x)=3x+1$

We know that the slope of the line formed by the function is 3. We also know that the *y*-intercept is (0,1). Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.
$g(x) = 3x+6$ $h(x) = 3x+1$ $p(x) = 3x+ 2\ 3$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1,7)$. This type of problem is often described as a point-slope problem because we have a point and a slope. In our example, we know that the slope is 3. We need to determine which value of b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.
$y- y_1 = m(x- x_1)$ $y-7 = 3(x-1)$ $y-7 = 3x-3$ $y = 3x+4$

So $g(x)=3x+4$ is parallel to $f(x)=3x+1$ and passes through the point $(1,7)$.

Given the equation of a function and a point

through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x)=3x+6$ that passes through the point $(3,0)$.

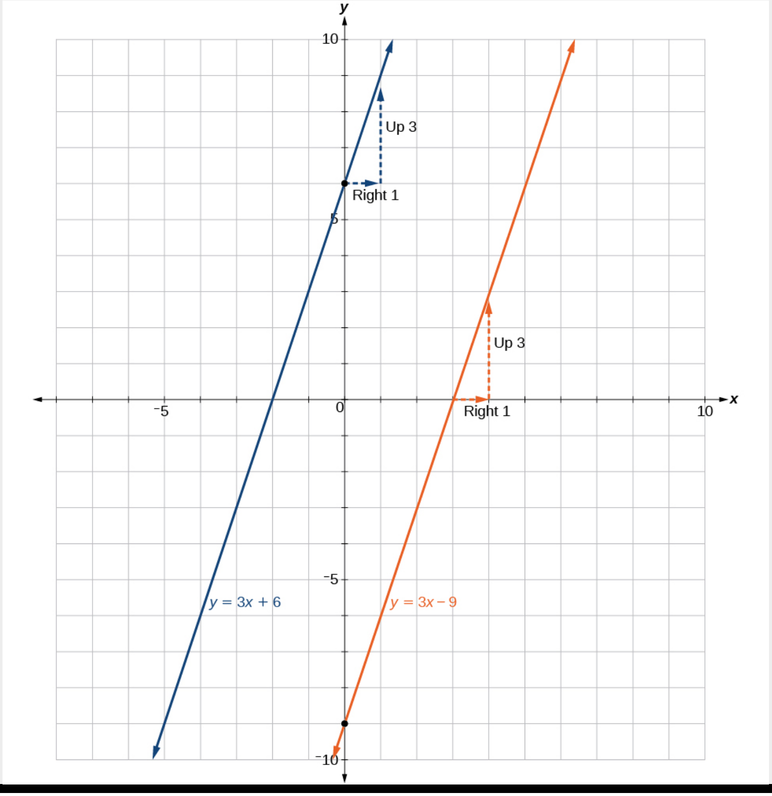
The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m=3, x=3$, and $f(x)=0$ into the slope-intercept form to find the *y*-intercept.

$$g(x) = 3x+b \quad 0 = 3(3)+b \quad b = -9$$

The line parallel to $f(x)$ that passes through $(3,0)$ is $g(x)=3x-9$.

Analysis

We can confirm that the two lines are parallel by graphing them. [link] shows that the two lines will never intersect.



Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the function shown.

$f(x) = 2x + 4$

The slope of the line is 2, and its negative reciprocal

is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$g(x) = -\frac{1}{2}x+4 \quad h(x) = -\frac{1}{2}x+2 \quad p(x) = -\frac{1}{2}x-\frac{1}{2}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4,0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the *y*-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

$$g(x) = mx+b \quad 0 = -\frac{1}{2}(4)+b \quad 0 = -2+b \quad 2 = b \quad b = 2$$

The equation for the function with a slope of $-\frac{1}{2}$ and a *y*-intercept of 2 is

$$g(x)=-\frac{1}{2}x+2$$

So $g(x)=-\frac{1}{2}x+2$ is perpendicular to $f(x)=2x+4$ and passes through the point $(4,0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x)=mx+b$.
4. Solve for b.
5. Write the equation of the line.

Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to

$f(x)=3x+3$ that passes through the point $(3,0)$.

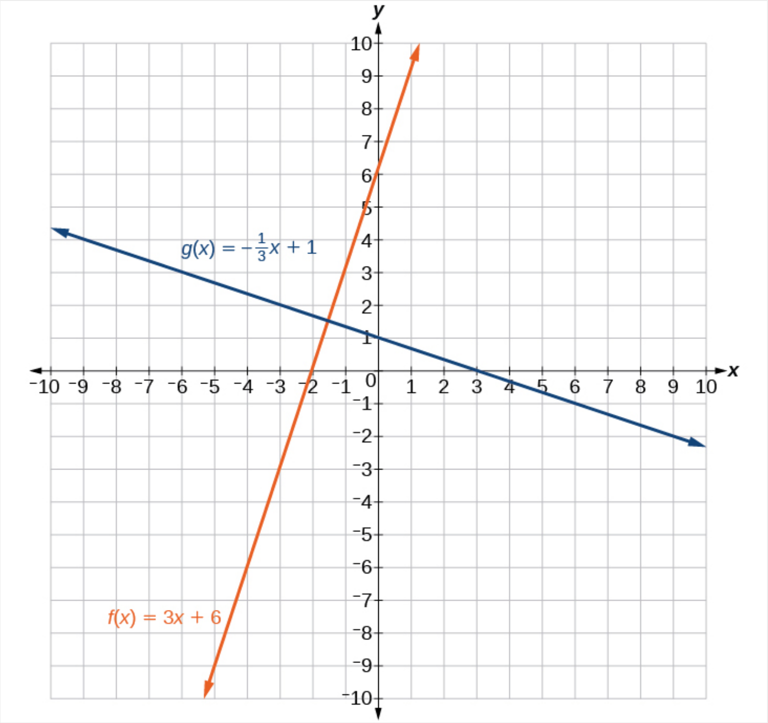
The original line has slope $m=3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation of the line.

$$\begin{aligned} g(x) &= -\frac{1}{3}x+b \\ 0 &= -\frac{1}{3}(3)+b \\ 1 &= b \\ b &= 1 \end{aligned}$$

The line perpendicular to $f(x)$ that passes through $(3,0)$ is $g(x)=-\frac{1}{3}x+1$.

Analysis

A graph of the two lines is shown in [link].



Note that that if we graph perpendicular lines on a graphing calculator using standard zoom, the lines may not appear to be perpendicular. Adjusting the window will make it possible to zoom in further to see the intersection more closely.

Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0,0)$ that is

1. parallel to $h(x)$

2. perpendicular to $h(x)$

a. $f(x)=2x$; b. $g(x)=-\ 1\ 2\ x$

Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2,6)$ and $(4,5)$. Find the equation of a perpendicular line that passes through the point $(4,5)$.

From the two points of the given line, we can calculate the slope of that line.
$m_1 = \frac{5-6}{4-(-2)} = \frac{-1}{6} = -\frac{1}{6}$

Find the negative reciprocal of the slope.
$m_2 = \frac{-1}{-\frac{1}{6}} = -1\left(-\frac{6}{1}\right) = 6$

We can then solve for the *y*-intercept of the line passing through the point $(4,5)$.
$g(x) = 6x+b$
$5 = 6(4)+b$
$5 = 24+b$
$-19 = b$
$b = -19$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4,5)$ is $y=6x-19$

A line passes through the points, $(-2,-15)$ and $(2,-3)$. Find the equation of a perpendicular line that passes through the point, $(6,4)$.

$y=-\frac{1}{3}x+6$

Access this online resource for additional instruction and practice with linear functions.

- Linear Functions
- Finding Input of Function from the Output and Graph
- Graphing Functions using Tables

Key Concepts

- Linear functions can be represented in words, function notation, tabular form, and graphical form. See [link].
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope. A decreasing linear function results in a graph that slants downward from left to right and has a negative slope. A constant linear function results in a graph that is a horizontal line. See [link].
- Slope is a rate of change. The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line. See [link] and [link].
- An equation for a linear function can be written from a graph. See [link].

- The equation for a linear function can be written if the slope m and initial value b are known. See [link] and [link].
- A linear function can be used to solve real-world problems given information in different forms. See [link], [link], and [link].
- Linear functions can be graphed by plotting points or by using the *y*-intercept and slope. See [link] and [link].
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See [link].
- The equation for a linear function can be written by interpreting the graph. See [link].
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See [link].
- Horizontal lines are written in the form, $f(x)=b.$ See [link].
- Vertical lines are written in the form, $x=b.$ See [link].
- Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See [link].
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x)=mx+b,$ and using the b that results. Similarly, the point-slope form of an equation can also be

used. See [link].
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See [link] and [link].

Section Exercises

Verbal

Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t)=3000-70t$. Write a complete sentence describing Terry’s starting elevation and how it is changing over time.

Terry starts at an elevation of 3000 feet and descends 70 feet per second.

Jessica is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

A boat is 100 miles away from the marina,

sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after *t* hours.

$d(t) = 100 - 10t$

If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the *y*-intercepts.

If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

The point of intersection is (a, a). This is because for the horizontal line, all of the y coordinates are a and for the vertical line, all of the x coordinates are a. The point of intersection is on both lines and therefore will have these two characteristics.

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

$y = 1\ 4\ x + 6$

$y = 3x - 5$

Yes

$y = 3\ x\ 2\ - 2$

$3x + 5y = 15$

Yes

$3\ x\ 2\ + 5y = 15$

$3x + 5\ y\ 2\ = 15$

No

$-2\ x\ 2\ + 3\ y\ 2\ = 6$

$-\ x - 3\ 5\ = 2y$

Yes

For the following exercises, determine whether each function is increasing or decreasing.

$f(x)=4x+3$

$g(x)=5x+6$

Increasing

$a(x)=5-2x$

$b(x)=8-3x$

Decreasing

$h(x)=-2x+4$

$k(x)=-4x+1$

Decreasing

$j(x)= 1\ 2\ x-3$

$p(x)= 1\ 4\ x-5$

Increasing

$n(x)=-\ 1\ 3\ x-2$

$m(x)=-\ 3\ 8\ x+3$

Decreasing

For the following exercises, find the slope of the line that passes through the two given points.

$(2,4)$ and $(4,10)$

$(1,5)$ and $(4,11)$

2

$(–1,4)$ and $(5,2)$

$(8,–2)$ and $(4,6)$

–2

$(6,11)$ and $(–4,3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

$f(-5)=-4,$ and $f(5)=2$

$y=\ 3\ 5\ x-1$

$f(-1)=4,$ and $f(5)=1$

Passes through $(2,4)$ and $(4,10)$

$y=3x-2$

Passes through $(1,5)$ and $(4,11)$

Passes through $(-1,4)$ and $(5,2)$

$y = - \ 1 \ 3 \ x + \ 11 \ 3$

Passes through $(-2,8)$ and $(4,6)$

x intercept at $(-2,0)$ and y intercept at $(0,-3)$

$y = -1.5x - 3$

x intercept at $(-5,0)$ and y intercept at $(0,4)$

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither.

$4x - 7y = 10 \ 7x + 4y = 1$

---

perpendicular

$3y + x = 12 \ \ -y = 8x + 1$

$3y + 4x = 12 \ \ -6y = 8x + 1$

---

parallel

$6x-9y=10$ $3x+2y=1$

For the following exercises, find the *x*- and *y*-intercepts of each equation.

$f(x)=-x+2$

$f(0)=-(0)+2$ $f(0)=2$ $y-\text{int:}(0,2)$ $0=-x+2$ $x-\text{int:}(2,0)$

$g(x)=2x+4$

$h(x)=3x-5$

$h(0)=3(0)-5$ $h(0)=-5$ $y-\text{int:}(0,-5)$ $0=3x-5$ $x-\text{int:}\left(\frac{5}{3},0\right)$

$k(x)=-5x+1$

$-2x+5y=20$

$-2x+5y=20$ $-2(0)+5y=20$ $5y=20$ $y=4$ $y-\text{int:}(0,4)$ $-2x+5(0)=20$ $x=-10$ $x-\text{int:}(-10,0)$

$7x + 2y = 56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(0,6)$ and $(3,-24)$

Line 2: Passes through $(-1,19)$ and $(8,-71)$

Line 1: $m = -10$ Line 2: $m = -10$ Parallel

Line 1: Passes through $(-8,-55)$ and $(10,89)$

Line 2: Passes through $(9,-44)$ and $(4,-14)$

Line 1: Passes through $(2,3)$ and $(4,-1)$

Line 2: Passes through $(6,3)$ and $(8,5)$

Line 1: $m = -2$ Line 2: $m = 1$ Neither

Line 1: Passes through $(1,7)$ and $(5,5)$

Line 2: Passes through $(-1,-3)$ and $(1,1)$

Line 1: Passes through $(2,5)$ and $(5,-1)$

Line 2: Passes through $(-3,7)$ and $(3,-5)$

Line 1: $m=–2$ Line 2: $m=–2$ Parallel

For the following exercises, write an equation for the line described.

Write an equation for a line parallel to $f(x)=-5x-3$ and passing through the point $(2,–12)$.

Write an equation for a line parallel to $g(x)=3x-1$ and passing through the point $(4,9)$.

$y=3x-3$

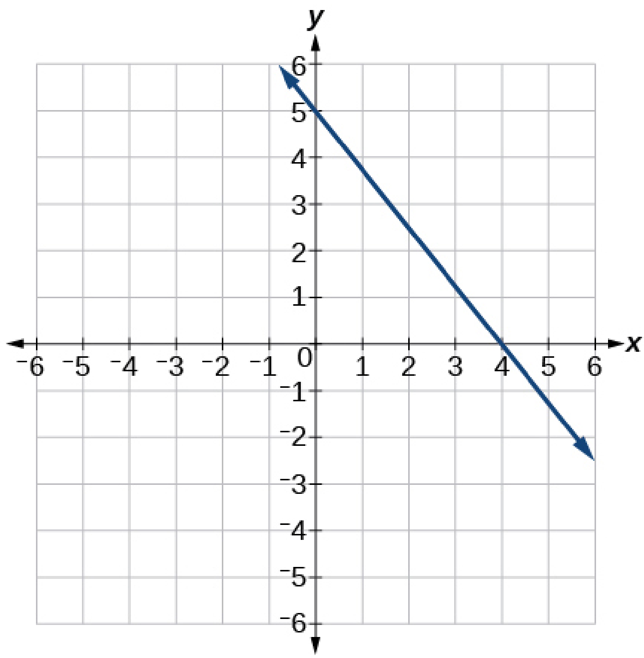
Write an equation for a line perpendicular to $h(t)=-2t+4$ and passing through the point $(-4,–1)$.

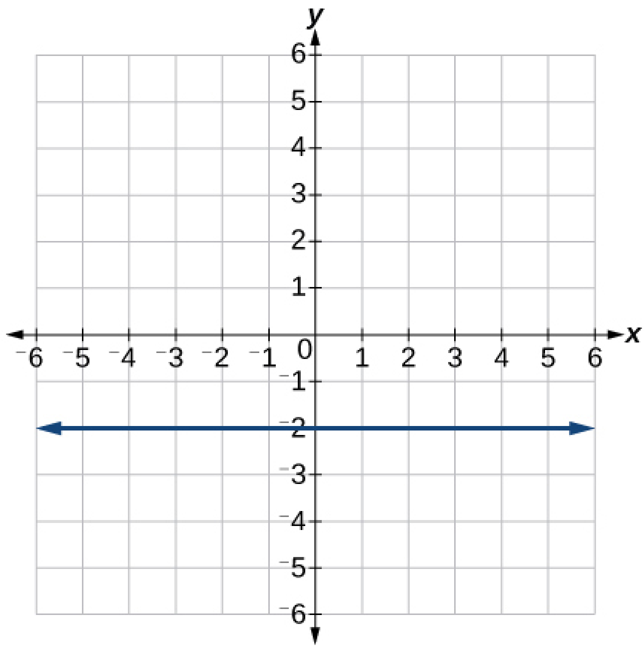
Write an equation for a line perpendicular to $p(t)=3t+4$ and passing through the point $(3,1)$.

$y = -\ 1\ 3\ t+2$

Graphical

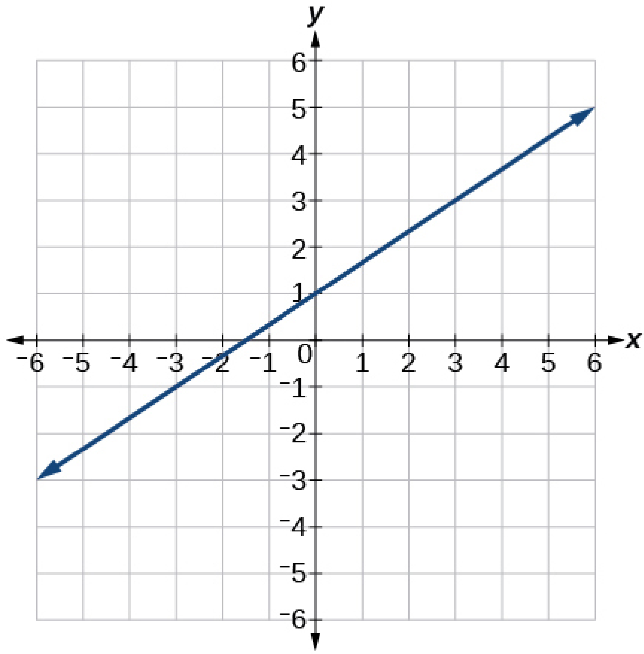
For the following exercises, find the slope of the line graphed.



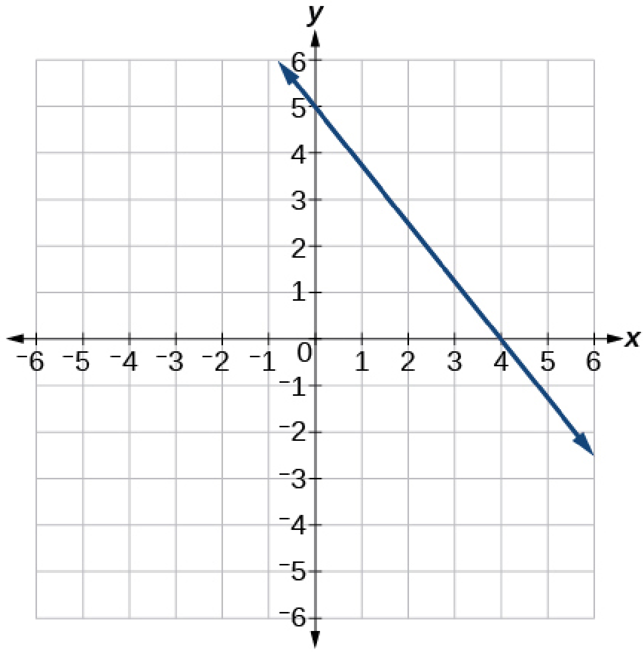


0

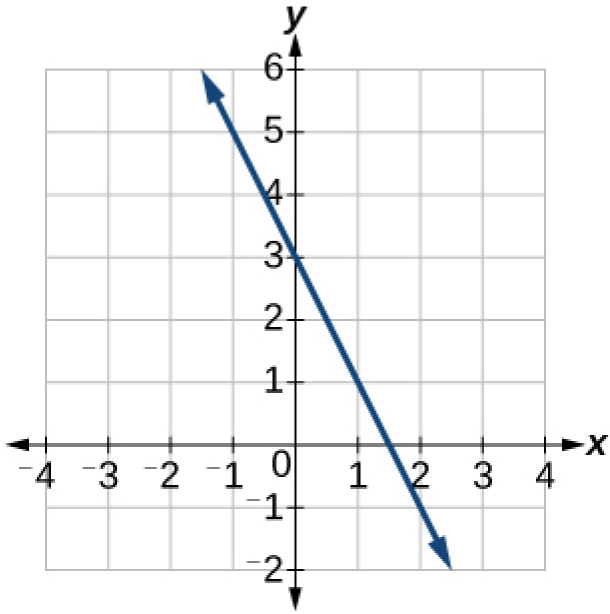
For the following exercises, write an equation for the line graphed.



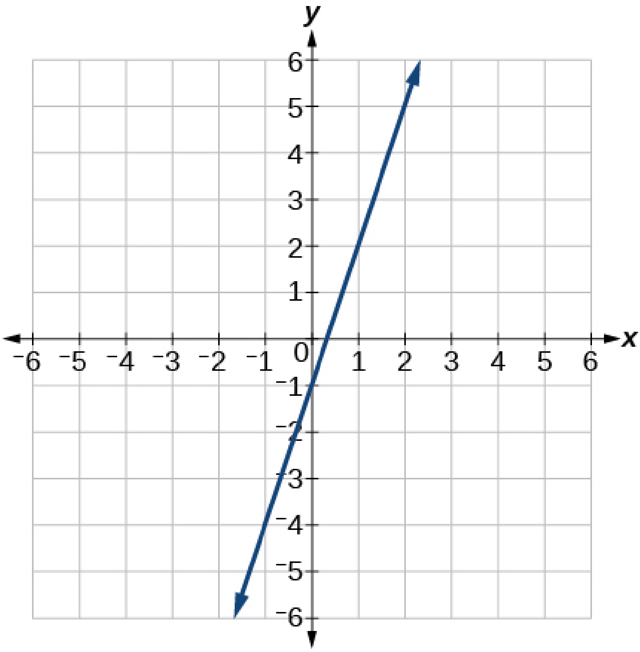
y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6



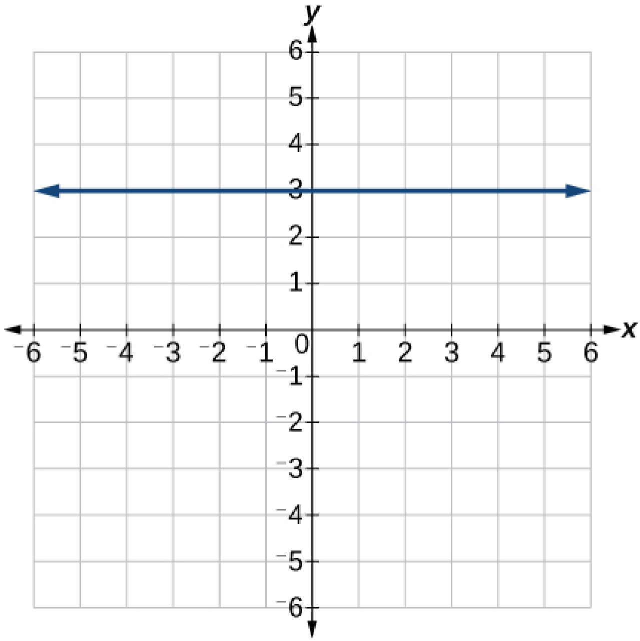
$$y=-\frac{5}{4}x+5$$



y
x
6
5
4
3
2
1
0
−1
−2
−4
−3
−2
−1
1
2
3
4



$y = 3x - 1$



y
x
6
5
4
3
2
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0
−1
−2
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−5
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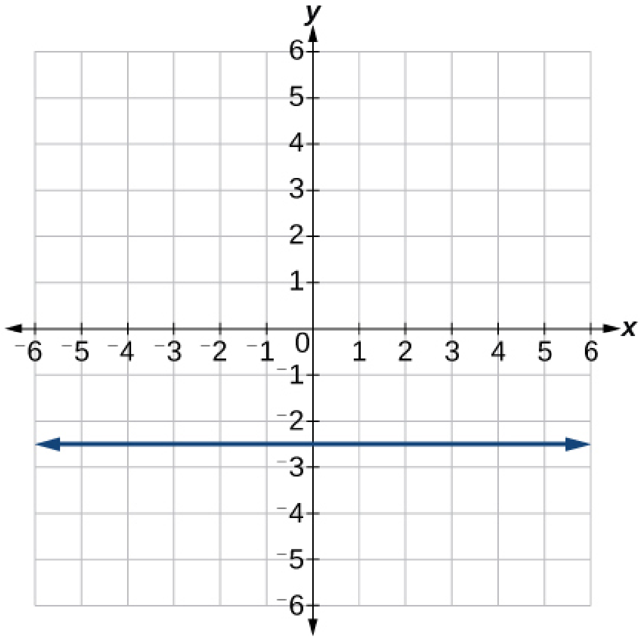
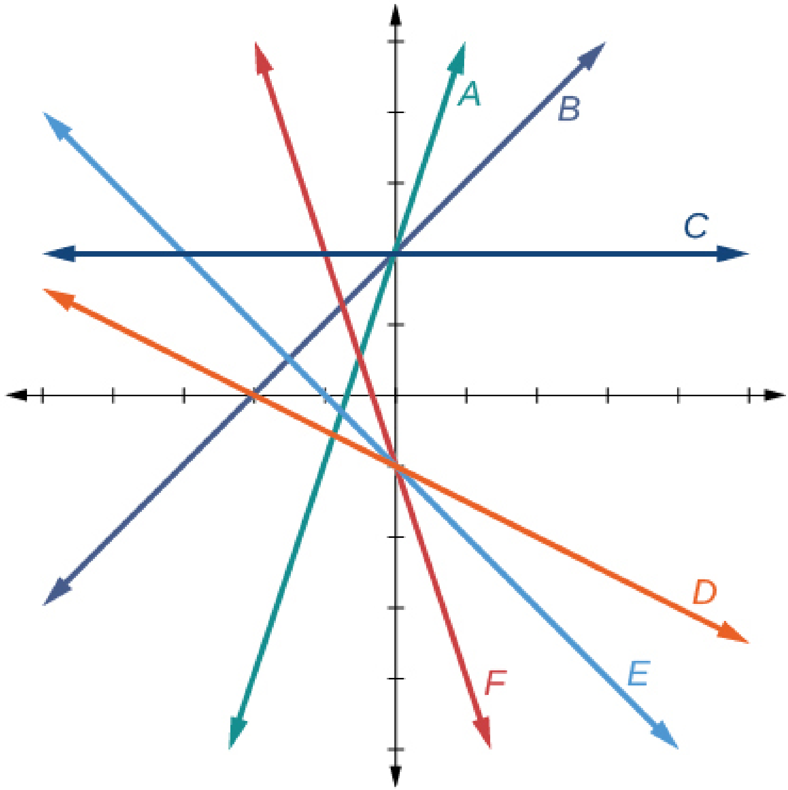
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6



$y = -2.5$

For the following exercises, match the given linear equation with its graph in [link].



$f(x)=-x-1$

$f(x)=-2x-1$

F

$f(x)=-\frac{1}{2}x-1$

$f(x)=2$

C

$f(x) = 2 + x$

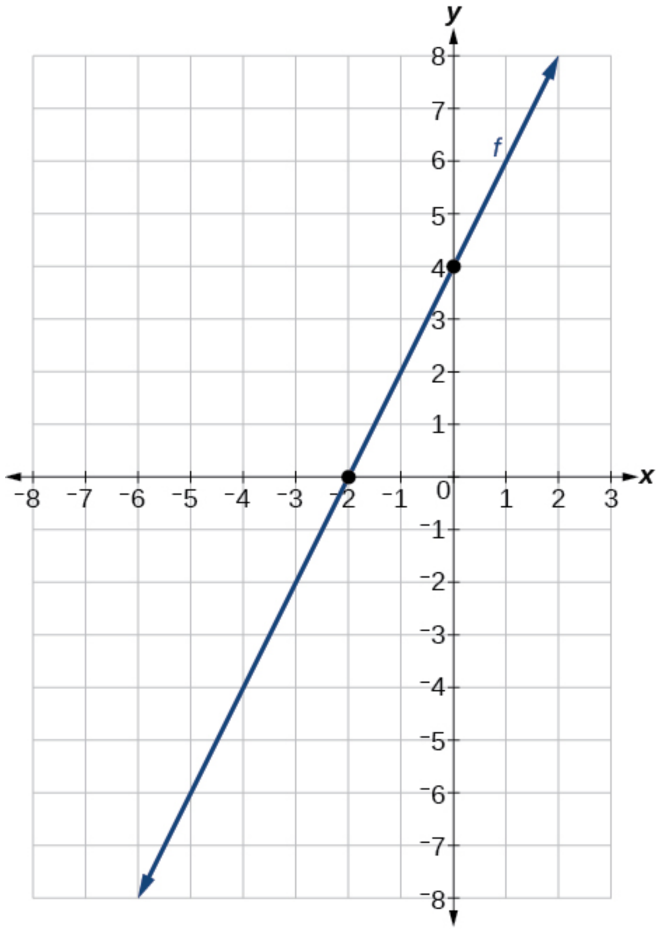
$f(x) = 3x + 2$

A

For the following exercises, sketch a line with the given features.

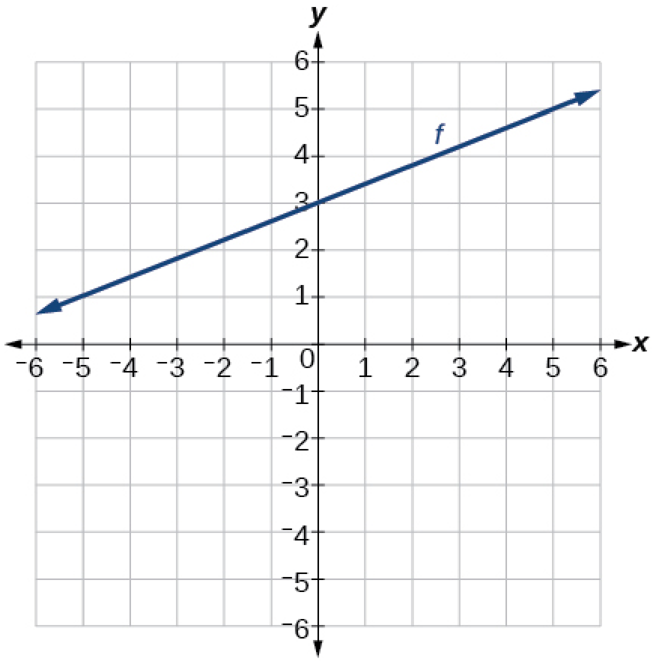
An *x*-intercept of $(–4,0)$ and *y*-intercept of $(0,–2)$

An *x*-intercept $(–2,0)$ and *y*-intercept of $(0,4)$



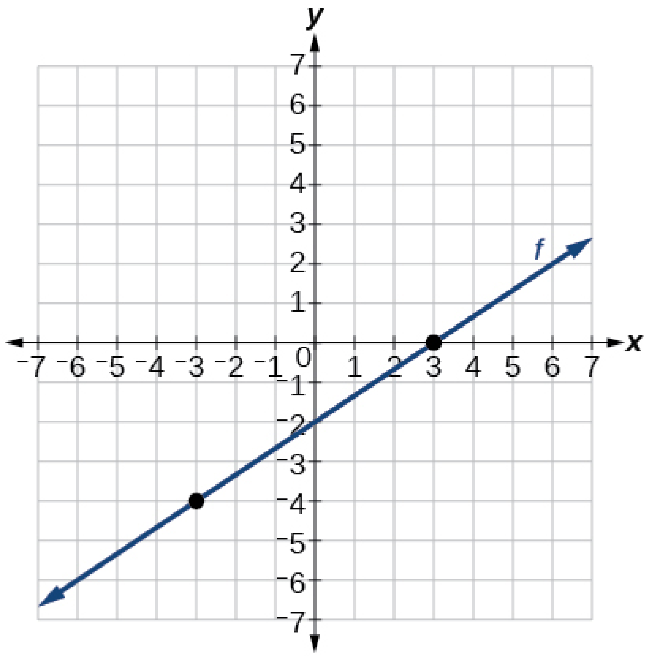
A y-intercept of $(0,7)$ and slope $-\ 3\ 2$

A y-intercept of $(0,3)$ and slope $2\ 5$



Passing through the points $(–6,–2)$ and $(6,–6)$

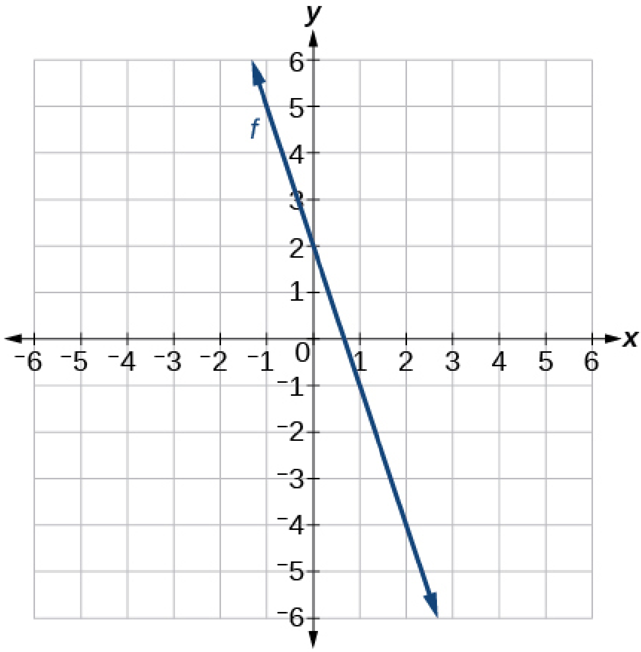
Passing through the points $(–3,–4)$ and $(3,0)$



For the following exercises, sketch the graph of each equation.

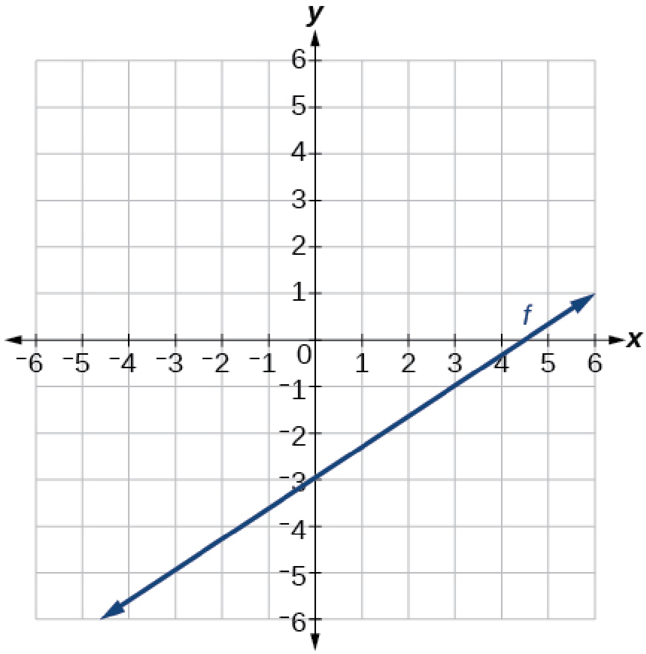
$$f(x) = -2x - 1$$

$$f(x) = -3x + 2$$



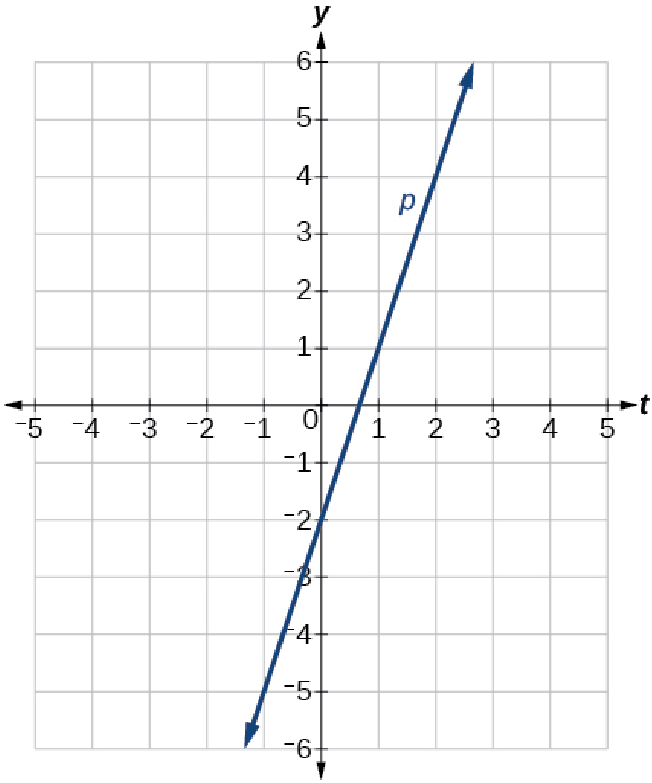
$f(x)=1\ 3\ x+2$

$f(x)=2\ 3\ x-3$



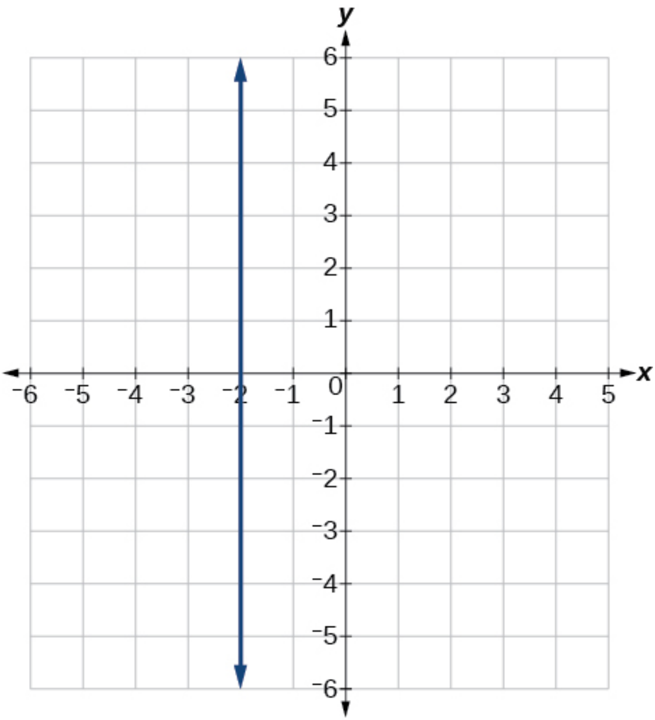
$f(t) = 3 + 2t$

$p(t) = -2 + 3t$



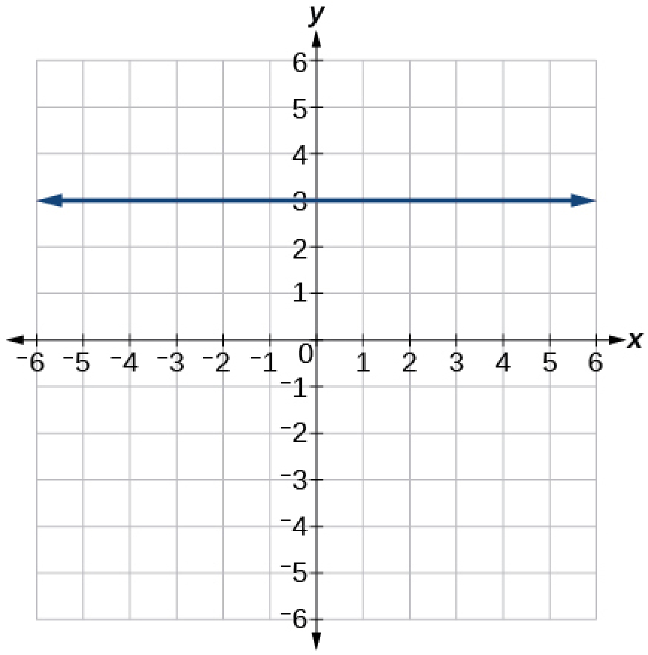
$x=3$

$x=-2$

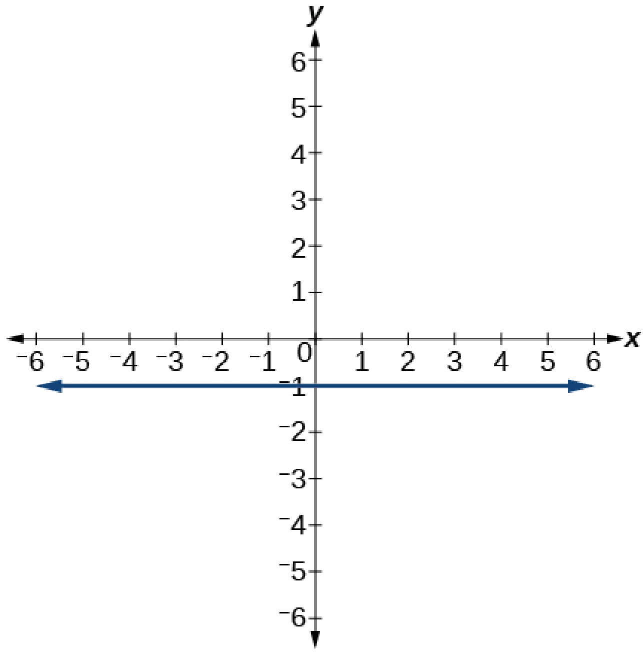


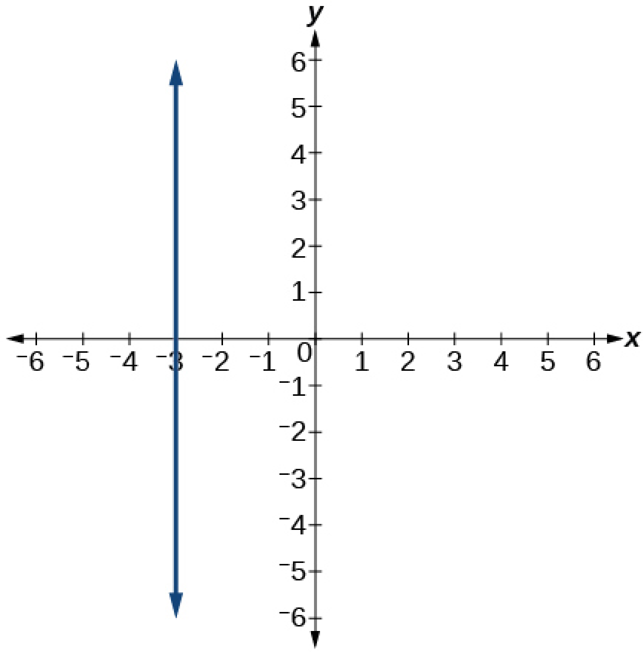
$r(x)=4$

For the following exercises, write the equation of the line shown in the graph.

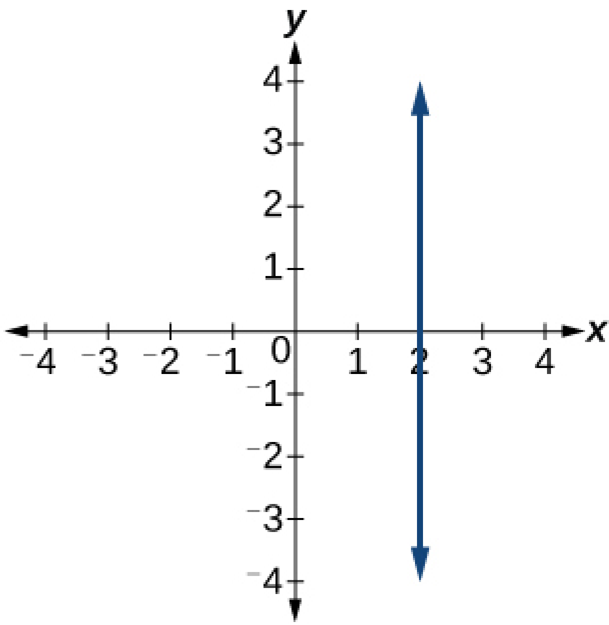


$y = 3$


y
x
0
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6



$x = -3$



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

x	0	5	10	15
g(x)	5	–10	–25	–40

Linear, $g(x) = -3x + 5$

x	0	5	10	15
h(x)	5	30	105	230

x	0	5	10	15
f(x)	–5	20	45	70

Linear, $f(x) = 5x - 5$

x	5	10	20	25
k(x)	13	28	58	73

x	0	2	4	6
g(x)	6	–19	–44	–69

Linear, $g(x) = -\ 25\ 2\ x + 6$

x	2	4	8	10
h(x)	13	23	43	53

x	2	4	6	8
f(x)	–4	16	36	56

Linear, $f(x) = 10x - 24$

x	0		2		6		8		
k(x)	6		31		106		231		

Technology

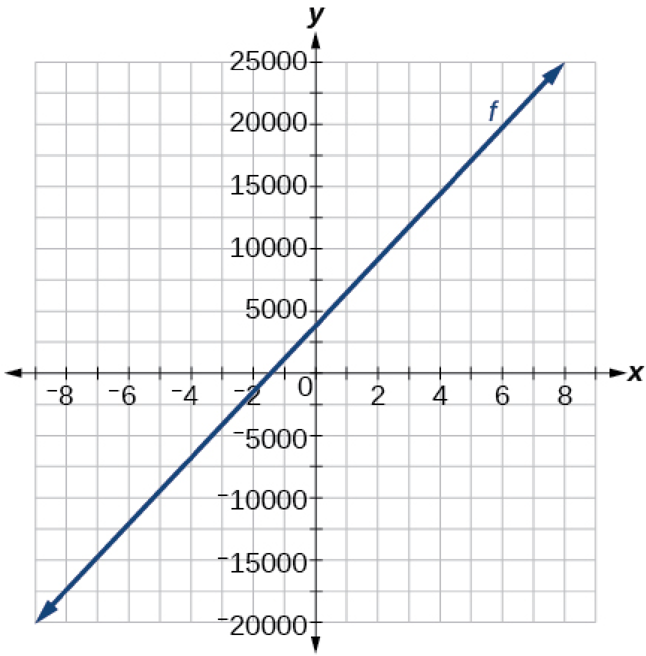
For the following exercises, use a calculator or graphing technology to complete the task.

If f is a linear function, $f(0.1) = 11.5$, and $f(0.4) = –5.9$, find an equation for the function.

$f(x) = -58x + 17.3$

Graph the function f on a domain of $[–10,10]: f(x) = 0.02x - 0.01.$ Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be $10.$

Graph the function f on a domain of [–10,10]:fx)$=2{,}500x+4{,}000$



[link] shows the input, w, and output, k, for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k, round to 3 decimal places.

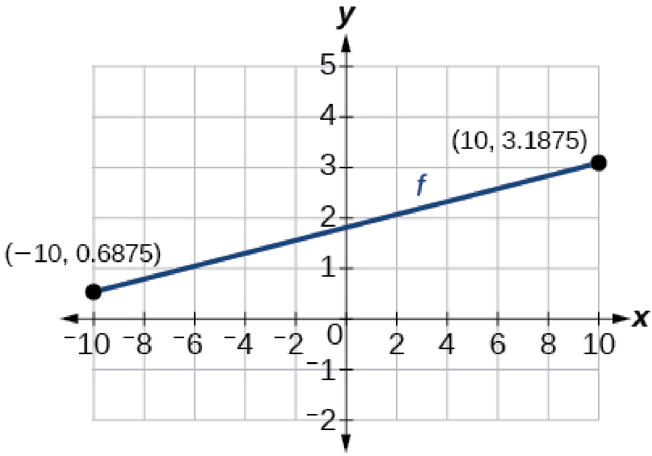
w	10	5.5	67.5	b
k	30	–26	a	–44

$y = 3.613x - 6.129$

[link] shows the input, p, and output, q, for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

p	0.5	0.8	12	b
q	400	700	a	1,000,000

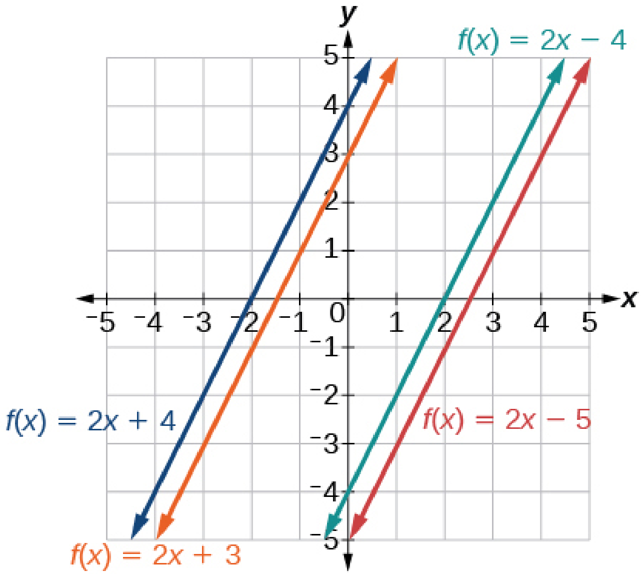
Graph the linear function f on a domain of $[-10, 10]$ for the function whose slope is $1\ 8$ and *y*-intercept is $31\ 16$. Label the points for the input values of -10 and 10.



Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and *y*-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

1. $a = 2; b = 3$
2. $a = 2; b = 4$
3. $a = 2; b = -4$
4. $a = 2; b = -5$



Extensions

Find the value of x if a linear function goes through the following points and has the following slope: $(x,2),(-4,6), m=3$

Find the value of y if a linear function goes through the following points and has the following slope: $(10,y),(25,100), m=-5$

$y = 175$

Find the equation of the line that passes through the following points:

(a, b) and $(a, b+1)$

Find the equation of the line that passes through the following points:

$(2a,b)$ and $(a,b+1)$

$y = -\ 1\ 2\ x + b + 2$

Find the equation of the line that passes through the following points:

$(a,0)$ and (c,d)

Find the equation of the line parallel to the line $g(x) = -0.01x + 2.01$ through the point $(1,2)$.

$y = -0.01x + 2.01$

Find the equation of the line perpendicular to the line $g(x) = -0.01x + 2.01$ through the point $(1,2)$.

For the following exercises, use the functions $f(x) = -0.1x + 200$ and $g(x) = 20x + 0.1$.

Find the point of intersection of the lines f and g.

$\left(\frac{1999}{201}, \frac{400,001}{2010} \right)$

Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

Real-World Applications

At noon, a barista notices that she has \$20 in her tip jar. If she makes an average of \$0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

$20 + 0.5n$

A gym membership with two personal training sessions costs \$125, while gym membership with five personal training sessions costs \$260. What is cost per session?

A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$, while 3,000 shirts can be sold at a price of \$22. Find a linear equation in the form $p(n)=mn+b$ that gives the price p they can charge for n shirts.

$p(n)=-0.004n+34$

A phone company charges for service according to the formula: $C(n)=24+0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y=mn+b$ that gives the yield when n stalks are planted.

$y=-0.5n+45$

A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

$P(t) = 1700t + 45{,}000$

Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23{,}286$, where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

1. As of 1990, average annual income was \$23,286.
2. In the ten-year period from 1990–1999, average annual income increased by a total of \$1,054.
3. Each year in the decade of the 1990s, average annual income increased by \$1,054.

4. Average annual income rose to a level of \$23,286 by the end of 1999.

When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C, the Celsius temperature, $F(C)$.

1. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
2. Find and interpret $F(28)$.
3. Find and interpret $F(–40)$.

---

1. Rate of change $= \Delta F\ \Delta C = 212-32$
   $100-0$
   $=1.8$ degrees F for one degree change in C
2. $F(28)=1.8(28)+32=82.4$ degrees F is 28 degrees C
3. $F(-40)=1.8(-40)+32=$
   -40 degrees F is -40 degrees C

Glossary

decreasing linear function
: a function with a negative slope: If $f(x)=mx$

$+ b$, then $m < 0$.

horizontal line
a line defined by $f(x) = b$, where b is a real number. The slope of a horizontal line is 0.

increasing linear function
a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function
a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

parallel lines
two or more lines with the same slope

perpendicular lines
two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form
the equation for a line that represents a linear function of the form $y - y_1 = m(x - x_1)$

slope
the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x)=mx+b$

vertical line
: a line defined by $x=a$, where a is a real number. The slope of a vertical line is undefined.

Solve Geometry Applications: Triangles, Rectangles, and the Pythagorean Theorem

By the end of this section, you will be able to:

- Solve applications using properties of triangles
- Use the Pythagorean Theorem
- Solve applications using rectangle properties

Before you get started, take this readiness quiz.

1. Simplify: $12(6h)$.
   If you missed this problem, review [link].
2. The length of a rectangle is three less than the width. Let *w* represent the width. Write an expression for the length of the rectangle.
   If you missed this problem, review [link].
3. Solve: $A=12bh$ for *b* when $A=260$ and $h=52$.
   If you missed this problem, review [link].
4. Simplify: 144.
   If you missed this problem, review [link].

Triangle ABC has vertices A, B, and C. The lengths of the sides are a, b, and c. The formula for the area of $\triangle ABC$ is $A=12bh$, where *b* is the base and *h* is the height.

Solve Applications Using Properties of Triangles

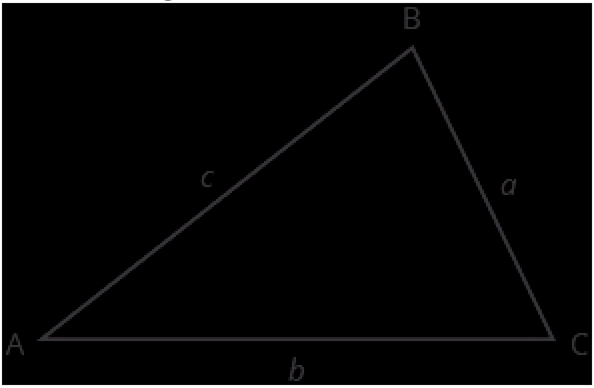
In this section we will use some common geometry formulas. We will adapt our problem-solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve. In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

Solve Geometry Applications.

Read the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information. **Identify** what we are looking for. **Label** what we are looking for by choosing a variable to represent it. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve** the equation using good algebra techniques. **Check** the answer by substituting it back into the equation solved in step 5 and by making sure it makes sense in the context of the problem. **Answer** the question with a complete sentence.

We will start geometry applications by looking at the properties of triangles. Let's review some basic facts about triangles. Triangles have three sides and three interior angles. Usually each side is labeled with a lowercase letter to match the uppercase letter of the opposite vertex.

The plural of the word *vertex* is *vertices*. All triangles have three vertices. Triangles are named by their vertices: The triangle in [link] is called $\triangle ABC$.



The three angles of a triangle are related in a special way. The sum of their measures is $180°$. Note that we read $m\angle A$ as "the measure of angle A." So in $\triangle ABC$ in [link],

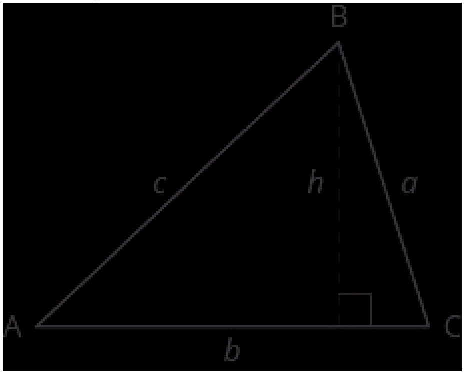
$$m\angle A + m\angle B + m\angle C = 180°$$

Because the perimeter of a figure is the length of its boundary, the perimeter of $\triangle ABC$ is the sum of the lengths of its three sides.

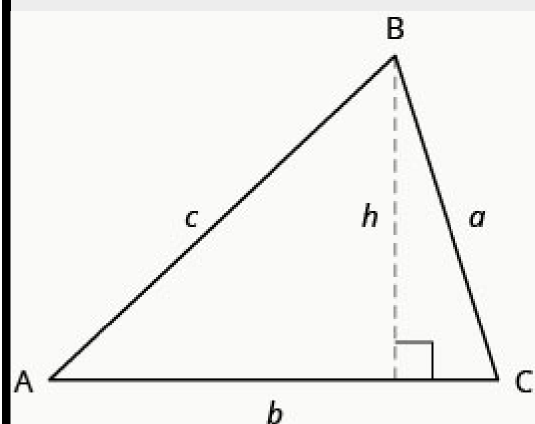
$$P = a + b + c$$

To find the area of a triangle, we need to know its

base and height. The height is a line that connects the base to the opposite vertex and makes a 90° angle with the base. We will draw △ABC again, and now show the height, *h*. See [link].



Triangle Properties



For △ABC

Angle measures:

$m\angle A + m\angle B + m\angle C = 180$

- The sum of the measures of the angles of a triangle is 180°.

Perimeter:
$P = a + b + c$

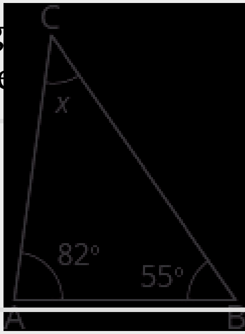
- The perimeter is the sum of the lengths of the sides of the triangle.

Area:
$A = 12bh, b = \text{base}, h = \text{height}$

- The area of a triangle is one-half the base times the height.

The measures of two angles of a triangle are 55 and 82 degrees. Find the measure of the third angle.

Solution

Step 1. Read the problem. Draw the fig t with the ation.	
	
Step 2. Identify what you are looking for.	the measure of the third angle in a triangle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of the angle.
Step 4. Translate. Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. Solve the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x =$
Step 6. Check. $55 + 82 + 43 \stackrel{?}{=} 180$ $180 = 180 \checkmark$	
Step 7. Answer the question.	The measure of the third angle is 43 degrees.

The measures of two angles of a triangle are 31 and 128 degrees. Find the measure of the third angle.

21 degrees

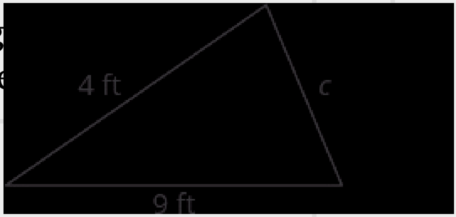
The measures of two angles of a triangle are 49 and 75 degrees. Find the measure of the third angle.

56 degrees

The perimeter of a triangular garden is 24 feet. The lengths of two sides are four feet and nine feet. How long is the third side?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.



[redacted]

Step 2. Identify what you are looking for.	length of the third side of a triangle
Step 3. Name. Choose a variable to represent it.	Let c = the third side.
Step 4. Translate. Write the appropriate formula and substitute.	[redacted]
Substitute in the given information.	[redacted]
Step 5. Solve the equation.	[redacted]
Step 6. Check.	

P = a + b

$+c \quad 24 \stackrel{?}{=} 4+9+11 \quad 24=24\checkmark$

Step 7. Answer the question.	The third side is 11 feet long.

The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

8 feet

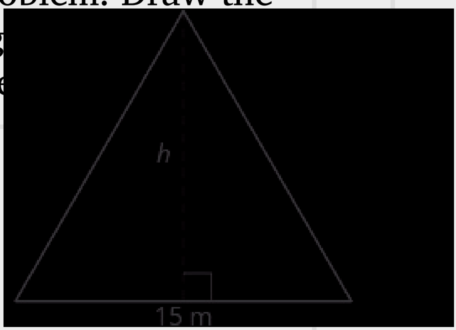
The lengths of two sides of a triangular window are seven feet and five feet. The perimeter is 18 feet. How long is the third side?

6 feet

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Solution

Step 1. Read the problem. Draw the fig the



Area $= 90m2$

Step 2. Identify what you are looking for. height of a triangle

Step 3. Name. Choose a variable to represent it. Let $h =$ the height.

Step 4. Translate. Write the appropriate formula.

$$A = \frac{1}{2} \cdot b \cdot h$$

Substitute in the given information.

[redacted]

Step 5. Solve the equation.

[redacted]

[redacted]

Step 6. Check.

$A = 12bh90 \stackrel{?}{=} 12 \cdot 15 \cdot 1290 = 90 ✓$

Step 7. Answer the question. The height of the triangle is 12 meters.

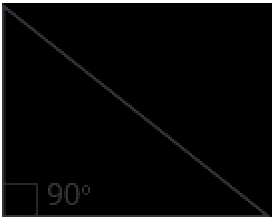
The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

14 inches

A triangular tent door has area 15 square feet. The height is five feet. What is the base?

6 feet

The triangle properties we used so far apply to all triangles. Now we will look at one specific type of triangle—a right triangle. A right triangle has one 90° angle, which we usually mark with a small square in the corner.



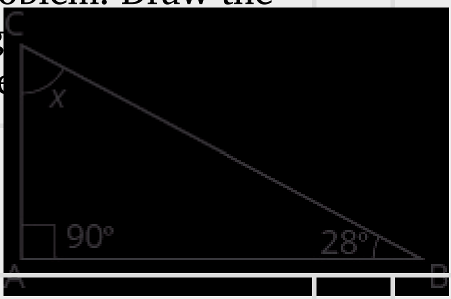
Right Triangle

A **right triangle** has one 90° angle, which is often marked with a square at the vertex.

One angle of a right triangle measures 28°. What is the measure of the third angle?

Solution

Step 1. Read the problem. Draw the fig the	
Step 2. Identify what you are looking for.	the measure of an angle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of an angle.
Step 4. Translate.	$m\angle A + m\angle B + m\angle C = 180$
Write the appropriate formula and substitute.	$x + 90 + 28 = 180$
Step 5. Solve the equation.	$x + 118 = 180$ $x = 62$
Step 6. Check.	



$180 \stackrel{?}{=} 90 + 28 + 62$ $180 = 180$ ✓

Step 7. Answer the question.	The measure of the third angle is 62°.

One angle of a right triangle measures 56°. What is the measure of the other small angle?

34°

One angle of a right triangle measures 45°. What is the measure of the other small angle?

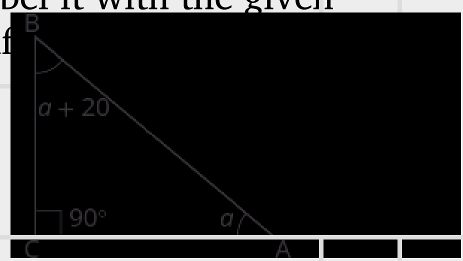
45°

In the examples we have seen so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

The measure of one angle of a right triangle is 20 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let a = 1st angle. a + 20 = 2nd angle 90 = 3rd angle (the right angle)
Draw the figure and label it with the given inf	



Step 4. Translate

$$m\angle A + m\angle B + m\angle C = 180$$

Write the appropriate formula.

Substitute into the formula.

[redacted]

Step 5. Solve the equation.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

55

90 third angle

Step 6. Check.

$35 + 55 + 90 \stackrel{?}{=} 180$ $180 = 180$ ✓

Step 7. Answer the question.

The three angles measure 35°, 55°, and

90°.

The measure of one angle of a right triangle is 50° more than the measure of the smallest angle. Find the measures of all three angles.

$20^\circ, 70^\circ, 90^\circ$

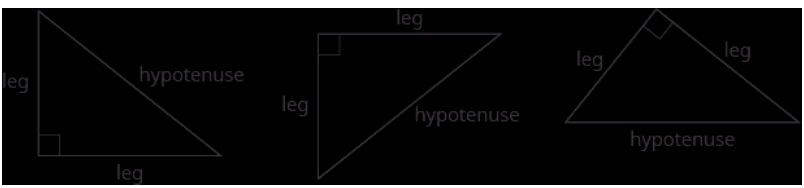
The measure of one angle of a right triangle is 30° more than the measure of the smallest angle. Find the measures of all three angles.

$30^\circ, 60^\circ, 90^\circ$

Use the Pythagorean Theorem

We have learned how the measures of the angles of a triangle relate to each other. Now, we will learn how the lengths of the sides relate to each other. An important property that describes the relationship among the lengths of the three sides of a right triangle is called the Pythagorean Theorem. This theorem has been used around the world since ancient times. It is named after the Greek philosopher and mathematician, Pythagoras, who lived around 500 BC.

Before we state the Pythagorean Theorem, we need to introduce some terms for the sides of a triangle. Remember that a right triangle has a 90° angle, marked with a small square in the corner. The side of the triangle opposite the 90° angle is called the *hypotenuse* and each of the other sides are called *legs*.

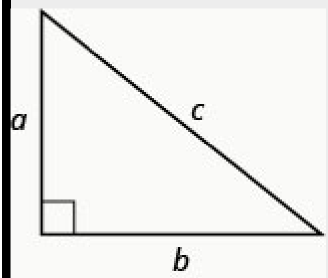


The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. In symbols we say: in any right triangle, $a2+b2=c2$, where aandb are the lengths of the legs and c is the length of the hypotenuse.

Writing the formula in every exercise and saying it aloud as you write it, may help you remember the Pythagorean Theorem.

The Pythagorean Theorem

In any right triangle, $a2+b2=c2$.



where *a* and *b* are the lengths of the legs, *c* is the length of the hypotenuse.

To solve exercises that use the Pythagorean Theorem, we will need to find square roots. We have used the notation m and the definition:

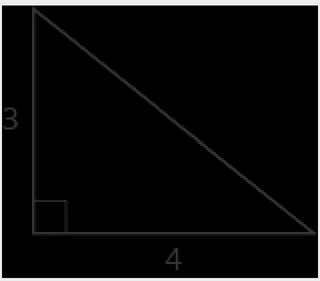
If $m=n2$, then $m=n$, for $n\geq 0$.

For example, we found that 25 is 5 because $25=52$.

Because the Pythagorean Theorem contains variables that are squared, to solve for the length of

a side in a right triangle, we will have to use square roots.

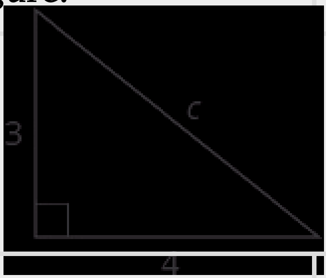
Use the Pythagorean Theorem to find the length of the hypotenuse shown below.



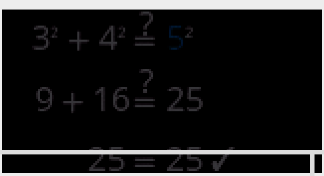
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the hypotenuse of the triangle
Step 3. Name. Choose	Let c = the length of

a variable to represent it. Label side *c* on the figure.	the hypotenuse.

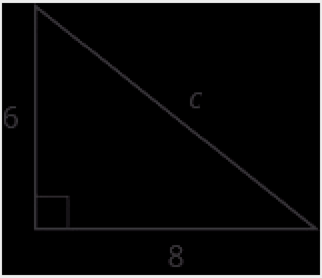


Step 4. Translate. Write the appropriate formula.	$a2+b2=c2$
Substitute.	$32+42=c2$
Step 5. Solve the equation.	$9+16=c2$
Simplify.	$25=c2$
Use the definition of square root.	$25=c$
Simplify.	$5=c$
Step 6. Check.	



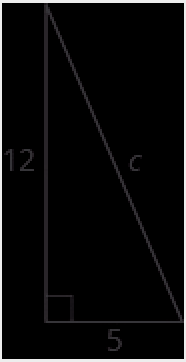
Step 7. Answer the question.	The length of the hypotenuse is 5.

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.



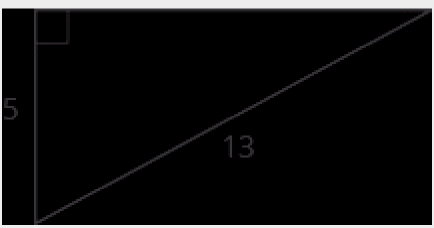
c = 10

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.



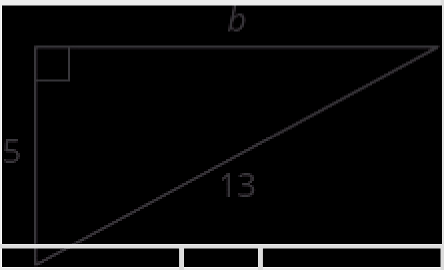
$c = 13$

Use the Pythagorean Theorem to find the length of the leg shown below.



Solution

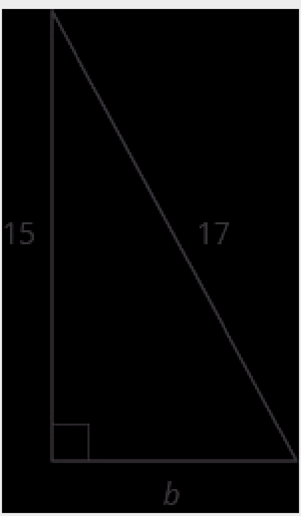
Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name. Choose a variable to represent it.	Let b = the leg of the triangle.
Lable side b.	



Step 4. Translate	
Write the appropriate formula.	$a2+b2=c2$
Substitute.	$52+b2=132$
Step 5. Solve the equation.	$25+b2=169$
Isolate the variable term.	$b2=144$
Use the definition of square root.	$b2=144$
Simplify.	$b=12$

Step 6. Check.	$5^2 + 12^2 \stackrel{?}{=} 13^2$ $25 + 144 \stackrel{?}{=} 169$ $169 = 169$ ✓	
Step 7. Answer the question.		The length of the leg is 12.

Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.



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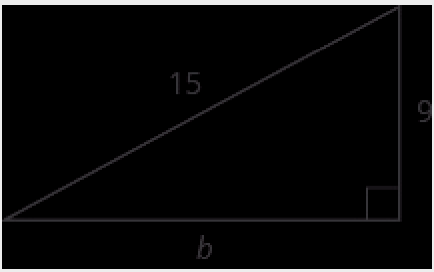
Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.



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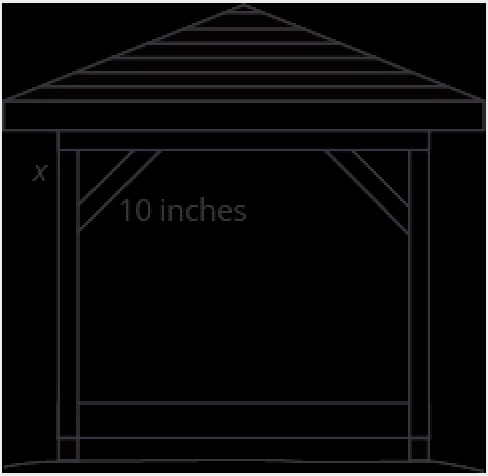


Kelvin is building a gazebo and wants to brace each corner by placing a 10″ piece of wood diagonally as shown above.

If he fastens the wood so that the ends of the brace are the same distance from the corner, what is the length of the legs of the right triangle formed? Approximate to the nearest tenth of an inch.

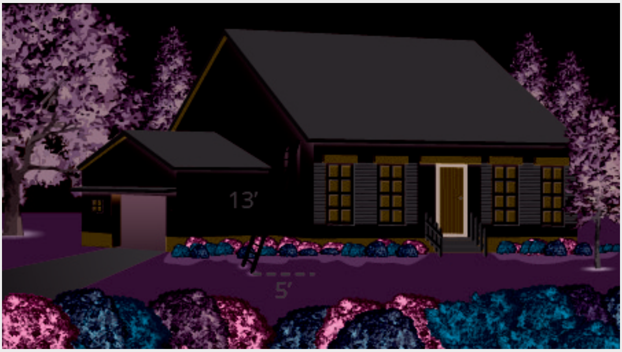
Solution

Step 1. Read the problem.

Step 2. Identify what the distance from the

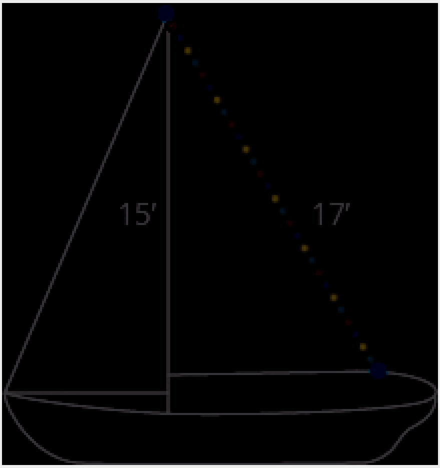
we are looking for.	corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let $x =$ the distance from the corner.
Step 4. Translate Write the appropriate formula and substitute.	$a^2 + b^2 = c^2$ $x^2 + x^2 = 10^2$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x^2 = 100$ $x^2 = 50$ $x = \sqrt{50}$ $x \approx 7.1$
Step 6. Check. $a^2 + b^2 = c^2$ $(7.1)^2 + (7.1)^2 \approx 10^2$ Yes.	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1" from the corner.

John puts the base of a 13-foot ladder five feet from the wall of his house as shown below. How far up the wall does the ladder reach?



12 feet

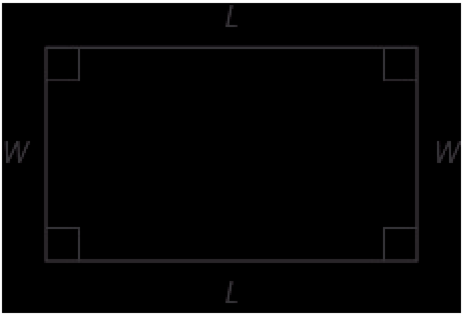
Randy wants to attach a 17 foot string of lights to the top of the 15 foot mast of his sailboat, as shown below. How far from the base of the mast should he attach the end of the light string?



8 feet

Solve Applications Using Rectangle Properties

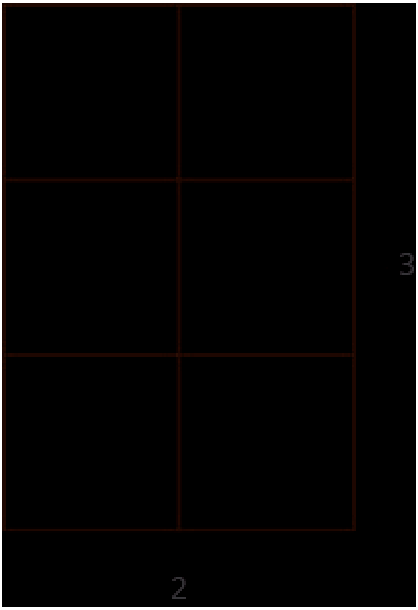
You may already be familiar with the properties of rectangles. Rectangles have four sides and four right (90°) angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, L, and its adjacent side as the width, W.



The distance around this rectangle is L + W + L + W, or 2L + 2W. This is the perimeter, *P*, of the rectangle.

$P = 2L + 2W$

What about the area of a rectangle? Imagine a rectangular rug that is 2-feet long by 3-feet wide. Its area is 6 square feet. There are six squares in the figure.



$A = 6 A = 2 \cdot 3 A = L \cdot W$

The area is the length times the width.

The formula for the area of a rectangle is $A = LW$.

Properties of Rectangles

Rectangles have four sides and four right (90°) angles.

The lengths of opposite sides are equal.

The perimeter of a rectangle is the sum of twice the length and twice the width.

$P = 2L + 2W$

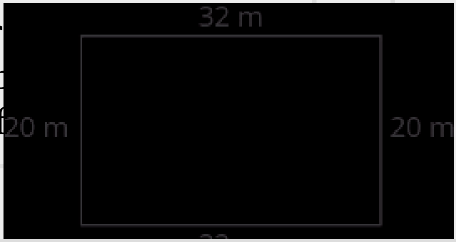
The area of a rectangle is the product of the length and the width.

$A = L \cdot W$

The length of a rectangle is 32 meters and the width is 20 meters. What is the perimeter?

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the perimeter of a rectangle
Step 3. Name. Choose a variable to represent it.	Let P = the perimeter.
Step 4. Translate. Write the appropriate formula. Substitute.	
Step 5. Solve the equation.	
Step 6. Check.	
Step 7. Answer the question.	The perimeter of the rectangle is 104



$$P = 2L + 2W$$

$$P = 2(32\text{ m}) + 2(20\text{ m})$$

$$P = 64 + 40$$

$$P = 104$$

$$P \stackrel{?}{=} 104$$
$$20 + 32 + 20 + 32 \stackrel{?}{=} 104$$
$$104 = 104 \checkmark$$

meters.

The length of a rectangle is 120 yards and the width is 50 yards. What is the perimeter?

340 yards

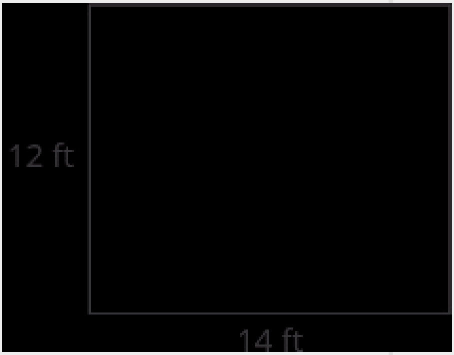
The length of a rectangle is 62 feet and the width is 48 feet. What is the perimeter?

220 feet

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution

Step 1. Read the problem. Dr lab inf	
	
Step 2. Identify what you are looking for.	the width of a rectangular room
Step 3. Name. Choose a variable to represent it.	Let $W =$ the width.
Step 4. Translate.	
Write the appropriate formula.	$A=LW$
Substitute.	$168=14W$
Step 5. Solve the equation.	$16814=14W14$ $12=W$
Step 6. Check.	



$A = LW$
$168 \stackrel{?}{=} 14 \cdot 12$
$168 = 168 ✓$

Step 7. Answer the question.	The width of the room is 12 feet.

The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?
26 feet

The width of a rectangle is 21 meters. The area is 609 square meters. What is the length?

29 meters

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1. Read the problem. Dr lab inf	
Step 2. Identify what you are looking for.	the length of the rectangle
Step 3. Name. Choose a variable to represent it.	Let $L =$ the length.

Step 4. Translate.
Write the appropriate formula.

[redacted]

Substitute.

[redacted]

Step 5. Solve the equation.

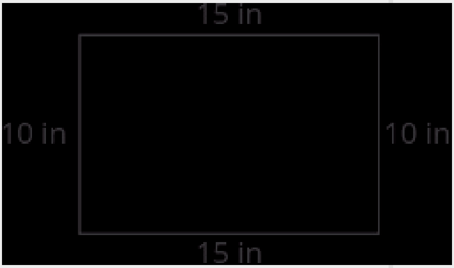
[redacted]

[redacted]

[redacted]

[redacted]

Step 6. Check.



$P = 5015 + 10 + 15 + 10 \stackrel{?}{=} 5050 = 50$ ✓

Step 7. Answer the question.	The length is 15 inches.

Find the length of a rectangle with: perimeter 80 and width 25.

15

Find the length of a rectangle with: perimeter 30 and width 6.

9

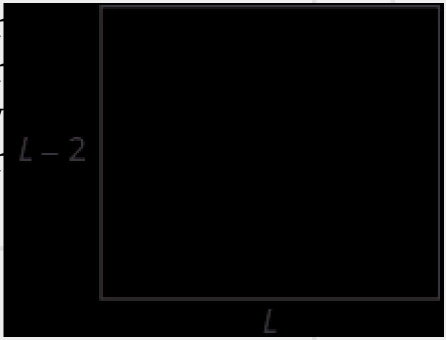
We have solved problems where either the length or width was given, along with the perimeter or area; now we will learn how to solve problems in which the width is defined in terms of the length. We will wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

The width of a rectangle is two feet less than the length. The perimeter is 52 feet. Find the length and width.

Solution

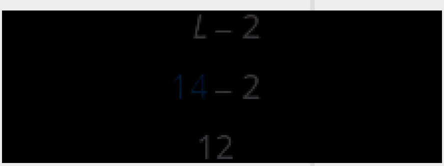
Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and width of a rectangle
Step 3. Name. Choose a variable to represent it. Since the width is defined in terms of the	

len
len
tw
len
=



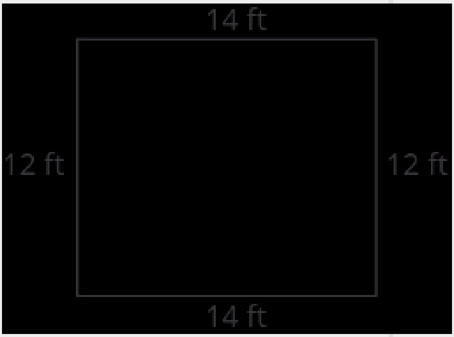
$P = 52$ ft

Step 4. Translate.	
Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information.	$P = 2L + 2W$
Substitute in the given information.	$52 = 2L + 2(L - 2)$
Step 5. Solve the equation.	$52 = 2L + 2L - 4$
Combine like terms.	$52 = 4L - 4$
Add 4 to each side.	$56 = 4L$
Divide by 4.	$564 = 4L4$
	$14 = L$
	The length is 14 feet.
Now we need to find the width.	The width is $L - 2$.



The width is 12 feet.

Step 6. Check.
Since
$14+12+14+12=52$,
this works!



Step 7. Answer the question.	The length is 14 feet and the width is 12 feet.

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

18 meters, 11 meters

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

19 feet, 11 feet

The length of a rectangle is four centimeters more than twice the width. The perimeter is 32 centimeters. Find the length and width.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the [redacted]	
The length is four more	

tha[redacted]

[redacted]

[redacted]

Step 4. Translate
Write the appropriate formula.

[redacted]

Substitute in the given information.

[redacted]

Step 5. Solve the equation.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

12

The length is 12 cm.

Step 6. Check.

[redacted]

$P = 2L + 2W$
$32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$
$32 = 32 \checkmark$

Step 7. Answer the question.	The length is 12 cm and the width is 4 cm.

The length of a rectangle is eight more than twice the width. The perimeter is 64. Find the length and width.

24, 8

The width of a rectangle is six less than twice the length. The perimeter is 18. Find the length and width.

5, 4

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1. Read the problem. Dr[redacted]	$P = 150$ ft
Step 2. Identify what you are looking for.	the length and the width of the pool
Step 3. Name. Choose a variable to represent the width. Th[redacted] more than the width.	

[redacted]

Step 4. Translate Write the appropriate formula.	

[redacted]

Substitute.	

[redacted]

Step 5. Solve the equation.	

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

Step 6. Check.

$P = 2L + 2W$
$150 \stackrel{?}{=} 2(45) + 2(30)$
$150 = 150 \checkmark$

Step 7. Answer the question.	The length of the pool is 45 feet and the width is 30 feet.

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.
70 feet, 30 feet

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.
90 yards, 60 yards

Key Concepts

- **Problem-Solving Strategy for Geometry Applications**

  Read the problem and make all the words and ideas are understood. Draw the figure and label it with the given information. **Identify** what we are looking for. **Name** what we are looking for

by choosing a variable to represent it. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve** the equation using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

- **Triangle Properties For $\triangle ABC$**
  Angle measures:
  - $m\angle A+m\angle B+m\angle C=180$

  Perimeter:
  - $P=a+b+c$

  Area:
  - $A=12bh, b=base, h=height$

  A right triangle has one 90° angle.
- **The Pythagorean Theorem** In any right triangle, $a2+b2=c2$ where *c* is the length of the hypotenuse and *a* and *b* are the lengths of the legs.
- **Properties of Rectangles**
  - Rectangles have four sides and four right (90°) angles.
  - The lengths of opposite sides are equal.

- The perimeter of a rectangle is the sum of twice the length and twice the width: $P = 2L + 2W$. The area of a rectangle is the length times the width: $A = LW$.

Practice Makes Perfect

Solving Applications Using Triangle Properties

In the following exercises, solve using triangle properties.

The measures of two angles of a triangle are 26 and 98 degrees. Find the measure of the third angle.

---

56 degrees

The measures of two angles of a triangle are 61 and 84 degrees. Find the measure of the third angle.

The measures of two angles of a triangle are 105 and 31 degrees. Find the measure of the third angle.

---

44 degrees

The measures of two angles of a triangle are 47 and 72 degrees. Find the measure of the third angle.

The perimeter of a triangular pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?

---

11 feet

A triangular courtyard has perimeter 120 meters. The lengths of two sides are 30 meters and 50 meters. How long is the third side?

If a triangle has sides 6 feet and 9 feet and the perimeter is 23 feet, how long is the third side?

---

8 feet

If a triangle has sides 14 centimeters and 18 centimeters and the perimeter is 49 centimeters, how long is the third side?

A triangular flag has base one foot and height 1.5 foot. What is its area?

0.75 sq. ft.

A triangular window has base eight feet and height six feet. What is its area?

What is the base of a triangle with area 207 square inches and height 18 inches?

---

23 inches

What is the height of a triangle with area 893 square inches and base 38 inches?

One angle of a right triangle measures 33 degrees. What is the measure of the other small angle?

57

One angle of a right triangle measures 51 degrees. What is the measure of the other small angle?

One angle of a right triangle measures 22.5 degrees. What is the measure of the other small angle?

67.5

One angle of a right triangle measures 36.5 degrees. What is the measure of the other small angle?

The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

---

13 ft., 12 ft., 14 ft.

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

One side of a triangle is twice the shortest side.

The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

---

3 ft., 6 ft., 8 ft.

One side of a triangle is three times the shortest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

$45^{\circ}, 45^{\circ}, 90^{\circ}$

The measure of the smallest angle of a right triangle is 20° less than the measure of the next larger angle. Find the measures of all three angles.

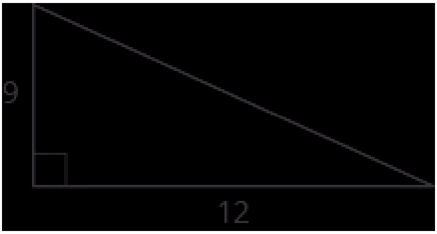
The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

$30°, 60°, 90°$

The angles in a triangle are such that one angle is $20°$ more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.



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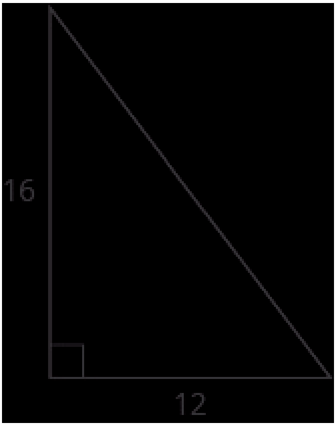

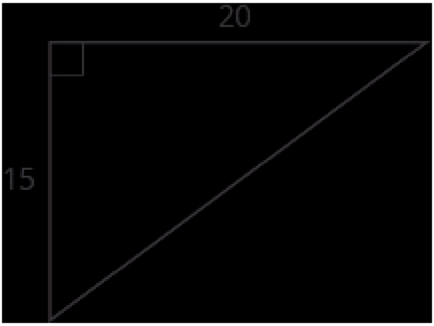

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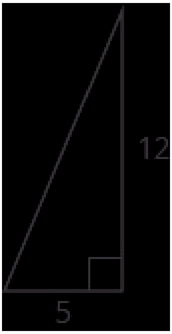
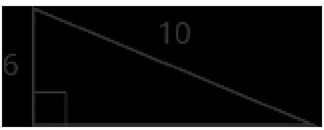


In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth, if necessary.



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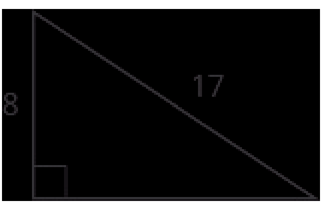

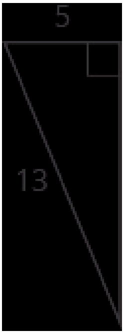

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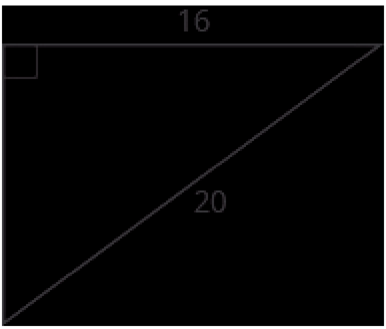
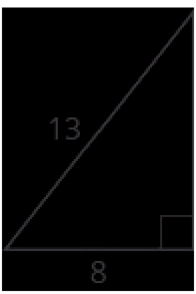
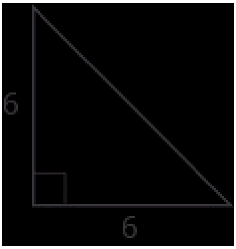
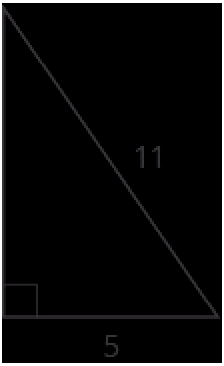




10.2





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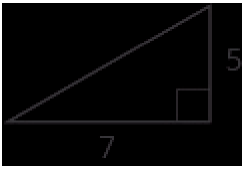
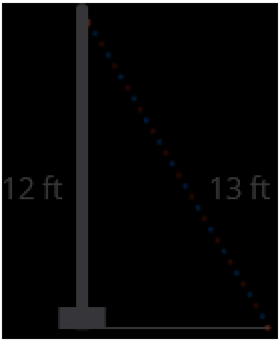


In the following exercises, solve using the Pythagorean Theorem. Approximate to the nearest tenth, if necessary.

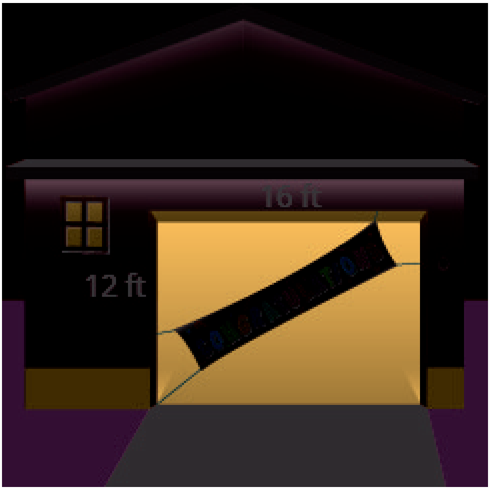
A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display, as shown below. How far from the base of the pole should the end of the string of lights be anchored?



---

5 feet

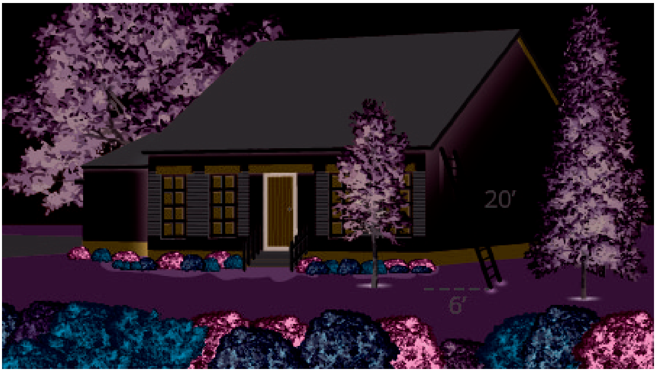
Pam wants to put a banner across her garage door, as shown below, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?



Chi is planning to put a path of paving stones through her flower garden, as shown below. The flower garden is a square with side 10 feet. What will the length of the path be?

14.1 feet

Brian borrowed a 20 foot extension ladder to use when he paints his house. If he sets the base of the ladder 6 feet from the house, as shown below, how far up will the top of the ladder reach?



Solve Applications Using Rectangle Properties

In the following exercises, solve using rectangle properties.

The length of a rectangle is 85 feet and the width is 45 feet. What is the perimeter?

---

260 feet

The length of a rectangle is 26 inches and the width is 58 inches. What is the perimeter?

A rectangular room is 15 feet wide by 14 feet long. What is its perimeter?

---

58 feet

A driveway is in the shape of a rectangle 20 feet wide by 35 feet long. What is its perimeter?

The area of a rectangle is 414 square meters. The length is 18 meters. What is the width?

---

23 meters

The area of a rectangle is 782 square centimeters. The width is 17 centimeters. What is the length?

The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?

---

26 inches

The length of a rectangular poster is 28 inches. The area is 1316 square inches. What is the

width?

Find the length of a rectangle with perimeter 124 and width 38.

24

Find the width of a rectangle with perimeter 92 and length 19.

Find the width of a rectangle with perimeter 16.2 and length 3.2.

4.9

Find the length of a rectangle with perimeter 20.2 and width 7.8.

The length of a rectangle is nine inches more than the width. The perimeter is 46 inches. Find the length and the width.

---

16 in., 7 in.

The width of a rectangle is eight inches more than the length. The perimeter is 52 inches. Find the length and the width.

The perimeter of a rectangle is 58 meters. The width of the rectangle is five meters less than the length. Find the length and the width of the rectangle.

---

17 m, 12 m

The perimeter of a rectangle is 62 feet. The width is seven feet less than the length. Find the length and the width.

The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

13.5 m length, 12.8 m width

The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

The perimeter of a rectangle is 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

---

50 ft., 25 ft.

The length of a rectangle is three times the width. The perimeter of the rectangle is 72 feet. Find the length and width of the rectangle.

The length of a rectangle is three meters less than twice the width. The perimeter of the rectangle is 36 meters. Find the dimensions of the rectangle.

---

7 m width, 11 m length

The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

---

160 yd., 120 yd.

The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

85 ft., 40 ft.

A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

Everyday Math

Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

24 feet

Jose just removed the children’s playset from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep out the dog. He has a 50 foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other length?

Writing Exercises

If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

---

area; answers will vary

If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

Look at the two figures below.

ⓐ Which figure looks like it has the larger area?
ⓑ Which looks like it has the larger perimeter?
ⓒ Now calculate the area and perimeter of each figure.
ⓓ Which has the larger area?
ⓔ Which has the larger perimeter?

ⓐ Answers will vary.
ⓑ Answers will vary.
ⓒ Answers will vary.
ⓓ The areas are the same.
ⓔ The 2x8 rectangle has a larger perimeter than the 4x4 square.

Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this

section.

I can…	Confidently	With some help	No-I don't get it!
solve applications using triangle properties.			
use the Pythagorean Theorem.			
solve applications using rectangle properties.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Models and Applications

In this section you will:

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

Credit: Kevin Dooley

Josh is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Josh to end the course with an A? A simple linear equation will give Josh his answer.

Many real-world applications can be modeled by

linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce x widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as $0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges

\$0.10/mi plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost C.

$C = 0.10x + 50$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. [link] lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by *a*	$x, x+a$
Twice a number	$2x$
One number is *a* more than another number	$x, x+a$
One number is *a* less than twice another number	$x, 2x-a$
The product of a number and *a*, decreased by *b*	$ax-b$
The quotient of a number and the number plus *a* is three times the number	$x\ x+a\ = 3x$
The product of three times a number and the	$3x(x-b)=c$

number decreased by b is c

Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is $31.$ Find the two numbers.

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x+17.$ The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.
$x+(x+17) = 31$ $2x+17 = 31$ Simplify and solve. $2x = 14$ $x = 7$ $x+17 = 7+17 = 24$

The two numbers are 7 and $24.$

Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is $36,$ find the numbers.

11 and 25

Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer

different packages. Company A charges a monthly service fee of \$34 plus \$.05/min talk-time. Company B charges a monthly service fee of \$40 plus \$.04/min talk-time.

1. Write a linear equation that models the packages offered by both companies.
2. If the average number of minutes used each month is 1,160, which company offers the better plan?
3. If the average number of minutes used each month is 420, which company offers the better plan?
4. How many minutes of talk-time would yield equal monthly statements from both companies?

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1. The model for Company *A* can be written as $A=0.05x+34.$ This includes the variable cost of $0.05x$ plus the monthly service charge of \$34. Company *B*'s package charges a higher monthly fee of \$40, but a lower variable cost of $0.04x.$ Company *B*'s model can be written as $B=0.04x+\$40.$

2. If the average number of minutes used each month is 1,160, we have the following:
   Company A $=$ $0.05(1.160)+34$ $=$

$58+34 = 92$ Company B $=$
$0.04(1,1600)+40 = 46.4+40 = 86.4$

So, Company *B* offers the lower monthly cost of \$86.40 as compared with the \$92 monthly cost offered by Company *A* when the average number of minutes used each month is 1,160.

3. If the average number of minutes used each month is 420, we have the following:
   Company A $= 0.05(420)+34 = 21+34$
   $= 55$ Company B $= 0.04(420)+40 =$
   $16.8+40 = 56.8$

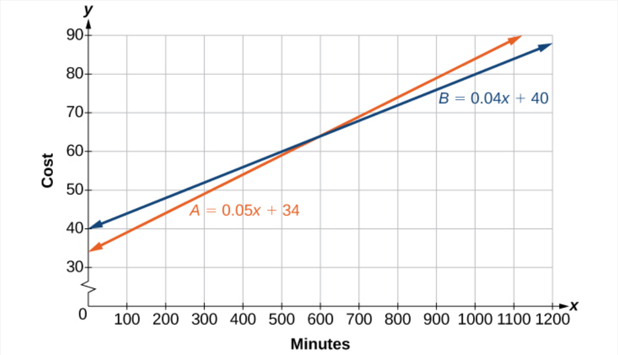
   If the average number of minutes used each month is 420, then Company *A* offers a lower monthly cost of \$55 compared to Company *B*'s monthly cost of \$56.80.

4. To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x,y) coordinates: At what point are both the *x*-value and the *y*-value equal? We can find this point by setting the equations equal to each other and solving for *x*.
   $0.05x+34 = 0.04x+40$ $0.01x = 6$ $x =$

600

Check the x-value in each equation.
$0.05(600)+34 = 64$ $0.04(600)+40 = 64$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See [link]



Find a linear equation to model this real-world application: It costs ABC electronics company $2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of $350 for utilities and $3,300 for salaries. What are the company's monthly expenses?

$C = 2.5x + 3{,}650$

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region, $A = LW$; the **perimeter** of a rectangle, $P = 2L + 2W$; and the **volume** of a rectangular solid, $V = LWH$. When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

Solving an Application Using a Formula

It takes Andrew 30 min to drive to work in the morning. He drives home using the same

route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

This is a distance problem, so we can use the formula $d=rt$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate r. His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance d. A table, such as [link], is often helpful for keeping track of information in these types of problems.

	d	r	t
To Work	d	r	$\frac{1}{2}$
To Home	d	$r-10$	$\frac{2}{3}$

Write two equations, one for each trip.

$d = r\left(\frac{1}{2}\right)$ To work $d = (r-10)\left(\frac{2}{3}\right)$

To home

As both equations equal the same distance, we set them equal to each other and solve for *r*.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ \frac{1}{2}r &= \frac{2}{3}r-\frac{20}{3} \\ \frac{1}{2}r-\frac{2}{3}r &= -\frac{20}{3} \\ -\frac{1}{6}r &= -\frac{20}{3} \\ r &= -\frac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for *d*.

$$d = 40\left(\frac{1}{2}\right) = 20$$

The distance between home and work is 20 mi.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6\times r\left(\frac{1}{2}\right) &= 6\times(r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ -r &= -40 \\ r &= 40 \end{aligned}$$

On Saturday morning, it took Jennifer 3.6 h to drive to her mother’s house for the weekend.

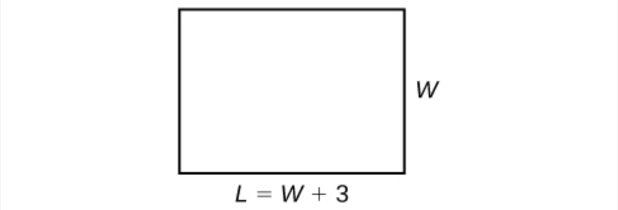
On Sunday evening, due to heavy traffic, it took Jennifer 4 h to return home. Her speed was 5 mi/h slower on Sunday than on Saturday. What was her speed on Sunday?

45 mi/h

Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

The perimeter formula is standard: $P=2L+2W.$ We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L=W+3.$ Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in [link].



Now we can solve for the width and then calculate the length.

$P = 2L + 2W$ $54 = 2(W+3) + 2W$ $54 = 2W + 6 + 2W$ $54 = 4W + 6$ $48 = 4W$ $12 = W$ $(12+3) = L$ $15 = L$

The dimensions are $L=15$ ft and $W=12$ ft.

Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

$L=37$ cm, $W=18$ cm

Solving an Area Problem

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

The standard formula for area is $A=LW$; however, we will solve the problem using the perimeter formula. The reason we use the

perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$P = 2L + 2W$ $48 = 2(W + 6) + 2W$ $48 = 2W + 12 + 2W$ $48 = 4W + 12$ $36 = 4W$ $9 = W$ $(9 + 6) = L$ $15 = L$

Now, we find the area given the dimensions of $L = 15$ in. and $W = 9$ in.

$A = LW$ $A = 15(9) = 135 \text{ in}.^2$

The area is 135 in.^{2}.

A game room has a perimeter of 70 ft. The length is five more than twice the width. How many ft^2 of new carpeting should be ordered?

$250\ ft^2$

Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is $1,600\ in.^3$.

The formula for the volume of a box is given as $V = LWH$, the product of length, width, and height. We are given that $L = 2W$, and $H = 8$. The volume is $1,600$ cubic inches.

$$V = LWH \quad 1,600 = (2W)W(8) \quad 1,600 = 16W^2 \quad 100 = W^2 \quad 10 = W$$

The dimensions are $L = 20$ in., $W = 10$ in., and $H = 8$ in.

Analysis

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

Access these online resources for additional

instruction and practice with models and applications of linear equations.

- Problem solving using linear equations
- Problem solving using equations
- Finding the dimensions of area given the perimeter
- Find the distance between the cities using the distance = rate * time formula
- Linear equation application (Write a cost equation)

Key Concepts

- A linear equation can be used to solve for an unknown in a number problem. See [link].
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See [link].
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d=rt$ formula. See [link].
- Many geometry problems are solved using the perimeter formula $P=2L+2W$, the area formula $A=LW$, or the volume formula

$V = LWH.$ See [link], [link], and [link].

Section Exercises

Verbal

To set up a model linear equation to fit real-world applications, what should always be the first step?

Answers may vary. Possible answers: We should define in words what our variable is representing. We should declare the variable. A heading.

Use your own words to describe this equation where *n* is a number:

$$5(n+3)=2n$$

If the total amount of money you had to invest was $2,000 and you deposit x amount in one investment, how can you represent the remaining amount?

$2{,}000 - x$

If a man sawed a 10-ft board into two sections and one section was n ft long, how long would the other section be in terms of n ?

If Bill was traveling v mi/h, how would you represent Daemon's speed if he was traveling 10 mi/h faster?

$v + 10$

Real-World Applications

For the following exercises, use the information to find a linear algebraic equation model to use to answer the question being asked.

Mark and Don are planning to sell each of their marble collections at a garage sale. If Don has 1 more than 3 times the number of marbles Mark has, how many does each boy have to sell if the total number of marbles is 113?

Beth and Ann are joking that their combined ages equal Sam's age. If Beth is twice Ann's age

and Sam is 69 yr old, what are Beth and Ann's ages?

Ann: 23; Beth: 46

Ben originally filled out 8 more applications than Henry. Then each boy filled out 3 additional applications, bringing the total to 28. How many applications did each boy originally fill out?

For the following exercises, use this scenario: Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of \$20 and charges of \$.05/min for calls. Company B has a monthly fee of \$5 and charges \$.10/min for calls.

Find the model of the total cost of Company A's plan, using m for the minutes.

$20+0.05m$

Find the model of the total cost of Company B's plan, using m for the minutes.

Find out how many minutes of calling would make the two plans equal.

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300 min

If the person makes a monthly average of 200 min of calls, which plan should for the person choose?

For the following exercises, use this scenario: A wireless carrier offers the following plans that a person is considering. The Family Plan: $90 monthly fee, unlimited talk and text on up to 8 lines, and data charges of $40 for each device for up to 2 GB of data per device. The Mobile Share Plan: $120 monthly fee for up to 10 devices, unlimited talk and text for all the lines, and data charges of $35 for each device up to a shared total of 10 GB of data. Use P for the number of devices that need data plans as part of their cost.

Find the model of the total cost of the Family Plan.

$90 + 40P$

Find the model of the total cost of the Mobile

Share Plan.

Assuming they stay under their data limit, find the number of devices that would make the two plans equal in cost.

---

6 devices

If a family has 3 smart phones, which plan should they choose?

For exercises 17 and 18, use this scenario: A retired woman has \$50,000 to invest but needs to make \$6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put in a CD that pays 7%.

If we let x be the amount the woman invests in the 15% bond, how much will she be able to invest in the CD?

$50,000 - x$

Set up and solve the equation for how much the woman should invest in each option to sustain a

$6,000 annual return.

Two planes fly in opposite directions. One travels 450 mi/h and the other 550 mi/h. How long will it take before they are 4,000 mi apart?

---

4 h

Ben starts walking along a path at 4 mi/h. One and a half hours after Ben leaves, his sister Amanda begins jogging along the same path at 6 mi/h. How long will it be before Amanda catches up to Ben?

Fiora starts riding her bike at 20 mi/h. After a while, she slows down to 12 mi/h, and maintains that speed for the rest of the trip. The whole trip of 70 mi takes her 4.5 h. For what distance did she travel at 20 mi/h?

She traveled for 2 h at 20 mi/h, or 40 miles.

A chemistry teacher needs to mix a 30% salt solution with a 70% salt solution to make 20 qt of a 40% salt solution. How many quarts of each solution should the teacher mix to get the

desired result?

Paul has \$20,000 to invest. His intent is to earn 11% interest on his investment. He can invest part of his money at 8% interest and part at 12% interest. How much does Paul need to invest in each option to make get a total 11% return on his \$20,000?

\$5,000 at 8% and \$15,000 at 12%

For the following exercises, use this scenario: A truck rental agency offers two kinds of plans. Plan A charges \$75/wk plus \$.10/mi driven. Plan B charges \$100/wk plus \$.05/mi driven.

Write the model equation for the cost of renting a truck with plan A.

Write the model equation for the cost of renting a truck with plan B.

$B = 100 + .05x$

Find the number of miles that would generate the same cost for both plans.

If Tim knows he has to travel 300 mi, which plan should he choose?

Plan A

For the following exercises, use the given formulas to answer the questions.

$A=P(1+rt)$ is used to find the principal amount *P* deposited, earning *r*% interest, for *t* years. Use this to find what principal amount *P* David invested at a 3% rate for 20 yr if $A=\$8,000$.

The formula $F=\frac{mv^2}{R}$ relates force (*F*), velocity (*v*), mass (*m*), and resistance (*R*). Find R when $m=45$, $v=7$, and $F=245$.

$R=9$

$F=ma$ indicates that force (*F*) equals mass (*m*) times acceleration (*a*). Find the acceleration of a mass of 50 kg if a force of 12 N is exerted on it.

$\text{Sum}=\frac{1}{1-r}$ is the formula for an infinite

series sum. If the sum is 5, find r.

$r=\frac{4}{5}$ or 0.8

For the following exercises, solve for the given variable in the formula. After obtaining a new version of the formula, you will use it to solve a question.

Solve for *W*: $P=2L+2W$

Use the formula from the previous question to find the width, W, of a rectangle whose length is 15 and whose perimeter is 58.

$W=\frac{P-2L}{2}=\frac{58-2(15)}{2}=14$

Solve for f: $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$

Use the formula from the previous question to find f when $p=8$ and $q=13$.

$f=\frac{pq}{p+q}=\frac{8(13)}{8+13}=\frac{104}{21}$

Solve for m in the slope-intercept formula:

$y=mx+b$

Use the formula from the previous question to find m when the coordinates of the point are $(4,7)$ and $b=12$.

$m=-\frac{5}{4}$

The area of a trapezoid is given by $A=\frac{1}{2}h(b_1+b_2)$. Use the formula to find the area of a trapezoid with $h=6$, $b_1=14$, and $b_2=8$.

Solve for *h:* $A=\frac{1}{2}h(b_1+b_2)$

$h=\frac{2A}{b_1+b_2}$

Use the formula from the previous question to find the height of a trapezoid with $A=150$, $b_1=19$, and $b_2=11$.

Find the dimensions of an American football field. The length is 200 ft more than the width, and the perimeter is 1,040 ft. Find the length and width. Use the perimeter formula $P=2L+2W$.

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length $=$ 360 ft; width $=$ 160 ft

Distance equals rate times time, $d=rt.$ Find the distance Tom travels if he is moving at a rate of 55 mi/h for 3.5 h.

Using the formula in the previous exercise, find the distance that Susan travels if she is moving at a rate of 60 mi/h for 6.75 h.

---

405 mi

What is the total distance that two people travel in 3 h if one of them is riding a bike at 15 mi/h and the other is walking at 3 mi/h?

If the area model for a triangle is $A=\frac{1}{2}bh,$ find the area of a triangle with a height of 16 in. and a base of 11 in.

$A=88\text{ in}.^2$

Solve for *h:* $A=\frac{1}{2}bh$

Use the formula from the previous question to find the height to the nearest tenth of a triangle with a base of 15 and an area of 215.

28.7

The volume formula for a cylinder is $V = \pi r 2$ h. Using the symbol π in your answer, find the volume of a cylinder with a radius, r, of 4 cm and a height of 14 cm.

Solve for *h:* $V = \pi r 2 h$

$h = V \pi r 2$

Use the formula from the previous question to find the height of a cylinder with a radius of 8 and a volume of 16π

Solve for *r:* $V = \pi r 2 h$

$r = V \pi h$

Use the formula from the previous question to find the radius of a cylinder with a height of 36

and a volume of 324π.

The formula for the circumference of a circle is $C = 2\pi r$. Find the circumference of a circle with a diameter of 12 in. (diameter $= 2r$). Use the symbol π in your final answer.

$C = 12\pi$

Solve the formula from the previous question for π. Notice why π is sometimes defined as the ratio of the circumference to its diameter.

Glossary

area
: in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A = LW$

perimeter
: in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P = 2L + 2W$

volume
: in cubic units, the volume measurement

includes length, width, and depth: $V = LWH$

Systems of Linear Equations: Two Variables
In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

(credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider

linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$2x + y = 15$ $3x - y = 5$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$2(4)+(7)=15 \quad \text{True} \quad 3(4)-(7)=5 \quad \text{True}$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts.

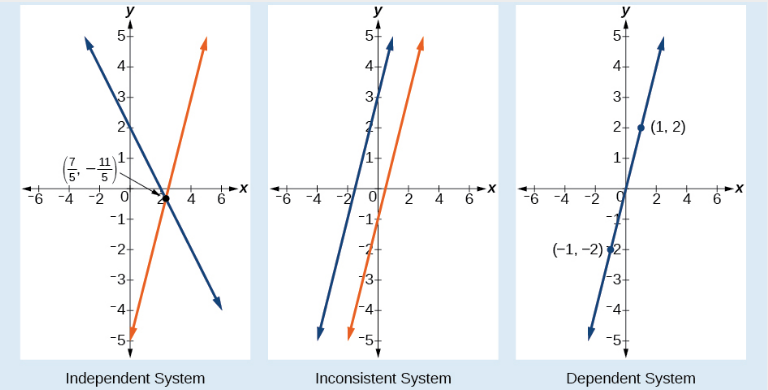
There are no points common to both lines; hence, there is no solution to the system.

Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair (x,y). The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

[link] compares graphical representations of each type of system.



Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5,1)$ is a solution to the given system of equations.

$x+3y=8 \quad 2x-9=y$

Substitute the ordered pair $(5,1)$ into both equations.

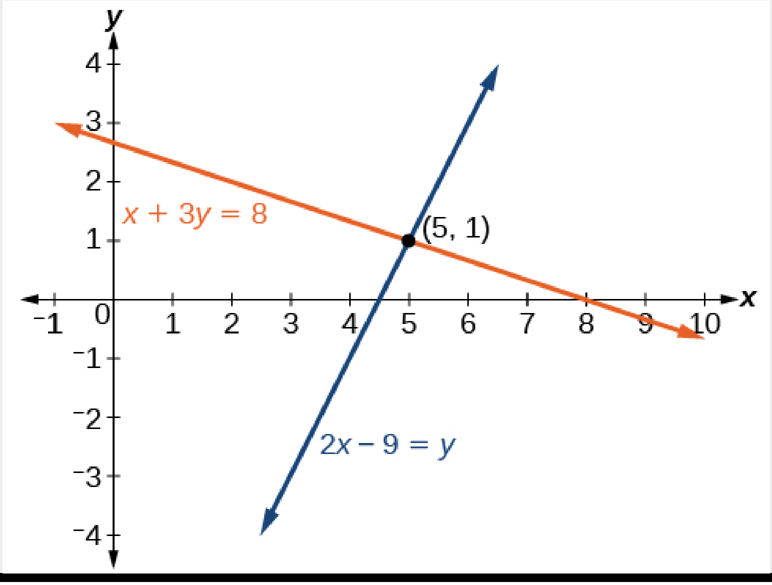
$$(5)+3(1)=8 \qquad 8=8 \text{ True}$$
$$2(5)-9=(1) \qquad 1=1 \text{ True}$$

The ordered pair $(5,1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the

graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See [link].



Determine whether the ordered pair $(8,5)$ is a solution to the following system.
$5x-4y=20 \quad 2x+1=3y$

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$2x+y=-8 \quad x-y=-1$

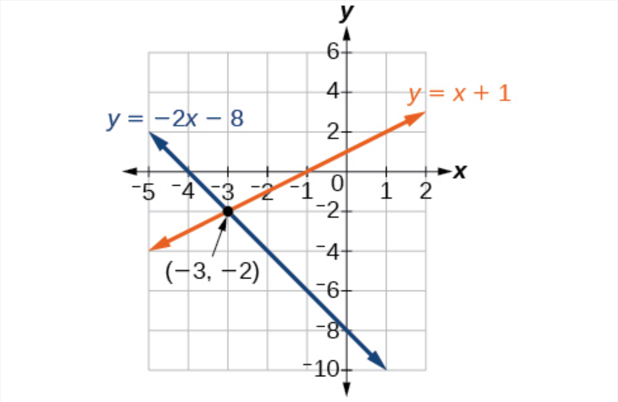
Solve the first equation for y.

$2x+y=-8 \qquad y=-2x-8$

Solve the second equation for y.

$x-y=-1 \qquad y=x+1$

Graph both equations on the same set of axes as in [link].



The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

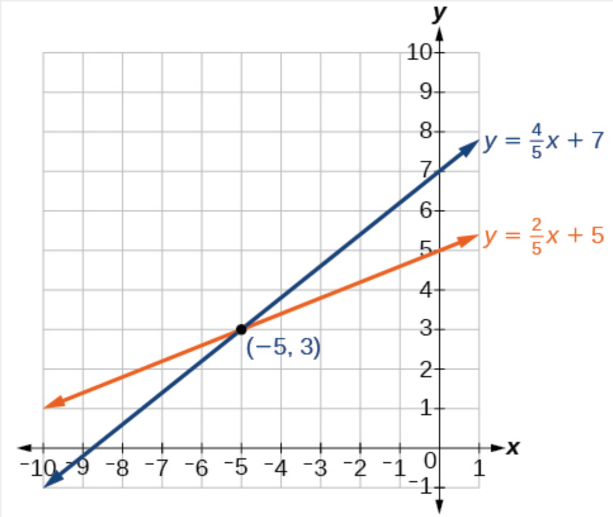
$$2(-3)+(-2)=-8 \qquad -8=-8 \text{ True}$$
$$(-3)-(-2)=-1 \qquad -1=-1 \text{ True}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

Solve the following system of equations by graphing.

$$2x-5y=-25 \quad -4x+5y=35$$

The solution to the system is the ordered pair $(-5,3)$.



Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable.

Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Solving a System of Equations in Two

Variables by Substitution

Solve the following system of equations by substitution.
$-x+y=-5 \quad 2x-5y=1$

First, we will solve the first equation for y.
$-x+y=-5 \qquad y=x-5$

Now we can substitute the expression $x-5$ for y in the second equation.
$2x-5y=1 \quad 2x-5(x-5)=1 \quad 2x-5x+25=1 \qquad -3x=-24 \qquad x=8$

Now, we substitute $x=8$ into the first equation and solve for y.
$-(8)+y=-5 \qquad y=3$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.
$-x+y=-5 \quad -(8)+(3)=-5$ True $\qquad 2x-5y=1 \quad 2(8)-5(3)=1$ True

Solve the following system of equations by

substitution.

$x = y + 3$ $4 = 3x - 2y$

$(-2, -5)$

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Solving a System by the Addition Method

Solve the given system of equations by addition.

$x+2y=-1 \quad -x+y=3$

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x+2y &= -1 \\ -x+y &= 3 \\ \hline 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

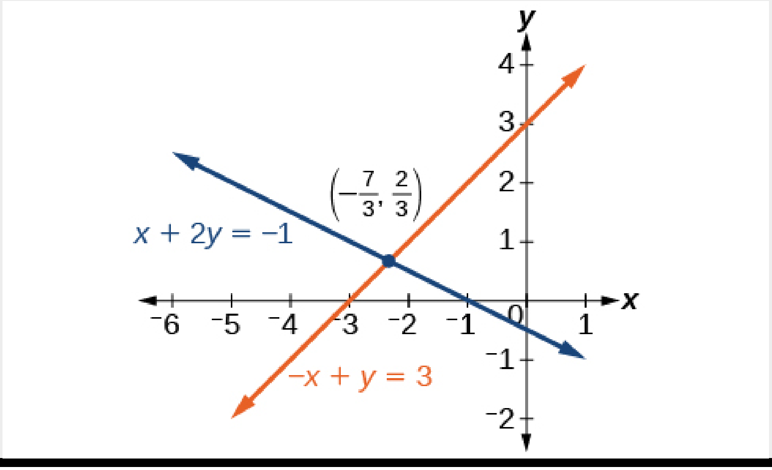
Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= \\ -\frac{3}{3} &= \\ -1 &= -1 \quad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See [link] to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has

exactly one solution.



Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$3x+5y=-11 \quad x-2y=11$

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the *x*-terms will add to zero.

$x-2y=11 \quad -3(x-2y)=-3(11)$

Multiply both sides by -3. $\quad -3x+6y=-33$

Use the distributive property.

Now, let’s add them.

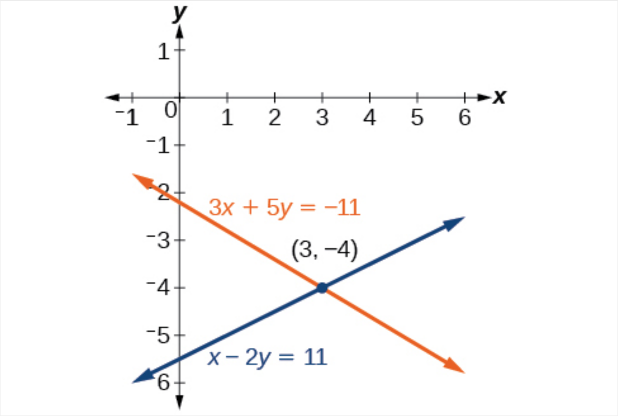
$$3x + 5y = -11 \quad -3x + 6y = -33 \quad \underline{\hspace{3cm}}$$
$$11y = -44 \qquad y = -4$$

For the last step, we substitute $y = -4$ into one of the original equations and solve for x.

$$3x + 5y = -11 \quad 3x + 5(-4) = -11 \quad 3x - 20 = -11 \quad 3x = 9 \quad x = 3$$

Our solution is the ordered pair $(3, -4)$. See [link]. Check the solution in the original second equation.

$$x - 2y = 11 \; (3) - 2(-4) = 3 + 8$$
$$11 = 11 \text{ True}$$



Solve the system of equations by addition.

$2x - 7y = 2 \qquad 3x + y = -20$

$(-6, -2)$

Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$2x + 3y = -16 \quad 5x - 10y = 30$

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$-5(2x + 3y) = -5(-16) \qquad -10x - 15y = 80$

$2(5x - 10y) = 2(30) \qquad 10x - 20y = 60$

Then, we add the two equations together.

$-10x - 15y = 80 \qquad 10x - 20y = 60$

$\underline{\hspace{3cm}} \qquad -35y = 140 \qquad y = -4$

Substitute $y = -4$ into the original first equation.

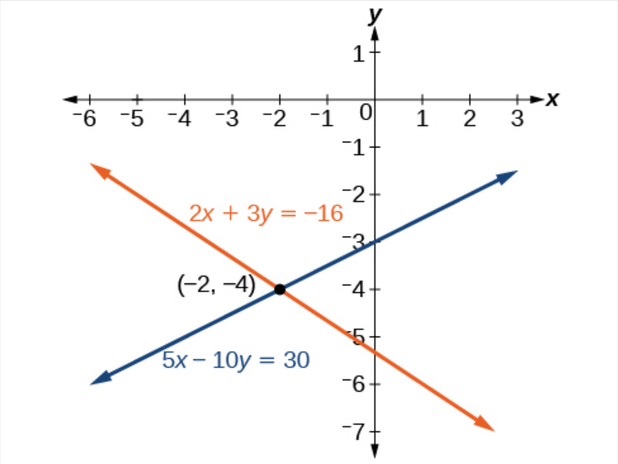
$2x + 3(-4) = -16 \qquad 2x - 12 = -16$

$2x = -4 \qquad x = -2$

The solution is $(-2, -4)$. Check it in the other equation.

$$5x - 10y = 30 \quad 5(-2) - 10(-4) = 30$$
$$-10 + 40 = 30 \qquad 30 = 30$$

See [link].



Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

x 3 + y 6 =3 x 2 − y 4 = 1

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$6(x\ 3\ +\ y\ 6\ )=6(3) \quad 2x+y=18\ 4(x\ 2\ -\ y\ 4\ )=4(1) \quad 2x-y=4$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.

$-1(2x-y)=-1(4) \quad -2x+y=-4$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.

$2x+y=18\ -2x+y=-4$ __________

$2y=14 \quad y=7$

Substitute $y=7$ into the first equation.

$2x+(7)=18 \quad 2x=11 \quad x=\ 11\ 2 \quad =5.5$

The solution is $(\ 11\ 2\ ,7\ )$. Check it in the other equation.

$x\ 2\ -\ y\ 4\ =1\ 11\ 2\ 2\ -\ 7\ 4\ =1\quad 11\ 4\ -\ 7\ 4\ =1 \quad 4\ 4\ =1$

Solve the system of equations by addition.

$2x+3y=8 \quad 3x+5y=10$

$(10, -4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Solving an Inconsistent System of Equations

Solve the following system of equations.

$$x = 9 - 2y \quad x + 2y = 13$$

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$x+2y=13 \quad (9-2y)+2y=13$$
$$9+0y=13 \quad 9=13$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$x=9-2y \quad 2y=-x+9 \quad y=-\frac{1}{2}x+\frac{9}{2}$$

We then convert the second equation expressed to slope-intercept form.

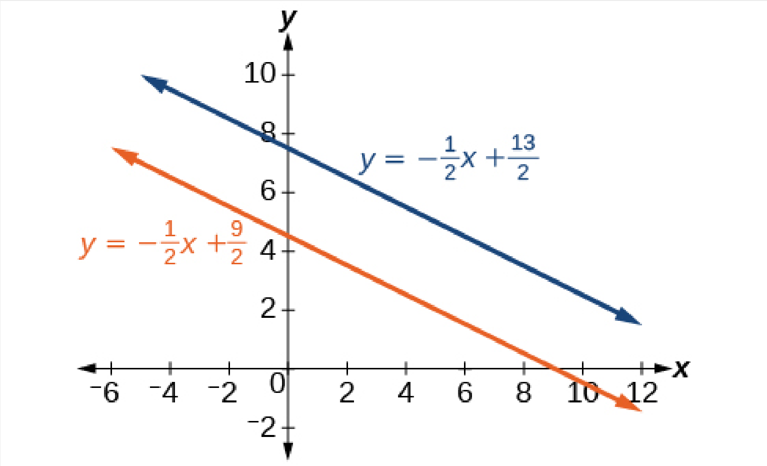
$$x+2y=13 \quad 2y=-x+13 \quad y=-\frac{1}{2}x+\frac{13}{2}$$

Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.

$$y=-\frac{1}{2}x+\frac{9}{2} \quad y=-\frac{1}{2}x+\frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in [link].



Solve the following system of equations in two variables.

$2y - 2x = 2 \quad 2y - 2x = 6$

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0=0$.

Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.

$x+3y=2$ $3x+9y=6$

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$x+3y=2 \quad (-3)(x+3y)=(-3)(2)$$
$$-3x-9y=-6$$

Now add the equations.

$$-3x-9y=-6 \quad + \quad 3x+9y=6 \quad \underline{\qquad\qquad}$$
$$0=0$$

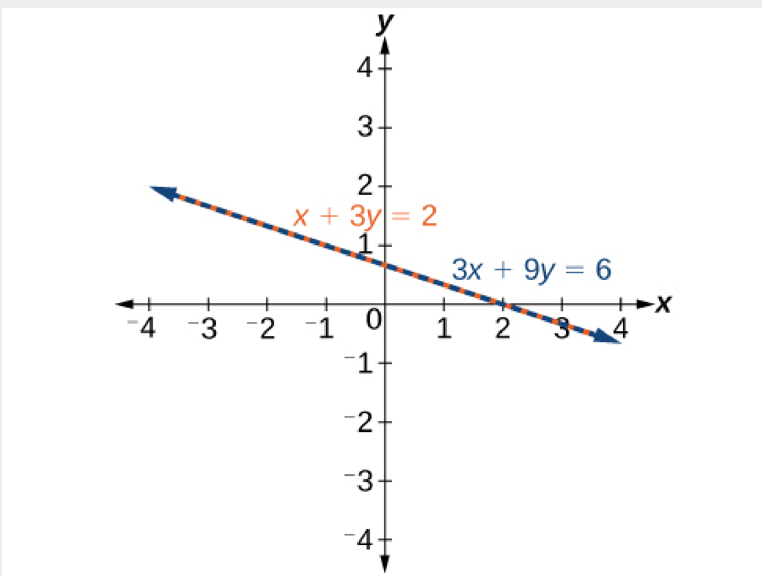
We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let’s look at what happens when we convert the system to slope-intercept form.

$$\begin{array}{lll} x+3y=2 & 3y=-x+2 & y=-\frac{1}{3}x+\frac{2}{3} \\ 3x+9y=6 & 9y=-3x+6 & y=-\frac{3}{9}x+\frac{6}{9} \\ & & y=-\frac{1}{3}x+\frac{2}{3} \end{array}$$

See [link]. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x+\frac{2}{3}\right)$, for x in the set of real numbers.



Solve the following system of equations in two variables.

$$y-2x=5 \quad -3y+6x=-15$$

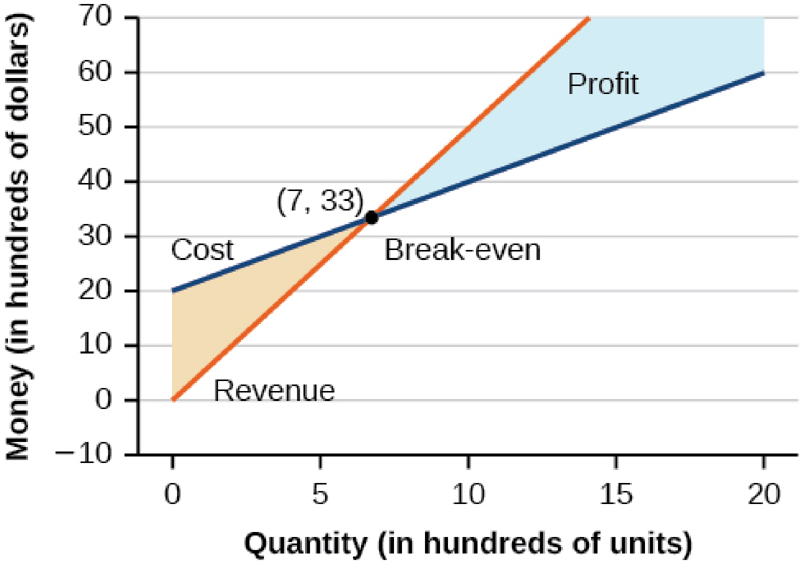
The system is dependent so there are infinite solutions of the form $(x,2x+5)$, for x in the set of real numbers.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R=xp$, where $x=$ quantity and $p=$ price. The revenue function is shown in orange in [link].

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as

utilities. The cost function is shown in blue in [link]. The x -axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.



The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is \$3,300 and the revenue is also \$3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x)=R(x)-C(x)$. Clearly, knowing the quantity for

which the cost equals the revenue is of great importance to businesses.

Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Write the system of equations using y to replace function notation.

$y = 0.85x + 35{,}000$ $y = 1.55x$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.

$0.85x + 35{,}000 = 1.55x$ $35{,}000 = 0.7x$
$50{,}000 = x$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

$1.55(50{,}000) = 77{,}500$

The break-even point is $(50{,}000, 77{,}500)$.

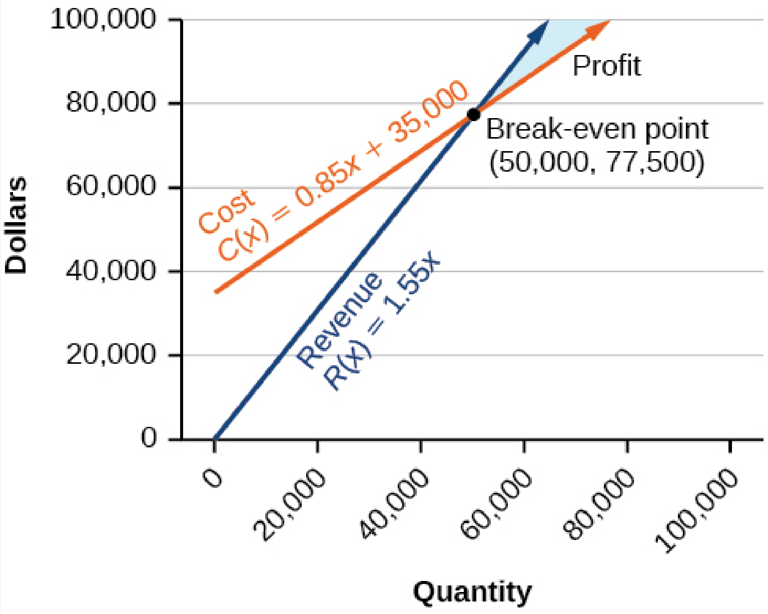
The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$P(x) = 1.55x - (0.85x + 35,000) \qquad = 0.7x - 35,000$$

The profit function is $P(x) = 0.7x - 35,000.$

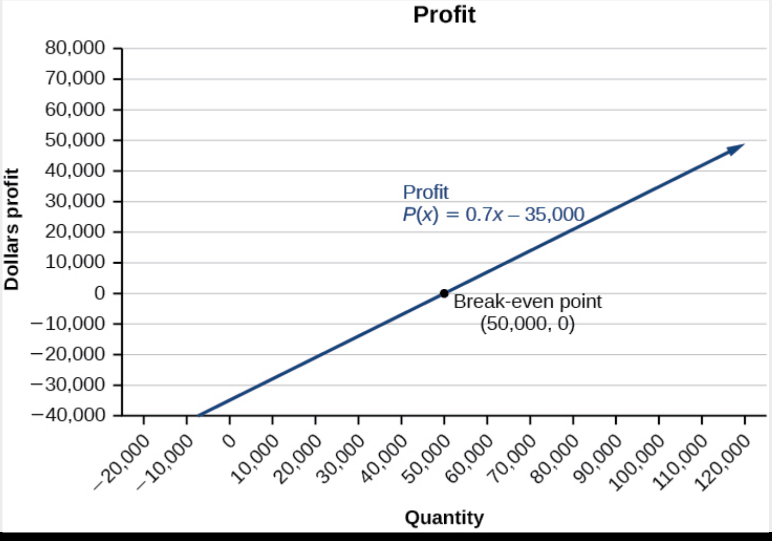
Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See [link].



We see from the graph in [link] that the profit function has a negative value until $x = 50,000,$ when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on

this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.



Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000.$ How many children and how many adults bought tickets?

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000.$ We can use this to write an equation for the number of people at the circus that day.

$c+a=2,000$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c.$ The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a.$ The total revenue is $\$70,000.$ We can use this to write an equation for the revenue.

$25c+50a=70,000$

We now have a system of linear equations in two variables.

$$c+a=2,000 \quad 25c+50a=70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or $a.$ We will solve for $a.$

$$c+a=2,000 \qquad\qquad a=2,000-c$$

Substitute the expression $2,000-c$ in the second equation for a and solve for $c.$

$$25c+50(2,000-c)=70,000 \quad 25c+100,000-50c=70,000$$
$$-25c=-30,000$$

$c = 1{,}200$

Substitute $c = 1{,}200$ into the first equation to solve for a.

$1{,}200 + a = 2{,}000$ $\quad\quad a = 800$

We find that $1{,}200$ children and 800 adults bought tickets to the circus that day.

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1{,}650$ meal tickets were bought for a total of $\$14{,}200$, how many children and how many adults bought meal tickets?

700 children, 950 adults

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See [link].
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See [link].
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See [link].
- A third method of solving a system of linear equations is by addition, in which we can

eliminate a variable by adding opposite coefficients of corresponding variables. See [link].
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See [link], [link], and [link].
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See [link].
- The solution to a system of dependent equations will always be true because both equations describe the same line. See [link].
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See [link] and [link].

Section Exercises

Verbal

Can a system of linear equations have exactly two solutions? Explain why or why not.

No, you can either have zero, one, or infinitely many. Examine graphs.

If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

Given a system of equations, explain at least two different methods of solving that system.

You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

$5x - y = 4 \;\; x + 6y = 2$ and $(4,0)$

$-3x - 5y = 13 \quad -x + 4y = 10$ and $(-6,1)$

Yes

$3x + 7y = 1 \;\; 2x + 4y = 0$ and $(2,3)$

$-2x + 5y = 7 \quad 2x + 9y = 7$ and $(-1,1)$

Yes

$x + 8y = 43 \;\; 3x - 2y = -1$ and $(3,5)$

For the following exercises, solve each system by substitution.

$x + 3y = 5 \;\; 2x + 3y = 4$

$(-1,2)$

$3x-2y=18\ 5x+10y=-10$

$4x+2y=-10\ 3x+9y=0$

$(-3,1)$

$2x+4y=-3.8\ 9x-5y=1.3$

$-2x+3y=1.2\ -3x-6y=1.8$

$(-\ 3\ 5\ ,0\ )$

$x-0.2y=1\ -10x+2y=5$

$3x+5y=9\ 30x+50y=-90$

No solutions exist.

$-3x+y=2\ 12x-4y=-8$

$1\ 2\ x+\ 1\ 3\ y=16\ 1\ 6\ x+\ 1\ 4\ y=9$

$(\ 72\ 5\ ,\ 132\ 5\ )$

$-\ 1\ 4\ x+\ 3\ 2\ y=11\ -\ 1\ 8\ x+\ 1\ 3\ y=3$

For the following exercises, solve each system by addition.

$-2x+5y=-42\quad 7x+2y=30$

$(\ 6,-6\ )$

$6x-5y=-34\ 2x+6y=4$

$5x-y=-2.6\ -4x-6y=1.4$

$(\ -\ 1\ 2\ ,1\ 10\ )$

$7x-2y=3\ 4x+5y=3.25$

$-x+2y=-1\ 5x-10y=6$

No solutions exist.

$$7x+6y=2 \quad -28x-24y=-8$$

$$5\ 6\ x+\ 1\ 4\ y=0\ 1\ 8\ x-\ 1\ 2\ y=-\ 43\ 120$$

(− 1 5 , 2 3)

$$1\ 3\ x+\ 1\ 9\ y=\ 2\ 9\ -\ 1\ 2\ x+\ 4\ 5\ y=-\ 1\ 3$$

$$-0.2x+0.4y=0.6 \qquad x-2y=-3$$

(x, x+3 2)

$$-0.1x+0.2y=0.6 \qquad 5x-10y=1$$

For the following exercises, solve each system by any method.

$$5x+9y=16 \quad x+2y=4$$

$(-4,4)$

$6x-8y=-0.6$ $3x+2y=0.9$

$5x-2y=2.25$ $7x-4y=3$

(1 2 , 1 8)

x− 5 12 y=− 55 12 −6x+ 5 2 y= 55 2

$7x-4y=$ 7 6 $2x+4y=$ 1 3

(1 6 ,0)

$3x+6y=11$ $2x+4y=9$

7 3 x− 1 6 y=2 − 21 6 x+ 3 12 y=−3

(x,2(7x−6))

1 2 x+ 1 3 y= 1 3 3 2 x+ 1 4 y=− 1 8

$2.2x+1.3y=-0.1$ $4.2x+4.2y=2.1$

$(-56, 43)$

$0.1x + 0.2y = 2$ $0.35x - 0.3y = 0$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

$3x - y = 0.6$ $x - 2y = 1.3$

Consistent with one solution

$-x + 2y = 4$ $2x - 4y = 1$

$x + 2y = 7$ $2x + 6y = 12$

Consistent with one solution

$3x - 5y = 7$ $x - 2y = 3$

$3x-2y=5 \quad -9x+6y=-15$

Dependent with infinitely many solutions

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

$0.1x+0.2y=0.3 \quad -0.3x+0.5y=1$

$-0.01x+0.12y=0.62 \quad 0.15x+0.20y=0.52$

$(-3.08, 4.91)$

$0.5x+0.3y=4 \quad 0.25x-0.9y=0.46$

$0.15x+0.27y=0.39 \quad -0.34x+0.56y=1.8$

$(-1.52, 2.29)$

$-0.71x+0.92y=0.13 \quad 0.83x+0.05y=2.1$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where A–F are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

$$\begin{aligned} x + y &= A \\ x - y &= B \end{aligned}$$

$$\left(\frac{A+B}{2}, \frac{A-B}{2} \right)$$

$$\begin{aligned} x + Ay &= 1 \\ x + By &= 1 \end{aligned}$$

$$\begin{aligned} Ax + y &= 0 \\ Bx + y &= 1 \end{aligned}$$

$$\left(\frac{-1}{A-B}, \frac{A}{A-B} \right)$$

$$\begin{aligned} Ax + By &= C \\ x + y &= 1 \end{aligned}$$

$$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

$$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE} \right)$$

Real-World Applications

For the following exercises, solve for the desired quantity.

A stuffed animal business has a total cost of production $C=12x+30$ and a revenue function $R=20x$. Find the break-even point.

A fast-food restaurant has a cost of production $C(x)=11x+120$ and a revenue function $R(x)=5x$. When does the company start to turn a profit?

They never turn a profit.

A cell phone factory has a cost of production $C(x)=150x+10{,}000$ and a revenue function $R(x)=200x$. What is the break-even point?

A musician charges $C(x)=64x+20{,}000$, where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

$(1{,}250, 100{,}000)$

A guitar factory has a cost of production $C(x) = 75x + 50{,}000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

Find two numbers whose sum is 28 and difference is 13.

The numbers are 7.5 and 20.5.

A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

24,000

A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

---

790 sophomores, 805 freshman

276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

---

56 men, 74 women

A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

---

10 gallons of 10% solution, 15 gallons of 60% solution

An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside

Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Swan Peak: $750,000, Riverside: $350,000

If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays 2 7 8 % interest, and the investor earns $737.50 annual interest, how much was invested in each account?

If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

$12,500 in the first account, $10,500 in the second account.

CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

A store clerk sold 60 pairs of sneakers. The high-tops sold for \$98.99 and the low-tops sold for \$129.99. If the receipts for the two types of sales totaled \$6,404.40, how many of each type of sneaker were sold?

High-tops: 45, Low-tops: 15

A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was \$12.50, and the price for an adult ticket was \$16.00. The register confirms that \$5,075 was taken in. How many student tickets and adult tickets were sold?

Admission into an amusement park for 4 children and 2 adults is \$116.90. For 6 children and 3 adults, the admission is \$175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

Infinitely many solutions. We need more information.

Glossary

addition method

an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

break-even point

the point at which a cost function intersects a revenue function; where profit is zero

consistent system

a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function

the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system

a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

inconsistent system

a system of linear equations with no common

solution because they represent parallel lines, which have no point or line in common

independent system
: a system of linear equations with exactly one solution pair (x, y)

profit function
: the profit function is written as $P(x) = R(x) - C(x)$, revenue minus cost

revenue function
: the function that is used to calculate revenue, simply written as $R = xp$, where $x =$ quantity and $p =$ price

substitution method
: an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations
: a set of two or more equations in two or more variables that must be considered simultaneously.

Systems of Linear Equations: Three Variables
In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

(credit: “Elembis,” Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John

invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution

(x,y,z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$Ax + By + Cz = D \qquad Ey + Fz = G \qquad Hz = K$$

The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

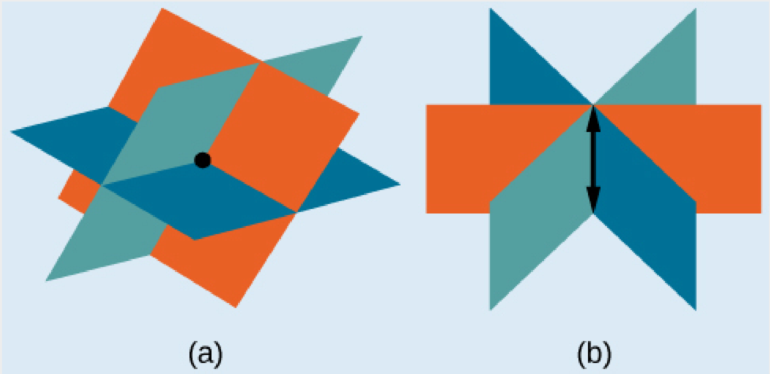
The **solution set** to a three-by-three system is an ordered triple $\{(x,y,z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Number of Possible Solutions

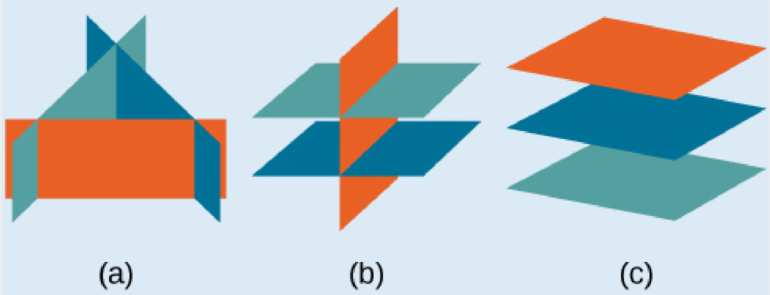
[link] and [link] illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{ (x,y,z) \}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0=0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3=0$. Graphically, a system with no solution is represented by three planes with no point in common.

(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.



Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$x+y+z=2 \quad 6x-4y+5z=31 \quad 5x+2y$$

$+2z=13$

We will check each equation by substituting in the values of the ordered triple for $x, y,$ and z.

$x+y+z=2$ $(3)+(-2)+(1)=2$ True $6x-4y+5z=31$ $6(3)-4(-2)+5(1)=31$ $18+8+5=31$ True $5x+2y+2z=13$ $5(3)+2(-2)+2(1)=13$ $15-4+2=13$ True

The ordered triple $(3,-2,1)$ is indeed a solution to the system.

Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$x-2y+3z=9 \ (1) \qquad -x+3y-z=-6 \ (2)$$
$$2x-5y+5z=17 \ (3)$$

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$x-2y+3z=9 \ (1) \qquad -x+3y-z=-6 \ \ (2)$$
$$y+2z=3 \qquad (3)$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$-2x+4y-6z=-18 \ (1) \text{ multiplied by } -2 \quad 2x$$
$$-5y+5z=17 \ (3) \ \underline{\qquad\qquad\qquad\qquad}$$
$$-y-z=-1 \ \ (5)$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$y+2z=3 \qquad (4) \ \ -y-z=-1 \ \ (5)$$
$$z=2 \qquad (6)$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$x-2y+3z=9 \ \ (1) \qquad y+2z=3 \ (4)$$

$z = 2 \quad (6)$

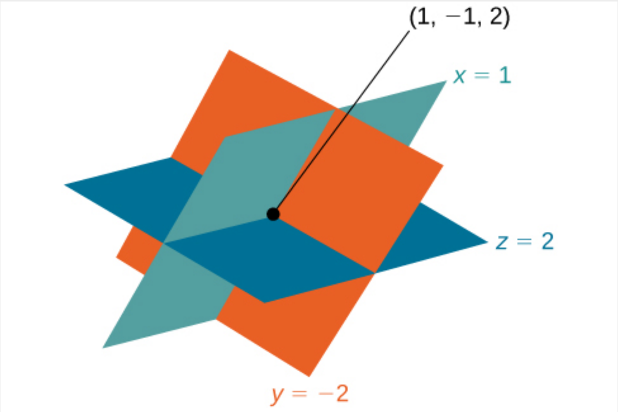
Next, we back-substitute $z = 2$ into equation (4) and solve for y.

$y + 2(2) = 3 \qquad y + 4 = 3 \qquad y = -1$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

$x - 2(-1) + 3(2) = 9 \qquad x + 2 + 6 = 9$
$x = 1$

The solution is the ordered triple $(1, -1, 2)$. See [link].



Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$x =$ amount invested in money-market fund
$y =$ amount invested in municipal bonds
$z =$ amount invested in mutual funds

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x + y + z = 12{,}000$$

We form the second equation according to the information that John invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z = y + 4{,}000$$

The third equation shows that the total

amount of interest earned from each fund equals $670.
$0.03x+0.04y+0.07z=670$

Then, we write the three equations as a system.

$$x+y+z=12,000 \quad -y+z=4,000 \quad 0.03x+0.04y+0.07z=670$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$x+\ y+z\ =12,000\ (1) \qquad -y+z\ =4,000\ (2) \quad 3x+4y+7z=67,000\ (3)$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$x+\ y\ +\ z=12,000 \quad 3x+4y\ +7z=67,000 \quad -y\ +\ z=4,000$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$x+y+z\ =12,000 \qquad y+4z=31,000 \qquad -y+z\ =4,000$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$x+y+\ z=12,000 \qquad y+4z=31,000 \qquad 5z\ =35,000$$

Step 4. Solve for z in equation (3). Back-

substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$5z = 35{,}000$$
$$z = 7{,}000 \qquad y$$
$$+ 4(7{,}000) = 31{,}000 \qquad y = 3{,}000$$
$$x + 3{,}000 + 7{,}000 = 12{,}000$$
$$x = 2{,}000$$

John invested $2,000 in a money-market fund, $3,000 in municipal bonds, and $7,000 in mutual funds.

Solve the system of equations in three variables.

$2x + y - 2z = -1$ $3x - 3y - z = 5$ $x - 2y + 3z = 6$

$(1, -1, 1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$x-3y+z=4$ (1) $-x+2y-5z=3$ (2) $5x-13y+13z=8$ (3)

Looking at the coefficients of x, we can see that we can eliminate x by adding equation (1) to equation (2).

$x-3y+z=4$ (1) $-x+2y-5z=3$ (2)
$-y-4z=7$ (4)

Next, we multiply equation (1) by -5 and add it to equation (3).

$-5x+15y-5z=-20$ (1) multiplied by -5 $5x-13y+13z=8$ (3)

$$2y + 8z = -12 \quad (5)$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{array}{rl} -2y - 8z = 14 & \text{(4) multiplied by 2} \\ 2y + 8z = -12 & (5) \\ \hline 0 = 2 & \end{array}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Solve the system of three equations in three variables.

$$\begin{aligned} x + y + z &= 2 \\ y - 3z &= 1 \\ 2x + y + 5z &= 0 \end{aligned}$$

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$2x + y - 3z = 0 \ (1) \ 4x + 2y - 6z = 0 \ (2) \quad x - y + z = 0 \ (3)$

First, we can multiply equation (1) by -2 and add it to equation (2).

$-4x-2y+6z=0$ equation (1) multiplied by -2 $4x+2y-6z=0$ (2)

$$0=0$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$2x+y-3z=0$ $x-y+z=0$ $3x-2z=0$

We then solve the resulting equation for z.

$3x-2z=0$ $z=3 2 x$

We back-substitute the expression for z into one of the equations and solve for y.

$2x+y-3(3 2 x)=0$ $2x+y-9 2 x=0$

$y=9 2 x-2x$

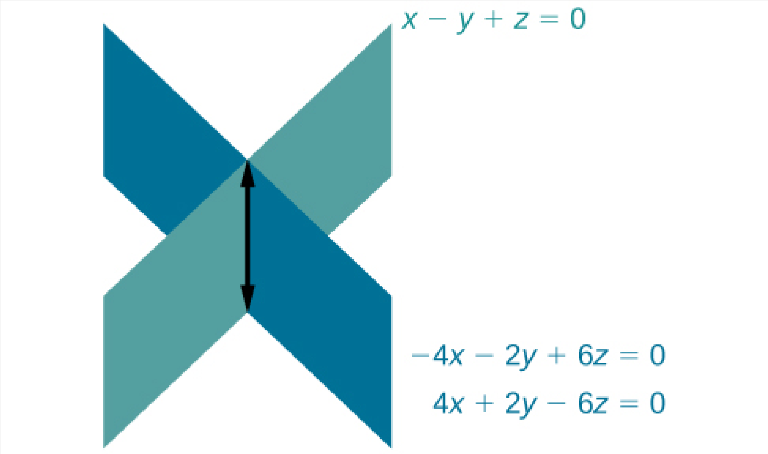
$y=5 2 x$

So the general solution is $(x, 5 2 x, 3 2 x)$. In

this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in [link], two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.



Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Solve the following system.

$$x+y+z=7 \quad 3x-2y-z=4 \quad x+6y+5z=24$$

Infinite number of solutions of the form $(x,4x-11,-5x+18)$.

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple $\{(x,y,z)\}$ that represents the intersection of three planes in space. See [link].
- A system of three equations in three variables

can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See [link].

- Systems of three equations in three variables are useful for solving many different types of real-world problems. See [link].
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See [link].
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See [link].
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Section Exercises

Verbal

Can a linear system of three equations have exactly two solutions? Explain why or why not

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No, there can be only one, zero, or infinitely many solutions.

If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$2x+3y-6z=1 \quad -4x-6y+12z=-2 \quad x+2y+5z=10$$

Using the method of addition, is there only one way to solve the system?

Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

$2x-6y+6z=-12 \quad x+4y+5z=-1 \quad -x+2y+3z=-1$ and $(0,1,-1)$

$6x-y+3z=6 \quad 3x+5y+2z=0 \quad x+y=0$ and $(3,-3,-5)$

No

$6x-7y+z=2$ $-x-y+3z=4$ $2x+y-z=1$ and $(4,2,-6)$

$x-y=0$ $x-z=5$ $x-y+z=-1$ and $(4,4,-1)$

Yes

$-x-y+2z=3$ $5x+8y-3z=4$ $-x+3y-5z=-5$ and $(4,1,-7)$

For the following exercises, solve each system by substitution.

$3x-4y+2z=-15$ $2x+4y+z=16$ $2x+3y+5z=20$

$(-1,4,2)$

$5x-2y+3z=20$ $2x-4y-3z=-9$ $x+6y-8z=21$

$5x+2y+4z=9$ $-3x+2y+z=10$ $4x-3y+5z=-3$

(− 85 107 , 312 107 , 191 107)

$4x-3y+5z=31$ $-x+2y+4z=20$ $x+5y-2z=-29$

$5x-2y+3z=4$ $-4x+6y-7z=-1$ $3x+2y-z=4$

(1, 1 2 ,0)

$4x+6y+9z=0$ $-5x+2y-6z=3$ $7x-4y+3z=-3$

For the following exercises, solve each system by Gaussian elimination.

$2x-y+3z=17$ $-5x+4y-2z=-46$ $2y+5z=-7$

(4,−6,1)

$5x-6y+3z=50$ $-x+4y=10$ $2x-z=10$

$2x+3y-6z=1\ \ -4x-6y+12z=-2\qquad x$
$+2y+5z=10$

$(\ x,\ 1\ 27\ (65-16x),\ x+28\ 27\ )$

$4x+6y-2z=8\ \ 6x+9y-3z=12\ \ -2x-3y$
$+z=-4$

$2x+3y-4z=5\ \ -3x+2y+z=11\ \ -x+5y$
$+3z=4$

$(\ -\ 45\ 13\ ,\ 17\ 13\ ,-2\ )$

$10x+2y-14z=8\quad -x-2y-4z=-1\ \ -12x$
$-6y+6z=-12$

$x+y+z=14\qquad 2y+3z=-14\ \ -16y$
$-24z=-112$

No solutions exist

$5x-3y+4z=-1\ \ -4x+2y-3z=0\ \ -x+5y$
$+7z=-11$

$x+y+z=0$ $2x-y+3z=0$ x
$-z=0$

(0,0,0)

$3x+2y-5z=6$ $5x-4y+3z=-12$ $4x+5y$
$-2z=15$

$x+y+z=0$ $2x-y+3z=0$ $x-z=1$

(4 7 ,− 1 7 ,− 3 7)

$3x-$ 1 2 $y-z=-$ 1 2 $4x+z=3$ $-x+$
3 2 $y=$ 5 2

$6x-5y+6z=38$ 1 5 $x-$ 1 2 $y+$ 3 5 $z=1$
$-4x-$ 3 2 $y-z=-74$

(7,20,16)

1 2 $x-$ 1 5 $y+$ 2 5 $z=-$ 13 10 1 4 $x-$ 2 5
$y-$ 1 5 $z=-$ 7 20 $-$ 1 2 $x-$ 3 4 $y-$ 1 2 $z=-$
5 4

− 1 3 x− 1 2 y− 1 4 z= 3 4 − 1 2 x− 1 4 y − 1 2 z=2 − 1 4 x− 3 4 y− 1 2 z=− 1 2

(−6,2,1)

1 2 x− 1 4 y+ 3 4 z=0 1 4 x− 1 10 y+ 2 5 z=−2 1 8 x+ 1 5 y− 1 8 z=2

4 5 x− 7 8 y+ 1 2 z=1 − 4 5 x− 3 4 y+ 1 3 z=−8 − 2 5 x− 7 8 y+ 1 2 z=−5

(5,12,15)

− 1 3 x− 1 8 y+ 1 6 z=− 4 3 − 2 3 x− 7 8 y+ 1 3 z=− 23 3 − 1 3 x− 5 8 y+ 5 6 z=0

− 1 4 x− 5 4 y+ 5 2 z=−5 − 1 2 x− 5 3 y+ 5 4 z= 55 12 − 1 3 x− 1 3 y+ 1 3 z= 5 3

(−5,−5,−5)

1 40 x+ 1 60 y+ 1 80 z= 1 100 − 1 2 x− 1 3 y− 1 4 z=− 1 5 3 8 x+ 3 12 y+ 3 16 z= 3 20

$0.1x-0.2y+0.3z=2$ $0.5x-0.1y+0.4z=8$ $0.7x-0.2y+0.3z=8$

$(10,10,10)$

$0.2x+0.1y-0.3z=0.2$ $0.8x+0.4y-1.2z=0.1$ $1.6x+0.8y-2.4z=0.2$

$1.1x+0.7y-3.1z=-1.79$ $2.1x+0.5y-1.6z=-0.13$ $0.5x+0.4y-0.5z=-0.07$

(1 2 , 1 5 , 4 5)

$0.5x-0.5y+0.5z=10$ $0.2x-0.2y+0.2z=4$ $0.1x-0.1y+0.1z=2$

$0.1x+0.2y+0.3z=0.37$ $0.1x-0.2y-0.3z=-0.27$ $0.5x-0.1y-0.3z=-0.03$

(1 2 , 2 5 , 4 5)

$0.5x-0.5y-0.3z=0.13$ $0.4x-0.1y-0.3z=0.11$ $0.2x-0.8y-0.9z=-0.32$

$0.5x+0.2y-0.3z=1$ $0.4x-0.6y+0.7z=0.8$
$0.3x-0.1y-0.9z=0.6$

(2,0,0)

$0.3x+0.3y+0.5z=0.6$ $0.4x+0.4y+0.4z=1.8$
$0.4x+0.2y+0.1z=1.6$

$0.8x+0.8y+0.8z=2.4$ $0.3x-0.5y+0.2z=0$
$0.1x+0.2y+0.3z=0.6$

(1,1,1)

Extensions

For the following exercises, solve the system for x,y, and z.

x+y+z=3 x−1 2 + y−3 2 + z+1 2 =0 x−2 3 + y+4 3 + z−3 3 = 2 3

5x−3y− z+1 2 = 1 2 6x+ y−9 2 +2z=−3 x+8 2 −4y+z=4

(128 557 , 23 557 , 28 557)

x+4 7 − y−1 6 + z+2 3 =1 x−2 4 + y+1 8 − z+8 12 =0 x+6 3 − y+2 3 + z+4 2 =3

x−3 6 + y+2 2 − z−3 3 =2 x+2 4 + y−5 2 + z+4 2 =1 x+6 2 − y−3 2 +z+1=9

(6,−1,0)

x−1 3 + y+3 4 + z+2 6 =1 4x +3y−2z=11 0.02x+0.015y−0.01z=0.065

Real-World Applications

Three even numbers sum up to 108. The smaller is half the larger and the middle number is 3 4 the larger. What are the three numbers?

24, 36, 48

Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

---

70 grandparents, 140 parents, 190 children

An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was \$82. She forgot to save the individual receipts but remembered that your groceries were \$0.05 cheaper than half of her groceries, and that your other roommate's groceries were \$2.10 more than your groceries. How much was each of your share of the groceries?

Your share was \$19.95, Sarah's share was \$40, and your other roommate's share was \$22.05.

Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

There are infinitely many solutions; we need more information

At a carnival, $2,914.25 in receipts were taken

at the end of the day. The cost of a child’s ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

---

500 students, 225 children, and 450 adults

In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

Last year, at Haven’s Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same

cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

The BMW was $49,636, the Jeep was $42,636, and the Toyota was $47,727.

A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid 3 1 8 % simple interest, and one that paid 2 1 2 % simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into

the account that pays 3% compared to 2%, how much did you invest in each account?

$400,000 in the account that pays 3% interest, $500,000 in the account that pays 4% interest, and $100,000 in the account that pays 2% interest.

You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?

[footnote]
“Oil reserves, production and consumption in 2001,” accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world’s oil.

The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world’s produced oil. Saudi Arabia and the United States combined for 22.1% of the world’s production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[footnote]
“Oil reserves, production and consumption in 2001,” accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from

Canada. What percent of the United States oil imports were from these three countries? [footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[footnote]
"USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[footnote]

"The United States Meat Industry at a Glance," accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

Glossary

solution set
: the set of all ordered pairs or triples that satisfy all equations in a system of equations

Matrices and Matrix Operations
In this section, you will:

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

(credit: “SD Dirk,” Flickr)


Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. [link] shows the needs of both teams.

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs \$300; a ball costs \$10; and a jersey costs \$30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named $A, B,$ and C are shown below.

A=[1 2 3 4],B=[1 2 7 0 −5 6 7 8 2],C=[−1 0 3 3 2 1]

Describing Matrices

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij}, we look for the entry in row i, column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A square matrix is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

A column matrix is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

Matrices

A **matrix** is a rectangular array of numbers that is usually named by a capital letter: A, B, C, and so on. Each entry in a matrix is referred to as a_{ij}, such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A:

1. What are the dimensions of matrix A?
2. What are the entries at a_{31} and a_{22}?

$A = [\ 2\ 1\ 0\ 2\ 4\ 7\ 3\ 1\ -2\ ]$

1. The dimensions are 3×3 because there are three rows and three columns.
2. Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

Adding and Subtracting Matrices

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or
matrix D of the same dimension.

$A+B=C$ such that $a_{ij} + b_{ij} = c_{ij}$

$A-B=D$ such that $a_{ij} - b_{ij} = d_{ij}$

Matrix addition is commutative.

$A+B=B+A$

It is also associative.

$(A+B)+C=A+(B+C)$

Finding the Sum of Matrices

Find the sum of A and B, given
$A = [\, a\ b\ c\ d \,]$ and $B = [\, e\ f\ g\ h \,]$

Add corresponding entries.
$A + B = [\, a\ b\ c\ d \,] + [\, e\ f\ g\ h \,] \quad = [\, a + e\ b + f\ c + g\ d + h \,]$

Adding Matrix *A* and Matrix *B*

Find the sum of A and B.
$A = [\, 4\ 1\ 3\ 2 \,]$ and $B = [\, 5\ 9\ 0\ 7 \,]$

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11}, of B. Continue the pattern until all entries have been added.
$A + B = [\, 4\ 1\ 3\ 2 \,] + [\, 5\ 9\ 0\ 7 \,] \quad = [\, 4 + 5\ 1 + 9\ 3 + 0\ 2 + 7 \,] \quad = [\, 9\ 10\ 3\ 9 \,]$

Finding the Difference of Two Matrices

Find the difference of A and B.

$A = [\ -2\ 3\ 0\ 1\ ]$ and $B = [\ 8\ 1\ 5\ 4\ ]$

We subtract the corresponding entries of each matrix.

$A - B = [\ -2\ 3\ 0\ 1\ ] - [\ 8\ 1\ 5\ 4\ ] \quad = [\ -2-8\ 3-1\ 0-5\ 1-4\ ] \quad = [\ -10\ 2\ -5\ -3\ ]$

Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B:

1. Find the sum.
2. Find the difference.

$A = [\ 2\ -10\ -2\ 14\ 12\ 10\ 4\ -2\ 2\ ]$ and $B = [\ 6\ 10\ -2\ 0\ -12\ -4\ -5\ 2\ -2\ ]$

1. Add the corresponding entries.
   $A + B = [\ 2\ \ -10\ \ -2\ 14\ \ 12\ \ 10\ 4\ \ -2\ \ 2\ ] + [\ 6\ \ 10\ \ -2\ 0\ \ -12\ \ -4\ -5\ \ 2\ \ -2\ ]$
   $= [\ 2+6\ \ -10+10\ \ -2-2\ 14+0\ 12-12\ \ 10-4\ 4-5\ \ -2+2\ \ 2-2\ ]$
   $= [\ 8\ \ 0\ \ -4\ 14\ \ 0\ \ 6\ -1\ \ 0\ \ 0\ ]$
2. Subtract the corresponding entries.

A − B = [2 −10 −2 14 12 10 4 −2 2] − [6 10 −2 0 −12 −4 −5 2 −2] = [2−6 −10−10 −2+2 14−0 12+12 10+4 4+5 −2−2 2+2] = [−4 −20 0 14 24 14 9 −4 4]

Add matrix A and matrix B.
A = [2 6 1 0 1 −3] and B = [3 −2 1 5 −4 3]

A + B = [2 1 1 6 0 −3] + [3 1 −4 −2 5 3] = [2 + 3 1 + 1 1 + (−4) 6 + (−2) 0 + 5 −3 + 3] = [5 2 −3 4 5 0]

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that

has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A **scalar multiple** is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15% more equipment is needed in both labs. The school's current inventory is displayed in [link].

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Converting the data to a matrix, we have

$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.

$(0.15)C_{2013} = [\ (0.15)15\ (0.15)16\ (0.15)16$

$(0.15)27 \; (0.15)34 \; (0.15)34 \,] = [\, 2.25 \; 2.4 \; 2.4 \quad 4.05 \; 5.1 \; 5.1 \,]$

We must round up to the next integer, so the amount of new equipment needed is

$[\, 3 \; 3 \; 3 \quad 5 \; 6 \; 6 \,]$

Adding the two matrices as shown below, we see the new inventory amounts.

$[\, 15 \; 16 \; 16 \quad 27 \; 34 \; 34 \,] + [\, 3 \; 3 \; 3 \quad 5 \; 6 \; 6 \,] = [\, 18 \; 19 \; 19 \quad 32 \; 40 \; 40 \,]$

This means

$C_{2014} = [\, 18 \; 19 \; 19 \quad 32 \; 40 \; 40 \,]$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

Scalar Multiplication

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$A = [\, a_{11} \; a_{12} \; a_{21} \; a_{22} \,]$

the scalar multiple cA is

$cA = c[\, a_{11} \; a_{12} \; a_{21} \; a_{22} \,] \quad = [\, ca_{11} \; ca_{12} \; ca_{21} \; ca_{22} \,]$

Scalar multiplication is distributive. For the matrices $A, B,$ and C with scalars a and $b,$

$a(A+B) = aA + aB \quad (a+b)A = aA + bA$

Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.

$A = [\ 8\ 1\ 5\ 4\ ]$

Multiply each entry in A by the scalar 3.

$3A = 3[\ 8\ \ 1\ 5\ \ 4\ ] \quad = [\ 3\cdot 8\ \ 3\cdot 1\ 3\cdot 5\ \ 3\cdot 4\ ]$

$= [\ 24\ 3\ 15\ 12\ ]$

Given matrix B, find $-2B$ where

$B = [\ 4\ 1\ 3\ 2\ ]$

$-2B = [\ -8\ \ -2\ \ -6\ \ -4\ ]$

Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$A = [\ 1\ \ -2\ 0\ 0\ \ -1\ 2\ 4\ 3\ \ -6\ ]$ and $B = [\ -1\ 2\ 1\ 0\ \ -3\ 2\ 0\ 1\ \ -4\ ]$

First, find $3A$, then $2B$.

$3A = [\ 3\cdot1 \ \ 3(-2) \ \ 3\cdot0 \ 3\cdot0 \ \ 3(-1) \ \ 3\cdot2 \ 3\cdot4 \ \ 3\cdot3 \ \ 3(-6)\ ] \quad = [\ 3 \ \ -6 \ \ 0 \ 0 \ \ -3 \ \ 6 \ 12 \ \ 9 \ \ -18\ ]$

$2B = [\ 2(-1) \ \ 2\cdot2 \ \ 2\cdot1 \ 2\cdot0 \ \ 2(-3) \ \ 2\cdot2 \ 2\cdot0 \ \ 2\cdot1 \ \ 2(-4)\ ] \quad = [\ -2 \ 4 \ 2 \ 0 \ -6 \ 4 \ 0 \ 2 \ -8\ ]$

Now, add $3A + 2B$.

$3A + 2B = [\ 3 \ -6 \ 0 \ 0 \ -3 \ 6 \ 12 \ 9 \ -18\ ] + [\ -2 \ 4 \ 2 \ 0 \ -6 \ 4 \ 0 \ 2 \ -8\ ] \quad = [\ 3-2 \ \ -6+4 \ \ 0+2 \ 0+0 \ \ -3-6 \ 6+4 \ 12+0 \ \ 9+2 \ -18-8\ ] \quad = [\ 1 \ \ -2 \ 2 \ 0 \ \ -9 \ 10 \ 12 \ \ 11 \ -26\ ]$

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product matrix AB is an $m \times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\begin{array}{ccc} A & \cdot & B \\ 2\times 3 & & 3\times 3 \end{array}$$

same

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of matrix multiplication becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3, the product of AB will be a 2×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB, multiply the first row in A by the first column in B, and add.
   $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$
2. To obtain the entry in row 1, column 2 of AB, multiply the first row of A by the second column in B, and add.
   $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11} \cdot b$$

$12 + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$

3. To obtain the entry in row 1, column 3 of AB, multiply the first row of A by the third column in B, and add.
   $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

Properties of Matrix Multiplication

For the matrices A, B, and C the following properties hold.

- Matrix multiplication is associative: $(AB)C = A(BC)$.
- Matrix multiplication is distributive: $C(A+B) = CA + CB$, $(A+B)C = AC + BC$.

Note that matrix multiplication is not commutative.

Multiplying Two Matrices

Multiply matrix A and matrix B.
$A = [\ 1\ 2\ 3\ 4\ ]$ and $B = [\ 5\ 6\ 7\ 8\ ]$

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} \\
&= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}
\end{aligned}
$$

Multiplying Two Matrices

Given A and B:

1. Find AB.
2. Find BA.

$A = [\ -1\ 2\ 3\ \ 4\ 0\ 5\ ]$ and $B = [\ 5\ -4\ 2\ \ -1\ 0\ 3\ ]$

---

1. As the dimensions of A are 2×3 and the dimensions of B are 3×2, these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.
   $AB = [\ -1\ 2\ 3\ 4\ 0\ 5\ ]\ \ [\ 5\ -1\ -4\ 0\ 2\ 3\ ]$
   $= [\ -1(5) + 2(-4) + 3(2)\ \ -1(-1) + 2(0) + 3(3)\ 4(5) + 0(-4) + 5(2)\ \ 4(-1) + 0(0) + 5(3)\ ]$ $= [\ -7\ 10\ 30\ 11\ ]$
2. The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.
   $BA = [\ 5\ -1\ -4\ 0\ 2\ 3\ ]\ \ [\ -1\ 2\ 3\ 4\ 0\ 5\ ]$
   $= [\ 5(-1) + -1(4)\ \ 5(2) + -1(0)\ \ 5(3) + -1(5)\ \ -4(-1) + 0(4)\ \ -4(2) + 0(0)\ \ -4(3) + 0(5)\ 2(-1) + 3(4)\ \ 2(2) + 3(0)\ \ 2(3) + 3(5)\ ]$ $= [\ -9\ 10\ 10\ 4\ -8\ -12\ 10\ 4\ 21\ ]$

Analysis

Notice that the products AB and BA are not equal.
$AB = [\ -7\ 10\ 30\ 11\ ] \neq [\ -9\ 10\ 10\ 4\ -8\ -12\ 10\ 4\ 21\ ] = BA$

This illustrates the fact that matrix multiplication is not commutative.

Is it possible for *AB* to be defined but not *BA*?
Yes, consider a matrix A with dimension 3×4 and matrix B with dimension 4×2. For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have [link], representing the equipment needs of two soccer teams.

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

We are also given the prices of the equipment, as shown in [link].

Goal	$300
Ball	$10
Jersey	$30

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E=\begin{bmatrix} 6 & 30 & 14 \\ 10 & 24 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C=\begin{bmatrix} 300 & 10 & 30 \end{bmatrix}$$

We perform matrix multiplication to obtain costs for the equipment.

$$CE=\begin{bmatrix} 300 & 10 & 30 \end{bmatrix}\cdot\begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}=\begin{bmatrix} 300(6)+10(30)+30(14) & 300(10)+10(24)+30(20) \end{bmatrix}=\begin{bmatrix} 2{,}520 & 3{,}840 \end{bmatrix}$$

The total cost for equipment for the Wildcats is \$2,520, and the total cost for equipment for the Mud Cats is \$3,840.

Given a matrix operation, evaluate using a calculator.

1. Save each matrix as a matrix variable $[\ A\ ],[\ B\ ],[\ C\ ],\ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

Using a Calculator to Perform Matrix Operations

Find $AB-C$ given $A=[\ -15\ 25\ 32\ 41\ -7\ -28\ 10\ 34\ -2\ ], B=[\ 45\ 21\ -37\ -24\ 52\ 19\ 6\ -48\ -31\ ]$, and $C=[\ -100\ -89\ -98\ 25\ -56\ 74\ -67\ 42\ -75\ ]$.

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[\ A\ ]$,

matrix B above as the matrix variable $[B]$, and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See [link].
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See [link], [link], [link], and [link].
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See [link].
- Scalar multiplication is often required before addition or subtraction can occur. See [link].
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B, is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B, and so on. See [link] and [link].
- Many real-world problems can often be solved using matrices. See [link].
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See [link].

Section Exercises

Verbal

Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a 2×2 matrix and the second is a 2×3 matrix. $[\ 1\ 2\ 3\ 4\ ] + [\ 6\ 5\ 4\ 3\ 2\ 1\ ]$ has no sum.

Can we multiply any column matrix by any row matrix? Explain why or why not.

Can both the products AB and BA be defined? If so, explain how; if not, explain why.

Yes, if the dimensions of A are $m \times n$ and the dimensions of B are $n \times m$, both products will be defined.

Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

Does matrix multiplication commute? That is, does $AB=BA$? If so, prove why it does. If not, explain why it does not.

Not necessarily. To find AB, we multiply the first row of A by the first column of B to get the first entry of AB. To find BA, we multiply the first row of B by the first column of A to get the first entry of BA. Thus, if those are unequal, then the matrix multiplication does not commute.

Algebraic

For the following exercises, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.

$A=[\ 1\ 3\ 0\ 7\ ],B=[\ 2\ 14\ 22\ 6\ ],C=[\ 1\ 5\ 8\ 92\ 12\ 6\ ],D=[\ 10\ 14\ 7\ 2\ 5\ 61\ ],E=[\ 6\ 12\ 14\ 5\ ],F=[\ 0\ 9\ 78\ 17\ 15\ 4\ ]$

$A+B$

$C+D$

$[\ 11\ 19\ 15\ 94\ 17\ 67\ ]$

$A+C$

$B-E$

$[\ -4\ 2\ 8\ 1\ ]$

$C+F$

$D-B$

Undidentified; dimensions do not match

For the following exercises, use the matrices below to perform scalar multiplication.
$A=[\ 4\ 6\ 13\ 12\ ], B=[\ 3\ 9\ 21\ 12\ 0\ 64\ ], C=[\ 16\ 3\ 7\ 18\ 90\ 5\ 3\ 29\ ], D=[\ 18\ 12\ 13\ 8\ 14\ 6\ 7\ 4\ 21\ ]$

$5A$

$3B$

[9 27 63 36 0 192]

−2B

−4C

[−64 −12 −28 −72 −360 −20 −12 −116]

1 2 C

100D

[1,800 1,200 1,300 800 1,400 600 700 400 2,100]

For the following exercises, use the matrices below to perform matrix multiplication.
A=[−1 5 3 2],B=[3 6 4 −8 0 12],C=[4 10 −2 6 5 9],D=[2 −3 12 9 3 1 0 8 −10]

AB

BC

$[\ 20\ 102\ 28\ 28\ ]$

CA

BD

$[\ 60\ 41\ 2\ -16\ 120\ -216\ ]$

DC

CB

$[\ -68\ 24\ 136\ -54\ -12\ 64\ -57\ 30\ 128\ ]$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.

$A=[\ 2\ -5\ 6\ 7\ ], B=[\ -9\ 6\ -4\ 2\ ], C=[\ 0\ 9\ 7\ 1\ ], D=[\ -8\ 7\ -5\ 4\ 3\ 2\ 0\ 9\ 2\ ], E=[\ 4\ 5\ 3\ 7\ -6\ -5\ 1\ 0\ 9\ ]$

$A+B-C$

$4A+5D$

Undefined; dimensions do not match.

$2C+B$

$3D+4E$

$[\ -8\ 41\ \ -3\ 40\ \ -15\ \ -14\ 4\ 27\ 42\ ]$

$C-0.5D$

$100D-10E$

$[\ -840\ 650\ \ -530\ 330\ 360\ 250\ \ -10\ 900\ 110\ ]$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2=A\cdot A$)
$A=[\ -10\ 20\ 5\ 25\ ], B=[\ 40\ 10\ \ -20\ 30\ ], C=[\ -1\ 0\ 0\ \ -1\ 1\ 0\ ]$

AB

BA

[−350 1,050 350 350]

CA

BC

Undefined; inner dimensions do not match.

A 2

B 2

[1,400 700 −1,400 700]

C 2

B 2 A 2

[332,500 927,500 −227,500 87,500]

$A^2 B^2$

$(AB)^2$

$\begin{bmatrix} 490,000 & 0 \\ 0 & 490,000 \end{bmatrix}$

$(BA)^2$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{bmatrix}$

AB

$\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$

BA

BD

$\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$

DC

D^2

$[\ -3\ -2\ -2\ -28\ 59\ 46\ -4\ 16\ 7\ ]$

A^2

D^3

$[\ 1\ -18\ -9\ -198\ 505\ 369\ -72\ 126\ 91\ ]$

$(AB)C$

$A(BC)$

$[\ 0\ 1.6\ 9\ -1\ ]$

Technology

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.

$A=\begin{bmatrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{bmatrix}, B=\begin{bmatrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

AB

BA

$\begin{bmatrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{bmatrix}$

CA

BC

$\begin{bmatrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{bmatrix}$

ABC

Extensions

For the following exercises, use the matrix below to perform the indicated operation on the given matrix.
$B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

B^2

[1 0 0 0 1 0 0 0 1]

B 3

B 4

[1 0 0 0 1 0 0 0 1]

B 5

Using the above questions, find a formula for B n . Test the formula for B 201 and B 202 , using a calculator.

B n = { [1 0 0 0 1 0 0 0 1], n even, [1 0 0 0 0 1 0 1 0], n odd.

Glossary

column
: a set of numbers aligned vertically in a matrix

entry
: an element, coefficient, or constant in a matrix

matrix
: a rectangular array of numbers

row
: a set of numbers aligned horizontally in a matrix

scalar multiple
: an entry of a matrix that has been multiplied by a scalar

Solve Systems of Equations Using Matrices

By the end of this section, you will be able to:

- Write the augmented matrix for a system of equations
- Use row operations on a matrix
- Solve systems of equations using matrices

Before you get started, take this readiness quiz.

Solve: $3(x+2)+4=4(2x-1)+9.$
If you missed this problem, review [link].

$x=1$

Solve: $0.25p+0.25(p+4)=5.20.$
If you missed this problem, review [link].

$p=8.4$

Evaluate when $x = -2$ and $y = 3$: $2x2 - xy + 3y2$.
If you missed this problem, review [link].

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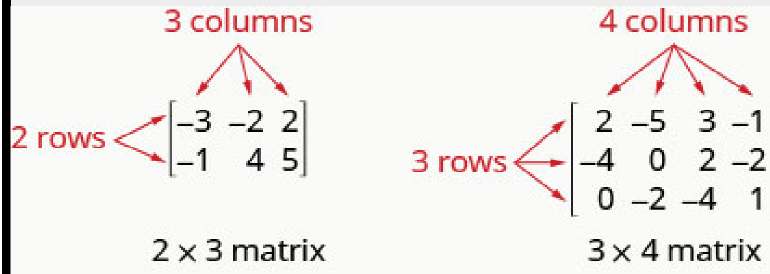
Write the Augmented Matrix for a System of Equations

Solving a system of equations can be a tedious operation where a simple mistake can wreak havoc on finding the solution. An alternative method which uses the basic procedures of elimination but with notation that is simpler is available. The method involves using a **matrix**. A matrix is a rectangular array of numbers arranged in rows and columns.

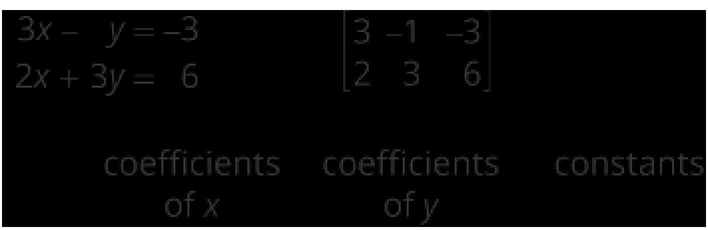
Matrix
A **matrix** is a rectangular array of numbers arranged in rows and columns.
A matrix with *m* rows and *n* columns has order

m × n. The matrix on the left below has 2 rows and 3 columns and so it has order 2 × 3. We say it is a 2 by 3 matrix.



Each number in the matrix is called an element or entry in the matrix.

We will use a matrix to represent a system of linear equations. We write each equation in standard form and the coefficients of the variables and the constant of each equation becomes a row in the matrix. Each column then would be the coefficients of one of the variables in the system or the constants. A vertical line replaces the equal signs. We call the resulting matrix the augmented matrix for the system of equations.



Notice the first column is made up of all the coefficients of *x*, the second column is the all the coefficients of *y*, and the third column is all the constants.

Write each system of linear equations as an augmented matrix:

ⓐ $\{5x-3y=-1y=2x-2$ ⓑ $\{6x-5y+2z=32x+y-4z=53x-3y+z=-1$

ⓐ The second equation is not in standard form. We rewrite the second equation in standard form.
$y=2x-2-2x+y=-2$

We replace the second equation with its standard form. In the augmented matrix, the first equation gives us the first row and the second equation gives us the second row. The vertical line replaces the equal signs.

$$\begin{array}{ll} 5x-3y=-1 \\ 2x-\ \ y=\ \ 2 \end{array} \qquad \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \end{array}$$

ⓑ All three equations are in standard form. In the augmented matrix the first equation gives us the first row, the second equation gives us the second row, and the third equation gives us the third row. The vertical line replaces the equal signs.

$$\begin{array}{l} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{array} \qquad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \end{array}$$

Write each system of linear equations as an augmented matrix:

ⓐ {3x + 8y = − 32x = − 5y − 3 ⓑ {2x − 5y + 3z = 83x − y + 4z = 7x + 3y + 2z = − 3

ⓐ [38 − 325 − 3]
ⓑ [2 − 5383 − 147132 − 3]

Write each system of linear equations as an augmented matrix:

ⓐ $\{11x = -9y - 57x + 5y = -1$ ⓑ $\{5x - 3y + 2z = -52x - y - z = 43x - 2y + 2z = -7$

ⓐ $[119 - 575 - 1]$
ⓑ $[5 - 32 - 52 - 1 - 143 - 22 - 7]$

It is important as we solve systems of equations using matrices to be able to go back and forth between the system and the matrix. The next example asks us to take the information in the matrix and write the system of equations.

Write the system of equations that corresponds to the augmented matrix:

$[4 - 3312 - 1 - 2 - 13 | -12 - 4]$.

We remember that each row corresponds to an equation and that each entry is a coefficient of a variable or the constant. The vertical line

replaces the equal sign. Since this matrix is a 4×3, we know it will translate into a system of three equations with three variables.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{rrr|r} 4 & -3 & 3 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & -1 & 3 & -4 \end{array}\right] \end{array} \qquad \begin{aligned} 4x - 3y + 3z &= -1 \\ x + 2y - z &= 2 \\ -2x - y + 3z &= -4 \end{aligned}$$

Write the system of equations that corresponds to the augmented matrix:
$[1-12321-214-120]$.

$\{x-y+2z=32x+y-2z=14x-y+2z=0$

Write the system of equations that corresponds to the augmented matrix:
$[111423-1811-13]$.

$\{x+y+z=42x+3y-z=8x+y-z=3$

Use Row Operations on a Matrix

Once a system of equations is in its augmented matrix form, we will perform operations on the rows that will lead us to the solution.

To solve by elimination, it doesn't matter which order we place the equations in the system. Similarly, in the matrix we can interchange the rows.

When we solve by elimination, we often multiply one of the equations by a constant. Since each row represents an equation, and we can multiply each side of an equation by a constant, similarly we can multiply each entry in a row by any real number except 0.

In elimination, we often add a multiple of one row to another row. In the matrix we can replace a row with its sum with a multiple of another row.

These actions are called row operations and will help us use the matrix to solve a system of equations.

Row Operations

In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

1. Interchange any two rows.
2. Multiply a row by any real number except 0.
3. Add a nonzero multiple of one row to another row.

Performing these operations is easy to do but all the arithmetic can result in a mistake. If we use a system to record the row operation in each step, it is much easier to go back and check our work.

We use capital letters with subscripts to represent each row. We then show the operation to the left of the new matrix. To show interchanging a row:

$$\begin{bmatrix} 5 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \\ R_1 \end{matrix} \begin{bmatrix} 2 & -1 & 2 \\ 5 & -3 & -1 \end{bmatrix}$$

To multiply row 2 by -3:

$$\begin{bmatrix} 5 & -3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \begin{matrix} \\ -3R_2 \end{matrix} \begin{bmatrix} 5 & -3 & -1 \\ -6 & 3 & -6 \end{bmatrix}$$

To multiply row 2 by -3 and add it to row 1:

$$\left[\begin{array}{cc|c} 5 & -3 & -1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{-3R_2 + R_1} \left[\begin{array}{cc|c} -1 & 0 & -7 \\ 2 & -1 & 2 \end{array}\right]$$

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 2 and 3.

ⓑ Multiply row 2 by 5.

ⓒ Multiply row 3 by -2 and add to row 1.
$[6-5221-43-31|35-1]$

ⓐ We interchange rows 2 and 3.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 3 & -3 & 1 & -1 \\ 2 & 1 & -4 & 5 \end{array}\right]$$

ⓑ We multiply row 2 by 5.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{5R_2} \left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 10 & 5 & -20 & 25 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

ⓒ We multiply row 3 by -2 and add to row

1.

$$\left[\begin{array}{ccc|c} 6 & -5 & 2 & 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right] \xrightarrow{-2R_3 + R_1} \left[\begin{array}{cccc} (-2 \cdot 3) + 6 & (-2 \cdot (-3)) + (-5) & (-2 \cdot 1) + 2 & (-2 \cdot (-1)) + 3 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_3 + R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 5 \\ 2 & 1 & -4 & 5 \\ 3 & -3 & 1 & -1 \end{array}\right]$$

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 3.

ⓑ Multiply row 3 by 3.

ⓒ Multiply row 3 by 2 and add to row 2.

$[5-2-24-1-4-230| -24-1]$

ⓐ $[-230-24-1-445-2-2-2]$
ⓑ $[-230-24-1-4415-6-6-6]$
ⓒ $[-230-234-13-16-815-6-6-6]$

Perform the indicated operations on the augmented matrix:

ⓐ Interchange rows 1 and 2,

ⓑ Multiply row 1 by 2,

ⓒ Multiply row 2 by 3 and add to row 1.

[2 − 3 − 241 − 3504| − 42 − 1]

ⓐ [41 − 322 − 3 − 2 − 4504 − 1]
ⓑ [82 − 642 − 3 − 2 − 4504 − 1]
ⓒ [14 − 7 − 12 − 82 − 3 − 2 − 4504 − 1]

Now that we have practiced the row operations, we will look at an augmented matrix and figure out what operation we will use to reach a goal. This is exactly what we did when we did elimination. We decided what number to multiply a row by in order that a variable would be eliminated when we added the rows together.

Given this system, what would you do to eliminate x?

$$\begin{aligned} x - y &= 2 \\ 4x - 8y &= 0 \end{aligned} \quad \xrightarrow{\text{multiply the first equation by } -4} \quad \begin{aligned} -4x + 4y &= -8 \\ 4x - 8y &= 0 \end{aligned} \quad \xrightarrow{\text{then add}} \quad \begin{aligned} -4x + 4y &= -8 \\ 4x - 8y &= 0 \\ \hline -4y &= -8 \end{aligned}$$

This next example essentially does the same thing, but to the matrix.

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $[1-14-8|20]$.

To make the 4 a 0, we could multiply row 1 by -4 and then add it to row 2.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right] \xrightarrow{-4R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -4 & -8 \end{array}\right]$$

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $[1-13-6|22]$.

$[1 - 1 2 0 - 3 - 4]$

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $[1 - 1 - 2 - 3 \| 3 2]$.
$[1 - 1 3 0 - 5 8]$

Solve Systems of Equations Using Matrices

To solve a system of equations using matrices, we transform the augmented matrix into a matrix in **row-echelon form** using row operations. For a consistent and independent system of equations, its augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

Row-Echelon Form

For a consistent and independent system of equations, its augmented matrix is in **row-echelon form** when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right] \qquad a, b, c, d, e, f \text{ are real numbers}$$

Once we get the augmented matrix into row-echelon form, we can write the equivalent system of equations and read the value of at least one variable. We then substitute this value in another equation to continue to solve for the other variables. This process is illustrated in the next example.

How to Solve a System of Equations Using a Matrix

Solve the system of equations using a matrix: $\{3x+4y=5x+2y=1.$

Step 1. Write the augmented matrix for the system of equations.		$3x + 4y = 5$ $x + 2y = 1$ $\left[\begin{array}{cc\|c} 3 & 4 & 5 \\ 1 & 2 & 1 \end{array}\right]$
Step 2. Using row operations get the entry in row 1, column 1 to be 1.	Interchange the rows, so 1 will be in row 1, column 1.	$\begin{array}{c} R_2 \\ R_1 \end{array}\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 3 & 4 & 5 \end{array}\right]$
Step 3. Using row operations, get zeros in column 1 below the 1.	Multiply row 1 by –3 and add it to row 2.	$\begin{array}{c} \\ -3R_1 + R_2 \end{array}\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & -2 & 2 \end{array}\right]$
Step 4. Using row operations, get the entry in row 2, column 2 to be 1.	Multiply row 2 by $-\frac{1}{2}$.	$\begin{array}{c} \\ -\frac{1}{2}R_2 \end{array}\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$
Step 5. Continue the process until the matrix is in row-echelon form.	The matrix is now in row-echelon form.	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$
Step 6. Write the corresponding system of equations.	$\begin{array}{ccc} x & y & \\ \end{array}$ $\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$	$x + 2y = 1$ $y = -1$
Step 7. Use substitution to find the remaining variables.	Substitute $y = -1$ into $x + 2y = 1$.	$y = -1$ $x + 2y = 1$ $x + 2(-1) = 1$ $x - 2 = 1$ $x = 3$

Step 8. Write the solution as an ordered pair or triple.	$(3, -1)$
Step 9. Check that the solution makes the original equations true.	We leave the check to you.

Solve the system of equations using a matrix: $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$.

The solution is $(4,-1)$.

Solve the system of equations using a matrix: $\begin{cases} 2x+y=-4 \\ x-y=-2 \end{cases}$.

The solution is $(-2,0)$.

The steps are summarized here.

Solve a system of equations using matrices.

Write the augmented matrix for the system of equations. Using row operations get the entry in row 1, column 1 to be 1. Using row operations, get zeros in column 1 below the 1. Using row operations, get the entry in row 2, column 2 to be 1. Continue the process until the matrix is in row-echelon form. Write the corresponding system of equations. Use substitution to find the remaining variables. Write the solution as an ordered pair or triple. Check that the solution makes the original equations true.

Here is a visual to show the order for getting the 1's and 0's in the proper position for row-echelon form.

2 × 3 matrix

Step 1	Step 2	Step 3
$\begin{bmatrix} 1 & & \\ & & \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ 0 & & \end{bmatrix}$	$\begin{bmatrix} 1 & & \\ 0 & 1 & \end{bmatrix}$

3 × 4 matrix

Step 1	Step 2	Step 3	Step 4
$\begin{bmatrix} 1 & & & \\ & & & \\ & & & \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ 0 & & & \\ & & & \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & & & \end{bmatrix}$

Step 5	Step 6
$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & & \end{bmatrix}$	$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{bmatrix}$

We use the same procedure when the system of equations has three equations.

Solve the system of equations using a matrix: $\{3x+8y+2z=-52x+5y-3z=0x+2y-2z=-1.$

$$\begin{aligned} 3x + 8y + 2z &= -5 \\ 2x + 5y - 3z &= 0 \\ x + 2y - 2z &= -1 \end{aligned}$$

Write the augmented matrix for the
eq

$$\left[\begin{array}{ccc|c} 3 & 8 & 2 & -5 \\ 2 & 5 & -3 & 0 \\ 1 & 2 & -2 & -1 \end{array}\right]$$

Interchange row 1 and 3 to get the entry in
ro
1.

$$\begin{array}{c} R_3 \\ \\ R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 2 & 5 & -3 & 0 \\ 3 & 8 & 2 & -5 \end{array}\right]$$

Using row operations, get zeros in column 1
be

$$\begin{array}{c} \\ -2R_1 + R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 3 & 8 & 2 & -5 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -3R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 8 & 2 \end{array}\right]$$

The entry in row 2, column 2 is now 1. Continue the process until the matrix is in row-echelon form.

$$\xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & -6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The matrix is now in row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Write the corresponding system of

$$\begin{aligned} x + 2y - 2z &= -1 \\ y + z &= 2 \\ z &= -1 \end{aligned}$$

Use substitution to find the remaining va

$$\begin{aligned} y + z &= 2 \\ y + (-1) &= 2 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} x + 2y - 2z &= -1 \\ x + 2(3) - 2(-1) &= -1 \\ x + 6 + 2 &= -1 \end{aligned}$$

[redacted]

Write the solution as an ordered pair or tri[redacted]

Check that the solution makes the original equations true. We leave the check for you.

Solve the system of equations using a matrix: $\{2x-5y+3z=83x-y+4z=7x+3y+2z=-3.$

$(6,-1,-3)$

Solve the system of equations using a matrix: $\{-3x+y+z=-4-x+2y-2z=12x-y-z=-1.$

$(5,7,4)$

So far our work with matrices has only been with systems that are consistent and independent, which means they have exactly one solution. Let’s now look at what happens when we use a matrix for a dependent or inconsistent system.

Solve the system of equations using a matrix: {x+y+3z=0x+3y+5z=02x+4z=1.

$$\begin{cases} x + y + 3z = 0 \\ x + 3y + 5z = 0 \\ 2x + 4z = 1 \end{cases}$$

Write the augmented matrix for the eq

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$$

The entry in row 1, column 1 is 1.

Using row operations,

ge be

$$\begin{array}{r} \\ -1R_1 + R_2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$$

Continue the process until the matrix is in ro

$$\begin{array}{r} \\ \frac{1}{2}R_2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$$

Multiply row 2 by 2 and add it to row 3.

$$\begin{array}{r} \\ \\ 2R_2 + R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

At this point, we have all zeros on the left of row 3.

Write the corresponding system of

$$\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \\ 0 &\neq 1 \end{aligned}$$

Since $0 \neq 1$ we have a false statement. Just as when we solved a

system using other methods, this tells us we have an inconsistent system. There is no solution.

Solve the system of equations using a matrix: $\{x-2y+2z=1-2x+y-z=2x-y+z=5.$

no solution

Solve the system of equations using a matrix: $\{3x+4y-3z=-22x+3y-z=-12x+y-2z=6.$

no solution

The last system was inconsistent and so had no

solutions. The next example is dependent and has infinitely many solutions.

Solve the system of equations using a matrix: $\{x - 2y + 3z = 1\ x + y - 3z = 7\ 3x - 4y + 5z = 7.$

$$\begin{cases} x - 2y + 3z = 1 \\ x + y - 3z = 7 \\ 3x - 4y + 5z = 7 \end{cases}$$

Write the augmented matrix for the eq

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & 1 & -3 & 7 \\ 3 & -4 & 5 & 7 \end{array}\right]$$

The entry in row 1, column 1 is 1.

Using row operations, get zeros in column 1 below the 1.

[redacted]

[redacted]

Continue the process until the matrix is in ro[redacted]

[redacted]

Multiply row 2 by -2 and add it to row 3.

[redacted]

At this point, we have all zeros in the bottom row.

Write the corresponding system of [redacted]

Since $0 = 0$ we have a true statement. Just as when we solved by substitution, this tells

us we have a dependent system. There are infinitely many solutions.

Solve for y in terms of z in the second eq[redacted]

Solve the first equation for x in terms of z.

[redacted]

Substitute $y = 2z + 2$.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

The system has infinitely many solutions (x,y,z), where $x = z + 5; y = 2z + 2$; z is any real number.

Solve the system of equations using a matrix: $\{x+y-z=0\ 2x+4y-2z=6\ 3x+6y-3z=9.$

infinitely many solutions (x,y,z), where $x=z-3; y=3; z$ is any real number.

Solve the system of equations using a matrix: $\{x-y-z=1\ -x+2y-3z=-4\ 3x-2y-7z=0.$

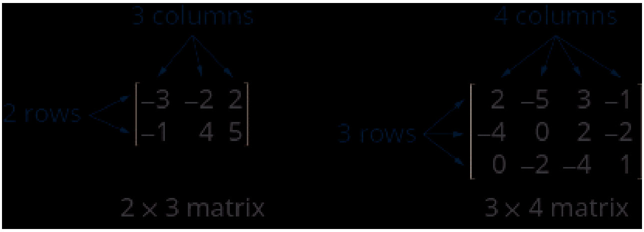
infinitely many solutions (x,y,z), where $x=5z-2; y=4z-3; z$ is any real number.

Access this online resource for additional instruction and practice with Gaussian Elimination.

- Gaussian Elimination

Key Concepts

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with *m* rows and *n* columns has *order* $m \times n$. The matrix on the left below has 2 rows and 3 columns and so it has order 2×3. We say it is a 2 by 3 matrix.



  Each number in the matrix is called an *element* or *entry* in the matrix.
- **Row Operations:** In a matrix, the following operations can be performed on any row and the resulting matrix will be equivalent to the original matrix.

  - Interchange any two rows
  - Multiply a row by any real number except 0
  - Add a nonzero multiple of one row to another row

- **Row-Echelon Form:** For a consistent and independent system of equations, its

augmented matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right] \qquad a, b, c, d, e, f \text{ are real numbers}$$

- **How to solve a system of equations using matrices.**

  Write the augmented matrix for the system of equations. Using row operations get the entry in row 1, column 1 to be 1. Using row operations, get zeros in column 1 below the 1. Using row operations, get the entry in row 2, column 2 to be 1. Continue the process until the matrix is in row-echelon form. Write the corresponding system of equations. Use substitution to find the remaining variables. Write the solution as an ordered pair or triple. Check that the solution makes the original equations true.

Practice Makes Perfect

Write the Augmented Matrix for a System of Equations

In the following exercises, write each system of linear equations as an augmented matrix.

ⓐ $\{3x-y=-12y=2x+5$
ⓑ $\{4x+3y=-2x-2y-3z=72x-y+2z=-6$

ⓐ $\{2x+4y=-53x-2y=2$
ⓑ $\{3x-2y-z=-2-2x+y=55x+4y+z=-1$

ⓐ $[24-53-22]$
ⓑ $[3-2-1-2-2105541-1]$

ⓐ $\{3x-y=-42x=y+2$
ⓑ $\{x-3y-4z=-24x+2y+2z=52x-5y$
$+7z=-8$

ⓐ $\{2x-5y=-34x=3y-1$
ⓑ $\{4x+3y-2z=-3-2x+y-3z=4-x-4y$
$+5z=-2$

ⓐ $[2-5-34-3-1]$

ⓑ $[43-2-3-21-34-1-45-2]$

Write the system of equations that corresponds to the augmented matrix.

$[2-11-3|42]$

$[2-43-3|-2-1]$

$\{2x-4y=-23x-3y=-1$

$[10-31-200-12|-1-23]$

$[2-2002-130-1|-12-2]$

$\{2x-2y=-12y-z=23x-z=-2$

Use Row Operations on a Matrix

In the following exercises, perform the indicated operations on the augmented matrices.

$[6-43-2|31]$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 2 by 3

ⓒ Multiply row 2 by -2 and add row 1 to it.

$[4-632|-31]$

ⓐ Interchange rows 1 and 2

ⓑ Multiply row 1 by 4

ⓒ Multiply row 2 by 3 and add row 1 to it.

ⓐ $[3214-6-3]$
ⓑ $[12844-6-3]$
ⓒ $[128424-10-5]$

$[4-12-84-2-3-62-1|16-1-1]$

ⓐ Interchange rows 2 and 3

ⓑ Multiply row 1 by 4

ⓒ Multiply row 2 by -2 and add to row 3.

$[6-5221-43-31|35-1]$

ⓐ Interchange rows 2 and 3

ⓑ Multiply row 2 by 5

ⓒ Multiply row 3 by -2 and add to row 1.

ⓐ $6-5233-31-121-45$
ⓑ $6-52315-155-521-45$
ⓒ $2-710-715-155-521-45$

Perform the needed row operation that will get the first entry in row 2 to be zero in the augmented matrix: $[12-3-4|5-1]$.

Perform the needed row operations that will get the first entry in both row 2 and row 3 to be zero in the augmented matrix: $[1-233-1-22-3-4|-45-1]$.

$[1-23-405-111701-107]$

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

$\{2x+y=2x-y=-2$

$\{3x+y=2x-y=2$

$(1,-1)$

$\{-x+2y=-2x+y=-4$

$\{-2x+3y=3x+3y=12$

$(3,3)$

In the following exercises, solve each system of equations using a matrix.

$\{2x-3y+z=19-3x+y-2z=-15x+y+z=0$

$\{2x-y+3z=-3-x+2y-z=10x+y+z=5$

$(-2,5,2)$

$\{2x-6y+z=33x+2y-3z=22x+3y-2z=3$

$\{4x-3y+z=72x-5y-4z=33x-2y-2z=-7$

$(-3,-5,4)$

$\{x+2z=04y+3z=-22x-5y=3$

$\{2x+5y=43y-z=34x+3z=-3$

$(-3,2,3)$

$\{2y+3z=-15x+3y=-67x+z=1$

$\{3x-z=-35y+2z=-64x+3y=-8$

$(-2,0,-3)$

$\{2x+3y+z=12x+y+z=93x+4y+2z=20$

$\{x+2y+6z=5-x+y-2z=3x-4y-2z=1$

---

no solution

$\{x+2y-3z=-1x-3y+z=12x-y-2z=2$

$\{4x - 3y + 2z = 0 - 2x + 3y - 7z = 12x - 2y + 3z = 6$

---

no solution

$\{x - y + 2z = -42x + y + 3z = 2 - 3x + 3y - 6z = 12$

$\{-x - 3y + 2z = 14 - x + 2y - 3z = -43x + y - 2z = 6$

---

infinitely many solutions (x,y,z) where $x = 12z + 4; y = 12z - 6;$ z is any real number

$\{x + y - 3z = -1y - z = 0 - x + 2y = 1$

$\{x + 2y + z = 4x + y - 2z = 3 - 2x - 3y + z = -7$

---

infinitely many solutions (x,y,z) where $x = 5z + 2; y = -3z + 1;$ z is any real number

Writing Exercises

Solve the system of equations $\{x+y=10x-y=6$ ⓐ by graphing and ⓑ by substitution. ⓒ Which method do you prefer? Why?

Solve the system of equations $\{3x+y=12x=y-8$ by substitution and explain all your steps in words.

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
write the augmented matrix for a system of equations.			
use row operations on a matrix.			
solve systems of equations using matrices.			
write the augmented matrix for a system of equations.			
use row operations on a matrix.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

matrix
: A matrix is a rectangular array of numbers arranged in rows and columns.

row-echelon form
: A matrix is in row-echelon form when to the left of the vertical line, each entry on the diagonal is a 1 and all entries below the diagonal are zeros.

Solving Systems with Inverses
In this section, you will:

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix.

Nancy plans to invest $10,500 into two different bonds to spread out her risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. In order to receive an 8.5% return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number a is a^{-1}, and $aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1$. For example, $2^{-1} = \frac{1}{2}$ and $\left(\frac{1}{2}\right)2 = 1$. The multiplicative inverse of a matrix is similar in concept, except that the product of matrix A and its

inverse A^{-1} equals the identity matrix. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. [link] and [link] are the identity matrices for a 2×2 matrix and a 3×3 matrix, respectively.

$$I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The identity matrix acts as a 1 in matrix algebra. For example, $AI=IA=A$.

A matrix that has a multiplicative inverse has the properties

$$AA^{-1}=I \quad A^{-1}A=I$$

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility, $AA^{-1}=A^{-1}A=I$, is a requirement. Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

The Identity Matrix and Multiplicative Inverse

The **identity matrix**, I_n, is a square matrix

containing ones down the main diagonal and zeros everywhere else.

$$I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A^{-1}$, the **multiplicative inverse of a matrix** A.

Showing That the Identity Matrix Acts as a 1

Given matrix *A*, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Use matrix multiplication to show that the product of A and the identity is equal to the product of the identity and *A*.

$$AI = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Given two matrices, show that one is the multiplicative inverse of the other.

1. Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB.
2. If $AB = I$, then find the product BA. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

Showing That Matrix *A* Is the Multiplicative Inverse of Matrix *B*

Show that the given matrices are multiplicative inverses of each other.

$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}, B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$

Multiply AB and BA. If both products equal the identity, then the two matrices are inverses of each other.

$$AB = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(-9)+5(2) & 1(-5)+5(1) \\ -2(-9)-9(2) & -2(-5)-9(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} = \begin{bmatrix} -9(1)-5(-2) & -9(5)-5(-9) \\ 2(1)+1(-2) & 2(-5)+1(-9) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A and B are inverses of each other.

Show that the following two matrices are inverses of each other.
$A=[\ 1\ 4\ -1\ -3\ ], B=[\ -3\ -4\ 1\ 1\ ]$

$AB=[\ 1\ 4\ -1\ -3\ ]\ [\ -3\ -4\ 1\ 1\ ]=[\ 1(-3)+4(1)\ 1(-4)+4(1)\ -1(-3)+-3(1)\ -1(-4)+-3(1)\ ]=[\ 1\ 0\ 0\ 1\ ]$ $BA=[\ -3\ -4\ 1\ 1\ ]\ [\ 1\ 4\ -1\ -3\ ]=[\ -3(1)+-4(-1)\ -3(4)+-4(-3)\ 1(1)+1(-1)\ 1(4)+1(-3)\ ]=[\ 1\ 0\ 0\ 1\ ]$

Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using matrix multiplication.

Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.
$A=[\ 1\ -2\ 2\ -3\ ]$

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a-2c & 1b-2d \\ 2a-3c & 2b-3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.

$$1a-2c=1 \quad R_1 \qquad 2a-3c=0 \quad R_2$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 \rightarrow R_2$. Add the equations, and solve for c.

$$1a-2c=1 \qquad 0+1c=-2 \qquad c=-2$$

Back-substitute to solve for a.

$$a-2(-2)=1 \qquad a+4=1 \qquad a=-3$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the

identity.

$1b-2d=0$ R 1 $2b-3d=1$ R 2

Using row operations, multiply and add as follows: $(-2)R1+R2=R2$. Add the two equations and solve for d.

$1b-2d=0$ $0+1d=1$ $d=1$

Once more, back-substitute and solve for b.

$b-2(1)=0$ $b-2=0$ $b=2$

$A^{-1}=[-3\ 2\ -2\ 1]$

Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the multiplicative inverse is by augmenting with the identity. When matrix A is transformed into I, the augmented matrix I transforms into A^{-1}.

For example, given

$A=[2\ 1\ 5\ 3]$

augment A with the identity

$[2\ 1\ 5\ 3\ |\ 1\ 0\ 0\ 1]$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.
   $\left[\begin{array}{cc|cc} 5 & 3 & 2 & 1 \end{array}\right] \;\; \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \end{array}\right]$

   [5 3 2 1 | 0 1 1 0]
2. Multiply row 2 by -2 and add to row 1.
   [1 1 2 1 | -2 1 1 0]
3. Multiply row 1 by -2 and add to row 2.
   [1 1 0 -1 | -2 1 5 -2]
4. Add row 2 to row 1.
   [1 0 0 -1 | 3 -1 5 -2]
5. Multiply row 2 by -1.
   [1 0 0 1 | 3 -1 -5 2]

The matrix we have found is A^{-1}.
$A^{-1} = [\ 3\ -1\ -5\ 2\ ]$

Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the multiplicative inverse of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as
$A=[\ a\ b\ c\ d\ ]$

the multiplicative inverse of A is given by the formula
$A^{-1} = 1\ ad-bc\ [\ d\ -b\ -c\ a\ ]$

where $ad-bc\neq0$. If $ad-bc=0$, then A has no inverse.

Using the Formula to Find the Multiplicative Inverse of Matrix *A*

Use the formula to find the multiplicative inverse of
$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

Using the formula, we have

$$\begin{aligned} A^{-1} &= \frac{1}{(1)(-3)-(-2)(2)} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{-3+4} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

Analysis

We can check that our formula works by using one of the other methods to calculate the inverse. Let’s augment A with the identity.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.
   $$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$
2. Multiply row 1 by 2 and add to row 1.
   $$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So, we have verified our original solution.

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Use the formula to find the inverse of matrix A. Verify your answer by augmenting with the identity matrix.
$A = [\ 1\ \ -1\ 2\ \ 3\ ]$

$A\ -1\ = [\ 3\ 5\ 1\ 5\ -\ \ 2\ 5\ 1\ 5\ ]$

Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.
$A = [\ 3\ 6\ 1\ 2\ ]$

We will use the method of augmenting with the identity.
$[\ 3\ 6\ 1\ 3\ \ |\ \ 1\ 0\ 0\ 1\ ]$

1. Switch row 1 and row 2.
   $[\ 1\ 3\ 3\ 6\ \ |\ \ 0\ 1\ 1\ 0\ ]$
2. Multiply row 1 by -3 and add it to row 2.
   $[\ 1\ 2\ 0\ 0\ \ |\ \ 1\ 0\ -3\ 1\ ]$
3. There is nothing further we can do. The zeros in row 2 indicate that this matrix

has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix
$A = [\ 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1\ ]$

augment A with the identity matrix
$A|I = [\ 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1 \quad | \quad 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$

To begin, we write the augmented matrix with the identity on the right and A on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $A A^{-1} = I$ and $A^{-1} A = I$.

Finding the Inverse of a 3 × 3 Matrix

Given the 3×3 matrix A, find the inverse.
$A = [\; 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1 \;]$

Augment A with the identity matrix, and then begin row operations until the identity matrix replaces A. The matrix on the right will be the inverse of A.
$[\; 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1 \;|\; 1\ 0\ 0\ 0\ 1\ 0\ \ 0\ 0\ 1 \;] \rightarrow$
Interchange $R2$ and $R1$ $[\; 3\ 3\ 1\ 2\ 3\ 1\ 2\ 4\ 1 \;|\; 0\ 1\ 0\ 1\ 0\ 0\ \ 0\ 0\ 1 \;]$
$-R2 + R1 = R1 \rightarrow [\; 1\ 0\ 0\ 2\ 3\ 1\ 2\ 4\ 1 \;|\; -1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \;]$
$-R2 + R3 = R3 \rightarrow [\; 1\ 0\ 0\ 2\ 3\ 1\ 0\ 1\ 0 \;|\; -1\ 1\ 0\ 1\ 0\ 0\ -1\ 0\ 1 \;]$
$R3 \leftrightarrow R2 \rightarrow [\; 1\ 0\ 0\ 0\ 1\ 0\ 2\ 3\ 1 \;|\; -1\ 1\ 0\ -1\ 0\ 1\ 1\ 0\ 0 \;]$
$-2R1 + R3 = R3 \rightarrow [\; 1\ 0\ 0\ 0\ 1\ 0\ 0\ 3\ 1 \;|$

$-1\ 1\ 0\ -1\ 0\ 1\ 3\ -2\ 0\ ]$
$-3R2 + R3 = R3 \rightarrow [\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ \ |$
$-1\ 1\ 0\ -1\ 0\ 1\ 6\ -2\ -3\ ]$

Thus,
$A^{-1} = B = [\ -1\ 1\ 0\ -1\ 0\ 1\ 6\ -2\ -3\ ]$

Analysis

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.
$AA^{-1} = [\ 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1\ ]\ \ [\ -1\ 1\ 0\ -1\ 0\ 1\ 6\ -2\ -3\ ]$ $= [\ 2(-1)+3(-1)+1(6)$ $2(1)+3(0)+1(-2)$ $2(0)+3(1)+1(-3)$ $3(-1)+3(-1)+1(6)$ $3(1)+3(0)+1(-2)$ $3(0)+3(1)+1(-3)$ $2(-1)+4(-1)+1(6)$ $2(1)+4(0)+1(-2)$ $2(0)+4(1)+1(-3)\ ]$ $= [\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$
$A^{-1}A = [\ -1\ 1\ 0\ -1\ 0\ 1\ 6\ -2\ -3\ ]\ \ [\ 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1\ ]$ $= [\ -1(2)+1(3)+0(2)$ $-1(3)+1(3)+0(4)$ $-1(1)+1(1)+0(1)$ $-1(2)+0(3)+1(2)$ $-1(3)+0(3)+1(4)$ $-1(1)+0(1)+1(1)$ $6(2)+-2(3)+-3(2)$ $6(3)+-2(3)+-3(4)$ $6(1)+-2(1)+-3(1)\ ]$ $= [\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$

Find the inverse of the 3×3 matrix.

$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$
$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system $AX = B$. For example, look at the following system of equations.

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $(2^{-1})2 = \left(\frac{1}{2}\right)2 = 1.$ To solve a single linear equation $ax = b$ for $x,$ we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of $a.$ Thus,

$$\begin{aligned} ax &= b \\ \left(\frac{1}{a}\right)ax &= \left(\frac{1}{a}\right)b \\ (a^{-1})ax &= (a^{-1})b \\ [(a^{-1})a]x &= (a^{-1})b \\ 1x &= (a^{-1})b \\ x &= (a^{-1})b \end{aligned}$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a 2×2 system and then move on to a 3×3 system.

Solving a System of Equations Using the Inverse of a Matrix

Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then

$AX = B$

Multiply both sides by the inverse of A to obtain the solution.

$(A^{-1})AX = (A^{-1})B \quad [(A^{-1})A]X = (A^{-1})B \quad IX = (A^{-1})B \quad X = (A^{-1})B$

If the coefficient matrix does not have an inverse, does that mean the system has no solution?

No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

Solving a 2 × 2 System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.

$3x + 8y = 5 \quad 4x + 11y = 7$

Write the system in terms of a coefficient

matrix, a variable matrix, and a constant matrix.

$$A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, B=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1}=\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.

$$\begin{aligned} (A^{-1})AX &= (A^{-1})B \\ \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 11(5)+(-8)7 \\ -4(5)+3(7) \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $(-1,1)$.

Can we solve for X by finding the product BA^{-1}?

No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. *Consider our steps for solving the matrix equation.*

$$(A^{-1})AX = (A^{-1})B \quad [(A^{-1})A]X = (A^{-1})B \quad IX = (A^{-1})B \quad X = (A^{-1})B$$

Notice in the first step we multiplied both sides of the equation by A^{-1}, *but the* A^{-1} *was to the left of* A *on the left side and to the left of* B *on the right side. Because matrix multiplication is not commutative, order matters.*

Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.

$$5x+15y+56z=35 \quad -4x-11y-41z=-26 \quad -x-3y-11z=-7$$

Write the equation $AX=B$.

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of A by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by 1 5 .
[1 3 56 5 −4 −11 −41 −1 −3 −11 | 1 5
0 0 0 1 0 0 0 1]

Multiply row 1 by 4 and add to row 2.
[1 3 56 5 0 1 19 5 −1 −3 −11 | 1 5 0 0 4
5 1 0 0 0 1]

Add row 1 to row 3.
[1 3 56 5 0 1 19 5 0 0 1 5 | 1 5 0 0 4 5 1 0
1 5 0 1]

Multiply row 2 by −3 and add to row 1.
[1 0 − 1 5 0 1 19 5 0 0 1 5 | − 11 5 −3 0
4 5 1 0 1 5 0 1]

Multiply row 3 by 5.
[1 0 − 1 5 0 1 19 5 0 0 1 | − 11 5 −3 0 4
5 1 0 1 0 5]

Multiply row 3 by 1 5 and add to row 1.
[1 0 0 0 1 19 5 0 0 1 | −2 −3 1 4 5 1 0 1 0
5]

Multiply row 3 by − 19 5 and add to row 2.
[1 0 0 0 1 0 0 0 1 | −2 −3 1 −3 1 −19 1
0 5]

So,
A −1 = [−2 −3 1 −3 1 −19 1 0 5]

Multiply both sides of the equation by A −1 .

We want $A^{-1}AX = A^{-1}B$:

$$\begin{bmatrix} -2 & -31 & -31 \\ -19 & 1 & 0 \\ 5 & 1 & 5 \end{bmatrix}$$

[−2 −3 1 −3 1 −19 1 0 5] [5 15 56 −4 −11 −41 −1 −3 −11] [x y z] = [−2 −3 1 −3 1 −19 1 0 5] [35 −26 −7]

Thus,
$A^{-1}B = [\ -70+78-7\ \ -105-26+133\ \ 35+0-35\ ] = [\ 1\ 2\ 0\ ]$

The solution is $(1,2,0)$.

Solve the system using the inverse of the coefficient matrix.
$2x-17y+11z=0 \quad -x+11y-7z=8 \quad 3y-2z=-2$

$X = [\ 4\ 38\ 58\ ]$

Given a system of equations, solve with matrix inverses using a calculator.

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator,

calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator
$2x+3y+z=32$ $3x+3y+z=-27$ $2x+4y+z=-2$

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.
$[A]=[\ 2\ 3\ 1\ 3\ 3\ 1\ 2\ 4\ 1\ ],\quad [B]=[\ 32\ -27\ -2\ ]$

On the home screen of the calculator, type in the multiplication to solve for X, calling up each matrix variable as needed.
$[A]^{-1}\times[B]$

Evaluate the expression.
$[\ -59\ -34\ 252\ ]$

Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix
- Determining Inverse Matrices
- Using a Matrix Equation to Solve a System of Equations

Key Equations

Identity matrix for a 2×2 matrix	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Identity matrix for a 3×3 matrix	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Multiplicative inverse of a 2×2 matrix	$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$

Key Concepts

- An identity matrix has the property $AI=IA=A$. See [link].
- An invertible matrix has the property $AA^{-1}=A^{-1}A=I$. See [link].
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See [link].
- The multiplicative inverse can be found using a formula. See [link].
- Another method of finding the inverse is by augmenting with the identity. See [link].
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See [link].
- Write the system of equations as $AX=B$, and multiply both sides by the inverse of A: $A^{-1}AX=A^{-1}B$. See [link] and [link].
- We can also use a calculator to solve a system of equations with matrix inverses. See [link].

Section Exercises

Verbal

In a previous section, we showed that matrix multiplication is not commutative, that is, $AB \neq BA$ in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$?

If A^{-1} is the inverse of A, then $AA^{-1} = I$, the identity matrix. Since A is also the inverse of A^{-1}, $A^{-1}A = I$. You can also check by proving this for a 2×2 matrix.

Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

Can you explain whether a 2×2 matrix with an entire row of zeros can have an inverse?

No, because ad and bc are both 0, so $ad - bc = 0$, which requires us to divide by 0 in the formula.

Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

Can a matrix with zeros on the diagonal have

an inverse? If so, find an example. If not, prove why not. For simplicity, assume a 2×2 matrix.

Yes. Consider the matrix $[\ 0\ 1\ 1\ 0\ ]$. The inverse is found with the following calculation: $A\ -1\ =\ 1\ 0(0)-1(1)\ [\ 0\ -1\ -1\ 0\ ]=[\ 0\ 1\ 1\ 0\ ]$.

Algebraic

In the following exercises, show that matrix A is the inverse of matrix B.

$A=[\ 1\ 0\ -1\ 1\ ], B=[\ 1\ 0\ 1\ 1\ ]$

$A=[\ 1\ 2\ 3\ 4\ ], B=[\ -2\ 1\ 3\ 2\ -\ 1\ 2\ ]$

$AB=BA=[\ 1\ 0\ 0\ 1\ ]=I$

$A=[\ 4\ 5\ 7\ 0\ ], B=[\ 0\ 1\ 7\ 1\ 5\ -\ 4\ 35\ ]$

$A=[\ -2\ 1\ 2\ 3\ -1\ ], B=[\ -2\ -1\ -6\ -4\ ]$

$AB=BA=[\ 1\ 0\ 0\ 1\ ]=I$

$$A=\begin{bmatrix}1&0&1\\0&1&-1\\0&1&1\end{bmatrix},\ B=\frac{1}{2}\begin{bmatrix}2&1&-1\\0&1&1\\0&-1&1\end{bmatrix}$$

$$A=\begin{bmatrix}1&2&3\\4&0&2\\1&6&9\end{bmatrix},\ B=\frac{1}{4}\begin{bmatrix}6&0&-2\\17&-3&-5\\-12&2&4\end{bmatrix}$$

$$AB=BA=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$A=\begin{bmatrix}3&8&2\\1&1&1\\5&6&12\end{bmatrix},\ B=\frac{1}{36}\begin{bmatrix}-6&84&-6\\7&-26&1\\-1&-22&5\end{bmatrix}$$

For the following exercises, find the multiplicative inverse of each matrix, if it exists.

$$\begin{bmatrix}3&-2\\1&9\end{bmatrix}$$

$$\frac{1}{29}\begin{bmatrix}9&2\\-1&3\end{bmatrix}$$

$$\begin{bmatrix}-2&2\\3&1\end{bmatrix}$$

$$\begin{bmatrix}-3&7\\9&2\end{bmatrix}$$

$$\frac{1}{69}\begin{bmatrix}-2&7\\9&3\end{bmatrix}$$

$[\ -4\ \ -3\ \ -5\ 8\ ]$

$[\ 1\ 1\ 2\ 2\ ]$

There is no inverse

$[\ 0\ 1\ 1\ 0\ ]$

$[\ 0.5\ 1.5\ 1\ \ -0.5\ ]$

---

4 7 $[\ 0.5\ 1.5\ 1\ \ -0.5\ ]$

$[\ 1\ 0\ 6\ \ -2\ 1\ 7\ 3\ 0\ 2\ ]$

$[\ 0\ 1\ \ -3\ 4\ 1\ 0\ 1\ 0\ 5\ ]$

---

1 17 $[\ -5\ 5\ \ -3\ 20\ \ -3\ 12\ 1\ \ -1\ \ 4\ ]$

$[\ 1\ 2\ \ -1\ \ -3\ 4\ 1\ \ -2\ \ -4\ \ -5\ ]$

$[\ 1\ 9\ \ -3\ 2\ 5\ 6\ 4\ \ -2\ 7\ ]$

$1\ 209\ [\ 47\ \ -57\ 69\ 10\ 19\ \ -12\ \ -24\ 38\ \ -13\ ]$

$[\ 1\ \ -2\ 3\ \ -4\ 8\ \ -12\ 1\ 4\ 2\ ]$

$[\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 3\ 1\ 4\ 1\ 5\ 1\ 6\ 1\ 7\ 1\ 8\ ]$

$[\ 18\ 60\ \ -168\ \ -56\ \ -140\ 448\ 40\ 80\ \ -280\ ]$

$[\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ ]$

For the following exercises, solve the system using the inverse of a 2×2 matrix.

$5x-6y=-61\ \ 4x+3y=-2$

$(\ -5,6\ )$

$8x+4y=-100\ \ 3x-4y=1$

$3x-2y=6\ \ -x+5y=-2$

$(\ 2,0\ )$

$5x - 4y = -5 \quad 4x + y = 2.3$

$-3x - 4y = 9 \quad 12x + 4y = -6$

$(1\ 3, -\ 5\ 2)$

$-2x + 3y = 3\ 10 \quad -x + 5y = 1\ 2$

$8\ 5\ x - 4\ 5\ y = 2\ 5 \quad -\ 8\ 5\ x + 1\ 5\ y = 7\ 10$

$(-\ 2\ 3, -\ 11\ 6)$

$1\ 2\ x + 1\ 5\ y = -\ 1\ 4 \quad 1\ 2\ x - 3\ 5\ y = -\ 9\ 4$

For the following exercises, solve a system using the inverse of a 3×3 matrix.

$3x - 2y + 5z = 21 \quad 5x + 4y = 37 \quad x - 2y - 5z = 5$

$(7, 1\ 2, 1\ 5)$

$4x + 4y + 4z = 40 \quad 2x - 3y + 4z = -12 \quad -x + 3y + 4z = 9$

$6x-5y-z=31$ $-x+2y+z=-6$ $3x+3y+2z=13$

$(5,0,-1)$

$6x-5y+2z=-4$ $2x+5y-z=12$ $2x+5y+z=12$

$4x-2y+3z=-12$ $2x+2y-9z=33$ $6y-4z=1$

---

1 34 $(-35,-97,-154)$

1 10 $x-$ 1 5 $y+4z=$ -41 2 1 5 $x-20y+$ 2 5 $z=-101$ 3 10 $x+4y-$ 3 10 $z=23$

1 2 $x-$ 1 5 $y+$ 1 5 $z=$ 31 100 $-$ 3 4 $x-$ 1 4 $y+$ 1 2 $z=$ 7 40 $-$ 4 5 $x-$ 1 2 $y+$ 3 2 $z=$ 1 4

---

1 690 $(65,-1136,-229)$

$0.1x+0.2y+0.3z=-1.4$ $0.1x-0.2y+0.3z=0.6$ $0.4y+0.9z=-2$

Technology

For the following exercises, use a calculator to solve the system of equations with matrix inverses.

$2x - y = -3 \quad -x + 2y = 2.3$

$(- \ 37\ 30\ , 8\ 15\ )$

$- \ 1\ 2\ x - \ 3\ 2\ y = - \ 43\ 20 \quad 5\ 2\ x + \ 11\ 5\ y = 31\ 4$

$12.3x - 2y - 2.5z = 2 \ 36.9x + 7y - 7.5z = -7$
$8y - 5z = -10$

$(\ 10\ 123\ , -1, \ 2\ 5\ )$

$0.5x - 3y + 6z = -0.8 \qquad 0.7x - 2y = -0.06$
$0.5x + 4y + 5z = 0$

Extensions

For the following exercises, find the inverse of the given matrix.

$[\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ ]$

1 2 [2 1 −1 −1 0 1 1 −1 0 −1 1 1 0 1 −1 1]

[−1 0 2 5 0 0 0 2 0 2 −1 0 1 −3 0 1]

[1 −2 3 0 0 1 0 2 1 4 −2 3 −5 0 1 1]

---

1 39 [3 2 1 −7 18 −53 32 10 24 −36 21 9 −9 46 −16 −5]

[1 2 0 2 3 0 2 1 0 0 0 3 0 1 0 2 0 0 1 0 0 1 2 0]

[1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 1 1 1 1 1]

[1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 −1 −1 −1 −1 −1 1]

Real-World Applications

For the following exercises, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.

2,400 tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $64,000, how much was the price of each ticket?

In the previous exercise, if you were told there were 400 more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

Infinite solutions.

A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was 350 and the total weight of all donated food was 348 lb, 12 oz. If the green bean cans weigh 2 oz less than the kidney bean cans, how many of each can was donated?

Students were asked to bring their favorite fruit to class. 95% of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were 5% less popular than bananas, what are the percentages of each individual fruit?

50% oranges, 25% bananas, 20% apples

A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $1 and the chocolate chip cookies at $0.75. They raised $700 and sold 850 items. How many brownies and how many cookies were sold?

A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at $13.99, the beanie at $7.99, and the cowboy hat at $14.49. If 100 hats were sold this past quarter, $1,119 was taken in by sales, and the amount of beanies sold was 10 more than cowboy hats, how many of each should the clothing store order to replace those already sold?

---

10 straw hats, 50 beanies, 40 cowboy hats

Anna, Ashley, and Andrea weigh a combined 370 lb. If Andrea weighs 20 lb more than Ashley, and Anna weighs 1.5 times as much as Ashley, how much does each girl weigh?

Three roommates shared a package of 12 ice cream bars, but no one remembers who ate how many. If Tom ate twice as many ice cream bars as Joe, and Albert ate three less than Tom, how many ice cream bars did each roommate eat?

Tom ate 6, Joe ate 3, and Albert ate 3.

A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost $2 per square foot, the wood $10 per square foot, and the plywood $5 per square foot. The farmer spent a total of $51, and the total amount of materials used was $14\ \text{ft}^2$. He used $3\ \text{ft}^2$ more chicken wire than plywood. How much of each material in did the farmer use?

Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs 8 oz, a lemon 5 oz, and a pomegranate 11 oz. Jay picked 142 pieces of fruit weighing a total of 70 lb, 10 oz. He picked 15.5 times more oranges than pomegranates. How many of each fruit did Jay pick?

---

124 oranges, 10 lemons, 8 pomegranates

Glossary

identity matrix
: a square matrix containing ones down the main diagonal and zeros everywhere else; it acts as a 1 in matrix algebra

multiplicative inverse of a matrix
: a matrix that, when multiplied by the original, equals the identity matrix

Solve Systems of Equations Using Determinants

By the end of this section, you will be able to:

- Evaluate the determinant of a 2×2 matrix
- Evaluate the determinant of a 3×3 matrix
- Use Cramer's Rule to solve systems of equations
- Solve applications using determinants

Before you get started, take this readiness quiz.

Simplify: $5(-2)-(-4)(1)$.
If you missed this problem, review [link].

-6

Simplify: $-3(8-10)+(-2)(6-3)-4(-3-(-4))$.
If you missed this problem, review [link].

-4

Simplify: $-12-8$.
If you missed this problem, review [link].

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In this section we will learn of another method to solve systems of linear equations called Cramer's rule. Before we can begin to use the rule, we need to learn some new definitions and notation.

Evaluate the Determinant of a 2×2 Matrix

If a matrix has the same number of rows and columns, we call it a **square matrix**. Each square matrix has a real number associated with it called its **determinant**. To find the determinant of the square matrix [abcd], we first write it as |abcd|. To get the real number value of the determinate we subtract the products of the diagonals, as shown.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant

The determinant of any square matrix [abcd], where *a*, *b*, *c*, and *d* are real numbers, is

$|abcd| = ad - bc$

Evaluate the determinate of ⓐ $[4-23-1]$ ⓑ $[-3-4-20]$.

ⓐ

[redacted]

Write the determinant.

[redacted]

Subtract the products of the diagonals.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

ⓑ

[redacted]

Write the determinant.

[redacted]

Subtract the products of the diagonals.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Evaluate the determinate of ⓐ $[5 - 3 2 - 4]$ ⓑ $[-4 - 6 0 7]$.

ⓐ -14; ⓑ -28

Evaluate the determinate of ⓐ $[-1 3 - 2 4]$ ⓑ $[-7 - 3 - 5 0]$.

ⓐ 2 ⓑ -15

Evaluate the Determinant of a 3×3 Matrix

To evaluate the determinant of a 3×3 matrix, we have to be able to evaluate the **minor of an entry** in the determinant. The minor of an entry is the

2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

Minor of an entry in 3×3 a Determinant

The **minor of an entry** in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

To find the minor of entry a1, we eliminate the row and column which contain it. So we eliminate the first row and first column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } a_1 \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

To find the minor of entry b2, we eliminate the row and column that contain it. So we eliminate the 2nd row and 2nd column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } b_2 \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

For the determinant $|4-2310-3-2-42|$, find and then evaluate the minor of ⓐ a1 ⓑ b3 ⓒ c2.

ⓐ

$$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$$

Eliminate the row and column that contains a1

$$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$$

Write the 2×2 determinant that re

$$\text{minor of } a_1 \begin{vmatrix} 0 & -3 \\ -4 & 2 \end{vmatrix}$$

Evaluate.

$$0(2) \quad (\ 3)(\ 4)$$

Simplify.

[redacted]

ⓑ

Eliminate the row and column that contains b3 [redacted]

Write the 2×2 determinant that re[redacted]

Evaluate.

[redacted]

Simplify.

[redacted]

©

[redacted]

Eliminate the row and column that contains c2[redacted]

Write the 2×2 determinant that re[redacted]

Evaluate.

[redacted]

Simplify.

[redacted]

For the determinant $\left|1 -1 4 0 2 -1 -2 -3 3\right|$, find and then evaluate the minor of ⓐ a_1 ⓑ b_2 ⓒ c_3.

ⓐ 3 ⓑ 11 ⓒ 2

For the determinant $\left|-2 -1 0 3 0 -1 -1 -2 3\right|$, find and then evaluate the minor of ⓐ a_2 ⓑ b_3 ⓒ c_2.

ⓐ -3 ⓑ 2 ⓒ 3

We are now ready to evaluate a 3×3 determinant. To do this we expand by minors, which allows us to evaluate the 3×3 determinant using 2×2 determinants—which we already know how to evaluate!

To evaluate a 3×3 determinant by expanding by minors along the first row, we use the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Remember, to find the minor of an entry we eliminate the row and column that contains the entry.

Expanding by Minors along the First Row to Evaluate a 3×3 Determinant

To evaluate a 3×3 determinant by **expanding by minors along the first row**, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Evaluate the determinant $|2 - 3 - 1320 - 1 - 1 - 2|$ by expanding by minors along the first row.

[redacted]

Expand by minors along the first row

[redacted]

Evaluate each determinant.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Evaluate the determinant |
3 – 240 – 1 – 223 – 1|, by expanding by minors along the first row.

37

Evaluate the determinant $|3-2-22-14-10-3|$, by expanding by minors along the first row.

7

To evaluate a 3×3 determinant we can expand by minors using any row or column. Choosing a row or column other than the first row sometimes makes the work easier.

When we expand by any row or column, we must be careful about the sign of the terms in the expansion. To determine the sign of the terms, we use the following sign pattern chart.
$|+-+-+-+-+|$

Sign Pattern
When expanding by minors using a row or column, the sign of the terms in the expansion follow the

following pattern.
$|+-+-+-+-+|$

Notice that the sign pattern in the first row matches the signs between the terms in the expansion by the first row.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Since we can expand by any row or column, how do we decide which row or column to use? Usually we try to pick a row or column that will make our calculation easier. If the determinant contains a 0, using the row or column that contains the 0 will make the calculations easier.

Evaluate the determinant $|4-1-33025-4-3|$ by expanding by minors.

To expand by minors, we look for a row or column that will make our calculations easier. Since 0 is in the second row and second column, expanding by either of those is a good choice. Since the second row has fewer negatives than the second column, we will expand by the second row.

[redacted]

Expand using the second row.
Be careful of the signs.

[redacted]

Evaluate each determinant.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Add.

[redacted]

Evaluate the determinant $|2-1-303-43-4-3|$ by expanding by minors.

-11

Evaluate the determinant $|-2-1-3-1224-40|$ by expanding by minors.

-12

Use Cramer’s Rule to Solve Systems of Equations

Cramer’s Rule is a method of solving systems of equations using determinants. It can be derived by solving the general form of the systems of equations by elimination. Here we will demonstrate the rule for both systems of two equations with two variables and for systems of three equations with three variables.

Let’s start with the systems of two equations with two variables.

Cramer’s Rule for Solving a System of Two Equations

For the system of equations {a1x+b1y=k1a2x+b2y=k2, the solution (x,y) can be determined by

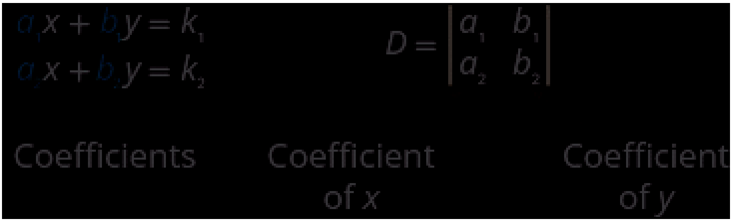
$$x=\frac{D_x}{D} \text{ and } y=\frac{D_y}{D}$$

where $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

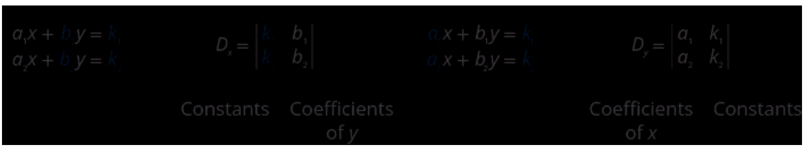
$D_x=\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y=\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant *D*, we use take the coefficients of the variables.

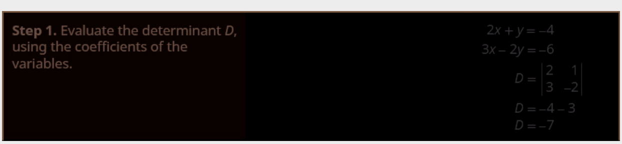


Notice that to form the determinant Dx and Dy, we substitute the constants for the coefficients of the variable we are finding.



How to Solve a System of Equations Using Cramer's Rule

Solve using Cramer's Rule: $\{2x + y = -43x - 2y = -6.$



Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.	We replace the coefficients of x, 2 and 3, with the constants, −4 and −6.	$D_x = \begin{vmatrix} -4 & 1 \\ -6 & -2 \end{vmatrix}$ $D_x = 8 - (-6)$ $D_x = 14$
Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.	We replace the coefficients of y, 1 and 2, with the constants, −4 and −6.	$D_y = \begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix}$ $D_y = -12 - (-12)$ $D_y = 0$
Step 4. Find x and y.	Substitute in the values of D, D_x and D_y.	$x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ $x = \frac{14}{-7}$ and $y = \frac{0}{-7}$ $x = -2$ and $y = 0$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(-2, 0)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $x = -2$, $y = 0$ into both equations and make sure they are both true.	$(-2, 0)$ is the solution to the system.

Solve using Cramer's rule: $\{3x+y=-32x+3y=6$.

$(-157,247)$

Solve using Cramer's rule: $\{-x+y=22x+y=-4.$
$(-2,0)$

Solve a system of two equations using Cramer's rule.

Evaluate the determinant *D*, using the coefficients of the variables. Evaluate the determinant Dx. Use the constants in place of the *x* coefficients. Evaluate the determinant Dy. Use the constants in place of the *y* coefficients. Find *x* and *y*. $x=DxD$, $y=DyD$ Write the solution as an ordered pair. Check that the ordered pair is a solution to both original equations.

To solve a system of three equations with three variables with Cramer's Rule, we basically do what we did for a system of two equations. However, we now have to solve for three variables to get the solution. The determinants are also going to be 3×3 which will make our work more interesting!

Cramer's Rule for Solving a System of Three Equations

For the system of equations {a1x+b1y +c1z=k1a2x+b2y+c2z=k2a3x+b3y+c3z=k3, the solution (x,y,z) can be determined by

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ replace the y coefficients with the constants.

$D_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$ replace the z coefficients with the constants.

Solve the system of equations using Cramer's Rule: {3x−5y+4z=55x+2y+z=02x+3y −2z=3.

Evaluate the determinant D.

[redacted]

Expand by minors using column 1.

[redacted]

Evaluate the determinants.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Evaluate the determinant Dx. Use the [redacted] the coefficients of x.

Expand by minors using column 1.

[redacted]

Evaluate the determinants.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Evaluate the determinant Dy. Use the [redacted] the coefficients of *y*.

[redacted]

Evaluate the determinants.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Evaluate the

de[redacted] the[redacted] constants to replace the coefficients of z.

[redacted]

Evaluate the determinants.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Find x, y, and z.

[redacted]

Substitute in the values.

[redacted]

Simplify.

[redacted]

Write the solution as an ordered triple.

[redacted]

Check that the ordered triple is a solution to **all three** original equations.	We leave the check to you.
	The solution is $(2, -3, -4)$.

Solve the system of equations using Cramer's Rule: $\{3x+8y+2z=-52x+5y-3z=0x+2y-2z=-1$.

$(-9,3,-1)$

Solve the system of equations using Cramer's Rule: $\{3x+y-6z=-32x+6y+3z=03x+2y-3z=-6$.

$(-6,3,-2)$

Cramer’s rule does not work when the value of the *D* determinant is 0, as this would mean we would be dividing by 0. But when $D=0$, the system is either inconsistent or dependent.

When the value of $D=0$ and Dx,Dy and Dz are all zero, the system is consistent and dependent and there are infinitely many solutions.

When the value of $D=0$ and Dx,Dy and Dz are not all zero, the system is inconsistent and there is no solution.

Dependent and Inconsistent Systems of Equations

For any system of equations, where the **value of the determinant** $D=0$,

Value of determinantsType of systemSolution$D=0$and$Dx,Dy$and$Dz$are all zeroconsistent and dependentinfinitely many solutions$D=0$and$Dx,Dy$and$Dz$are not all zeroinconsistentno solution

In the next example, we will use the values of the determinants to find the solution of the system.

Solve the system of equations using Cramer's rule : $\{x+3y=4 -2x-6y=3$.

$\{x+3y=4 -2x-6y=3$ Evaluate the determinantD,using thecoefficients of the variables.$D=|1 3 -2 -6|$ $D=-6-(-6) D=0$

We cannot use Cramer's Rule to solve this system. But by looking at the value of the determinants Dx and Dy, we can determine whether the system is dependent or inconsistent.

Evaluate the determinantDx.$Dx=|4 3 3 -6|$ $Dx=-24-9 Dx=15$

Since all the determinants are not zero, the system is inconsistent. There is no solution.

Solve the system of equations using Cramer's rule: $\{4x-3y=8 8x-6y=14$.

no solution

Solve the system of equations using Cramer's rule: $\{x=-3y+42x+6y=8$.

infinite solutions

Solve Applications using Determinants

An interesting application of determinants allows us to test if points are collinear. Three points (x1,y1), (x2,y2) and (x3,y3) are collinear if and only if the determinant below is zero.

$$|x1y11x2y21x3y31|=0$$

Test for Collinear Points

Three points (x1,y1), (x2,y2) and (x3,y3) are collinear if and only if

$$|x1y11x2y21x3y31|=0$$

We will use this property in the next example.

Determine whether the points $(5,-5)$, $(4,-3)$, and $(3,-1)$ are collinear.

[redacted]

Substitute the values into the determinant. (5, [redacted]

(3, [redacted]

Evaluate the determinant by ex[redacted] by minors using column 3.

Evaluate the determinants.

[redacted]

Simplify.

[redacted]

Simplify.

0

The value of the determinate is 0, so the points are collinear.

Determine whether the points $(3,-2)$, $(5,-3)$, and $(1,-1)$ are collinear.

yes

Determine whether the points $(-4,-1)$, $(-6,2)$, and $(-2,-4)$ are collinear.

yes

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Determinant:** The determinant of any square matrix [abcd], where *a, b, c,* and *d* are real numbers, is
  |abcd| = ad − bc
- **Expanding by Minors along the First Row to Evaluate a 3 × 3 Determinant:** To evaluate a 3×3 determinant by expanding by minors along the first row, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1\underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1\underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

- **Sign Pattern:** When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.
  |+ − + − + − + − +|
- **Cramer's Rule:** For the system of equations {a1x + b1y = k1a2x + b2y = k2, the solution (x,y) can be determined by

$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant *D*, we use take the coefficients of the variables.

- **How to solve a system of two equations using Cramer's rule.**

Evaluate the determinant *D*, using the coefficients of the variables. Evaluate the determinant Dx. Use the constants in place of the *x* coefficients. Evaluate the determinant Dy. Use the constants in place of the *y* coefficients. Find *x* and *y*. x=DxD, y=DyD. Write the solution as an ordered pair. Check that the ordered pair is a solution to **both** original equations. **Dependent and Inconsistent Systems of Equations:** For any system of equations, where the **value of the determinant** D=0,
Value of determinantsType of systemSolutionD=0andDx,DyandDzare all zeroconsistent and dependentinfinitely many solutionsD=0andDx,DyandDzare not all

zeroinconsistentno solution **Test for Collinear Points:** Three points $(x1,y1)$, $(x2,y2)$, and $(x3,y3)$ are collinear if and only if $|x1y11x2y21x3y31|=0$

Practice Makes Perfect

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercises, evaluate the determinate of each square matrix.

$[6-23-1]$

$[-48-35]$

4

$[-350-4]$

$[-207-5]$

10

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercises, find and then evaluate the indicated minors.

$|3 - 14 - 10 - 2 - 415|$
Find the minor ⓐ a1 ⓑ b2 ⓒ c3

$|-1 - 324 - 2 - 1 - 20 - 3|$
Find the minor ⓐ a1 ⓑ b1 ⓒ c2

ⓐ 6 ⓑ -14 ⓒ -6

$|2 - 3 - 4 - 12 - 30 - 1 - 2|$
Find the minor ⓐ a2 ⓑ b2 ⓒ c2

$|-2 - 231 - 30 - 23 - 2|$
Find the minor ⓐ a3 ⓑ b3 ⓒ c3

ⓐ 9 ⓑ -3 ⓒ 8

In the following exercises, evaluate each determinant by expanding by minors along the first row.

$|-23 - 1 - 12 - 231 - 3|$

$|4-1-2-3-21-2-57|$

-77

$|-2-3-45-67-120|$

$|13-25-640-2-1|$

49

In the following exercises, evaluate each determinant by expanding by minors.

$|-5-1-440-32-26|$

$|4-133-22-104|$

-24

$|354-130-261|$

$|2-4-35-1-4320|$

25

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's Rule.

$\{-2x+3y=3x+3y=12$

$\{x-2y=-52x-3y=-4$

$(7,6)$

$\{x-3y=-92x+5y=4$

$\{2x+y=-43x-2y=-6$

$(-2,0)$

$\{x-2y=-52x-3y=-4$

$\{x-3y=-92x+5y=4$

$(-3,2)$

$\{5x-3y=-12x-y=2$

$\{3x+8y=-32x+5y=-3$

$(-9,3)$

$\{6x-5y+2z=32x+y-4z=53x-3y+z=-1$

$\{4x-3y+z=72x-5y-4z=33x-2y-2z=-7$

$(-3,-5,4)$

$\{2x-5y+3z=83x-y+4z=7x+3y+2z=-3$

$\{11x+9y+2z=-97x+5y+3z=-74x+3y$
$+z=-3$

$(2,-3,-2)$

$\{x+2z=04y+3z=-22x-5y=3$

$\{2x+5y=43y-z=34x+3z=-3$

$(-3,2,3)$

$\{2y+3z=-15x+3y=-67x+z=1$

$\{3x-z=-35y+2z=-64x+3y=-8$

$(-2,0,-3)$

$\{2x+y=36x+3y=9$

$\{x-4y=-1-3x+12y=3$

---

infinitely many solutions

$\{-3x-y=46x+2y=-16$

$\{4x+3y=220x+15y=5$

---

inconsistent

$\{x+y-3z=-1y-z=0-x+2y=1$

$\{2x+3y+z=12x+y+z=93x+4y+2z=20$

---

inconsistent

$\{3x+4y-3z=-22x+3y-z=-12x+y-2z=6$

$\{x-2y+3z=1x+y-3z=73x-4y+5z=7$

---

infinitely many solutions

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

$(0,1)$, $(2,0)$, and $(-2,2)$.

$(0,-5)$, $(-2,-2)$, and $(2,-8)$.

---

yes

$(4,-3)$, $(6,-4)$, and $(2,-2)$.

$(-2,1)$, $(-4,4)$, and $(0,-2)$.

---

yes

Writing Exercises

Explain the difference between a square matrix and its determinant. Give an example of each.

Explain what is meant by the minor of an entry in a square matrix.

Answers will vary.

Explain how to decide which row or column you will use to expand a 3×3 determinant.

Explain the steps for solving a system of equations using Cramer's rule.

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the determinant of a 2×2 matrix.			
evaluate the determinant of a 3×3 matrix.			
use Cramer's rule to solve systems of equations.			
solve applications using determinants.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

determinant
: Each square matrix has a real number associated with it called its determinant.

minor of an entry in a 3×3 determinant
: The minor of an entry in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

square matrix
: A square matrix is a matrix with the same number of rows and columns.

Greatest Common Factor and Factor by Grouping

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

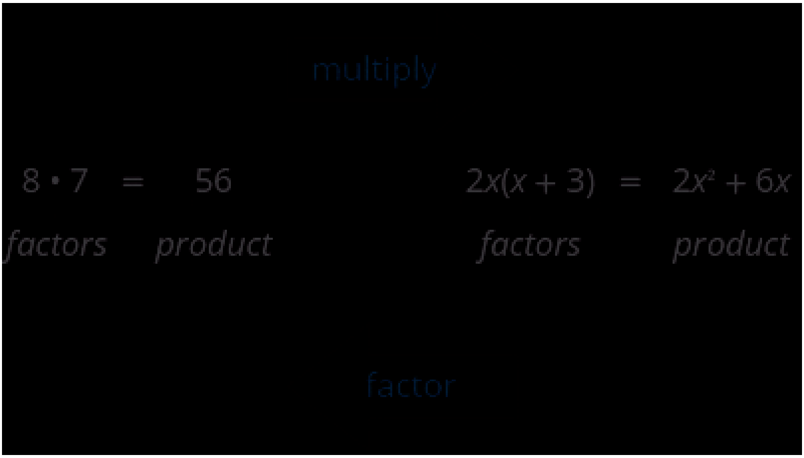
Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review [link].
2. Find the least common multiple of 18 and 24.
   If you missed this problem, review [link].
3. Simplify $-3(6a+11)$.
   If you missed this problem, review [link].

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we

will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

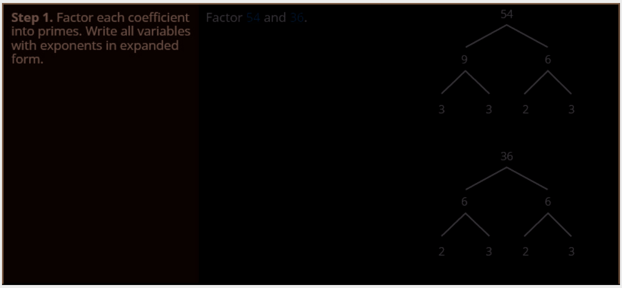
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

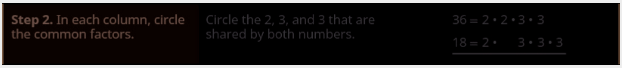
First we'll find the GCF of two numbers.

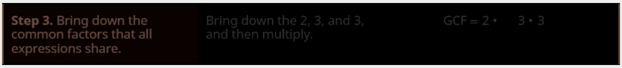
How to Find the Greatest Common Factor of Two or More Expressions

Find the GCF of 54 and 36.

Solution







Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.
$54 = 18 \cdot 3 \quad 36 = 18 \cdot 2$

Find the GCF of 48 and 80.

16

Find the GCF of 18 and 40.

2

We summarize the steps we use to find the GCF below.

Find the Greatest Common Factor (GCF) of two

expressions.

Factor each coefficient into primes. Write all variables with exponents in expanded form. List all factors—matching common factors in a column. In each column, circle the common factors. Bring down the common factors that all expressions share. Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

Find the greatest common factor of 27x3and18x4.

Solution

Factor each coefficient into primes and write the variables with		

ex
for
co
each column.

$$27x^3 = 3\cdot 3\cdot 3\cdot x\cdot x\cdot x$$
$$18x^4 = 2\cdot 3\cdot 3\cdot x\cdot x\cdot x\cdot x$$

Bring down the common factors.

$$GCF = 3\cdot 3\cdot x\cdot x\cdot x$$

Multiply the factors.

$$GCF = 9x^3$$

The GCF of 27x3 and 18x4 is 9x3.

Find the GCF: 12x2,18x3.

6x2

Find the GCF: 16y2,24y3.

8y2

Find the GCF of 4x2y,6xy3.

Solution

Factor each coefficient into primes and write the [redacted] common factors in each column.

Bring down the common factors.

[redacted]

Multiply the factors.

[redacted]

The GCF of 4x2y and 6xy3 is 2xy.

Find the GCF: $6ab^4, 8a^2b$.

2ab

Find the GCF: $9m^5n^2, 12m^3n$.

$3m^3n$

Find the GCF of: $21x^3, 9x^2, 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the		

co
ea[redacted]

Bring down the common factors.

[redacted]

Multiply the factors.

[redacted]

The GCF of 21x3, 9x2 and 15x is 3x.

Find the greatest common factor: 25m4,35m3,20m2.
5m2

Find the greatest common factor: 14x3,70x2,105x.

7x

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2\cdot6 \text{or} 3\cdot4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

$2(x+7) \text{factors} 2\cdot x+2\cdot7 2x+14 \text{product}$

Now we will start with a product, like $2x+14$, and end with its factors, $2(x+7)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If a,b,c are real numbers, then

$a(b+c)=ab+ac \text{and} ab+ac=a(b+c)$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

How to Factor the Greatest Common Factor from a Polynomial

Factor: $4x+12$.

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \quad \cdot x$ $12 = 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$

Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$

Step 3. Use the “reverse” Distributive Property to factor the expression.		$4(x + 3)$

Step 4. Check by multiplying the factors.	$4(x+3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

Factor: $6a + 24$.

$6(a + 4)$

Factor: $2b + 14$.

$2(b + 7)$

Factor the greatest common factor from a polynomial.

Find the GCF of all the terms of the polynomial. Rewrite each term as a product using the GCF. Use

the “reverse” Distributive Property to factor the expression. Check by multiplying the factors.

Factor as a Noun and a Verb

We use “factor” as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

Factor: $5a + 5$.

Solution

Find the GCF of $5a$ and 5.

[redacted]

$5a + 5$

Rewrite each term as a product using the GCF.

$5 \cdot a + 5 \cdot 1$

Use the Distributive Property "in reverse" to factor the GCF.

$5(a + 1)$

Check by mulitplying the factors to get the orginal polynomial.

$5(a+1)$

$5 \cdot a+5 \cdot 1$

$5a+5$✓

Factor: $14x+14$.

$14(x+1)$

Factor: $12p+12$.

$12(p+1)$

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

Factor: $12x-60$.
Solution

Find the GCF of $12x$ and 60.

[redacted]

[redacted]

Rewrite each term as a product using the GCF.

[redacted]

Factor the GCF.

[redacted]

Check by mulitplying the factors.

$12(x-5)$

$12{\cdot}x-12{\cdot}5$

$12x-60$✓

Factor: $18u-36$.

$8(u-2)$

Factor: $30y-60$.

$30(y-2)$

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

Factor: $4y2+24y+28$.
Solution   We start by finding the GCF of all three terms.

Find the GCF of $4y2$, $24y$ and 28.		

$$\begin{aligned} 4y^2 &= 2\cdot 2\cdot \qquad\quad y\cdot y \\ 24y &= 2\cdot 2\cdot 2\cdot 3\cdot \ y \\ 28 &= 2\cdot 2\cdot \qquad 7 \\ \hline \text{GCF} &= 2\cdot 2 \\ \text{GCF} &= 4 \end{aligned}$$

$$4y^2+24y+28$$

Rewrite each term as a		

pr[redacted]
Factor the GCF.

[redacted]

Check by mulitplying.

$4(y2+6y+7)$

$4\cdot y2+4\cdot 6y+4\cdot 7$

$4y2+24y+28$✓

Factor: $5x2-25x+15$.

$5(x2-5x+3)$

Factor: $3y2-12y+27$.

$3(y2-4y+9)$

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.

$$\begin{aligned} 5x^3 &= 5 \cdot x \cdot x \cdot x \\ 25x^2 &= 5 \cdot 5 \cdot x \cdot x \\ \hline \text{GCF} &= 5 \cdot x \cdot x \\ \text{GCF} &= 5x^2 \end{aligned}$$

$$5x^3 - 25x^2$$

Rewrite each term.

$$5x^2 \cdot x - 5x^2 \cdot 5$$

Factor the GCF.

$$5x^2(x - 5)$$

Check.

$5x^2(x - 5)$

$5x^2 \cdot x - 5x^2 \cdot 5$

$5x^3 - 25x^2$✓

Factor: $2x3+12x2$.

$2x2(x+6)$

Factor: $6y3-15y2$.

$3y2(2y-5)$

Factor: $21x3-9x2+15x$.

Solution

In a previous example we found the GCF of $21x3, 9x2, 15x$ to be $3x$.

[redacted]

Rewrite each term using the GCF, $3x$.

[redacted]

Factor the GCF.

[redacted]

Check.

$3x(7x2-3x+5)$

$3x\cdot 7x2-3x\cdot 3x+3x\cdot 5$

$21x3-9x2+15x$✓

Factor: $20x3-10x2+14x$.

$2x(10x2-5x+7)$

Factor: $24y3-12y2-20y$.

$4y(6y2-3y-5)$

Factor: $8m3 - 12m2n + 20mn2$.

Solution

Find the GCF of $8m3$, $12m2n$, $20mn2$.

[redacted]

Rewrite each term.

[redacted]

Factor the GCF.

[redacted]

Check.

$4m(2m^2 - 3mn + 5n^2)$

$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$

$8m^3 - 12m^2n + 20mn^2$ ✓

Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

$3y^2(3x + 2x^2 + 7y)$

Factor: $3p^3 - 6p^2q + 9pq^3$.

$3p(p^2 - 2pq + 3q^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Factor: $-8y-24$.

Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the ... is 8. ... n $-$ ... ne... coefficient, we use ... 8 as the GCF.

$$8y = 2\cdot2\cdot2\cdot y$$
$$24 = 2\cdot2\cdot2\cdot3$$
$$\text{GCF} = 2\cdot2\cdot2$$
$$\text{GCF} = 8$$

Rewrite each term using the GCF.

$$-8y-24$$

$$-8\cdot y+(-8)\cdot 3$$

Factor the GCF.

$$-8(y+3)$$

Check.

$-8(y+3)$

$-8\cdot y+(-8)\cdot 3$

$-8y-24$✓

Factor: $-16z-64$.

$-8(8z+8)$

Factor: $-9y-27$.

$-9(y+3)$

Factor: $-6a2+36a$.

Solution

The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the

$$6a^2 = 2 \cdot 3 \cdot a \cdot a$$
$$36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$$
$$\text{GCF} = 2 \cdot 3 \cdot a$$
$$\text{GCF} = 6a$$

$-6a^2 + 36a$

Rewrite each term using the GCF.

$6a \cdot a \quad (6a) \cdot 6$

Factor the GCF.

$6a(a \quad 6)$

Check.

$-6a(a-6)$

$-6a \cdot a + (-6a)(-6)$

$-6a2+36a$✓

Factor: $-4b2+16b$.

$-4b(b-4)$

Factor: $-7a2+21a$.

$-7a(a-3)$

Factor: $5q(q+7)-6(q+7)$.

Solution

The GCF is the binomial $q+7$.

[redacted]

Factor the GCF, $(q + 7)$.

[redacted]

Check on your own by multiplying.

Factor: $4m(m+3)-7(m+3)$.
$(m+3)(4m-7)$

Factor: $8n(n-4)+5(n-4)$.
$(n-4)(8n+5)$

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

How to Factor by Grouping

Factor: $xy + 3y + 2x + 6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy + 3y + 2x + 6$
	No, so let's separate the first two terms from the second two.	$\underbrace{xy + 3y}\underbrace{+ 2x + 6}$

Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x + 3)\underbrace{+ 2x + 6}$
	Factor the GCF from the second two terms.	$y(x + 3) + 2(x + 3)$

Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$. Factor out the common factor.	$y(x+3)+2(x+3)$ $(x+3)(y+2)$

Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

Factor: $xy+8y+3x+24.$

$(x+8)(y+3)$

Factor: $ab+7b+8a+56.$

$(a+7)(b+8)$

Factor by grouping.

Group terms with common factors. Factor out the common factor in each group. Factor the common factor from the expression. Check by multiplying the factors.

Factor: $x2+3x-2x-6$.

Solution

There is no GCF in all four terms.$x2+3x-2x-6$Separate into two parts.$x2+3x_\ -2x-6_$Factor the GCF from both parts. Be carefulwith the signs when factoring the GCF fromthe last two terms.$x(x+3)-2(x+3)(x+3)(x-2)$Check on your own by multiplying.

Factor: $x2+2x-5x-10$.

$(x-5)(x+2)$

Factor: $y2+4y-7y-28$.
$(y+4)(y-7)$

Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- Greatest Common Factor (GCF)
- Factoring Out the GCF of a Binomial
- Greatest Common Factor (GCF) of Polynomials

Key Concepts

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:

  Factor each coefficient into primes. Write all variables with exponents in expanded form. List all factors—matching common factors in a column. In each column, circle the common factors. Bring down the common factors that all

expressions share. Multiply the factors as in [link].

- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:

  Find the GCF of all the terms of the polynomial. Rewrite each term as a product using the GCF. Use the ‘reverse’ Distributive Property to factor the expression. Check by multiplying the factors as in [link].

- **Factor by Grouping:** To factor a polynomial with 4 four or more terms

  Group terms with common factors. Factor out the common factor in each group. Factor the common factor from the expression. Check by multiplying the factors as in [link].

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

8, 18

2

24, 40

72, 162

18

150, 275

10*a*, 50

10

5*b*, 30

3x,10x2

---

x

21b2,14b

8w2,24w3

8w2

30x2,18x3

10p3q,12pq2

2pq

8a2b3,10ab2

12m2n3,30m5n3

6m2n3

28x2y4,42x4y4

10a3,12a2,14a

$2a$

$20y3,28y2,40y$

$35x3,10x4,5x5$

$5x3$

$27p2,45p3,9p4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

$4x+20$

$4(x+5)$

$8y+16$

$6m+9$

$3(2m+3)$

$14p+35$

$9q+9$

$9(q+1)$

$7r+7$

$8m-8$

$8(m-1)$

$4n-4$

$9n-63$

$9(n-7)$

$45b-18$

$3x2 + 6x - 9$

$3(x2 + 2x - 3)$

$4y2 + 8y - 4$

$8p2 + 4p + 2$

$2(4p2 + 2p + 1)$

$10q2 + 14q + 20$

$8y3 + 16y2$

$8y2(y + 2)$

$12x3 - 10x$

$5x3 - 15x2 + 20x$

$5x(x2 - 3x + 4)$

$8m2 - 40m + 16$

$12xy2 + 18x2y2 - 30y3$

$6y2(2x + 3x2 - 5y)$

$21pq2 + 35p2q2 - 28q3$

$-2x - 4$

$-2(x + 2)$

$-3b + 12$

$5x(x + 1) + 3(x + 1)$

$(x + 1)(5x + 3)$

$2x(x - 1) + 9(x - 1)$

$3b(b - 2) - 13(b - 2)$

$(b-2)(3b-13)$

$6m(m-5)-7(m-5)$

Factor by Grouping

In the following exercises, factor by grouping.

$xy+2y+3x+6$

$(y+3)(x+2)$

$mn+4n+6m+24$

$uv-9u+2v-18$

$(u+2)(v-9)$

$pq-10p+8q-80$

$b2+5b-4b-20$

$(b-4)(b+5)$

$m2 + 6m - 12m - 72$

$p2 + 4p - 9p - 36$

$(p - 9)(p + 4)$

$x2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

$-20x - 10$

$-10(2x + 1)$

$5x3 - x2 + x$

$3x3 - 7x2 + 6x - 14$

$(x2 + 2)(3x - 7)$

$x3 + x2 - x - 1$

$x2+xy+5x+5y$

$(x+y)(x+5)$

$5x3-3x2-5x-3$

Everyday Math

Area of a rectangle The area of a rectangle with length 6 less than the width is given by the expression $w2-6w$, where $w=$ width. Factor the greatest common factor from the polynomial.

$w(w-6)$

Height of a baseball The height of a baseball *t* seconds after it is hit is given by the expression $-16t2+80t+4$. Factor the greatest common factor from the polynomial.

Writing Exercises

The greatest common factor of 36 and 60 is 12.

Explain what this means.

Answers will vary.

What is the GCF of y4,y5,andy10? Write a general rule that tells you how to find the GCF of ya,yb,andyc.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

factoring
: Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor
: The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

Factor Trinomials

By the end of this section, you will be able to:

- Factor trinomials of the form $x2+bx+c$
- Factor trinomials of the form $ax2+bx+c$ using trial and error
- Factor trinomials of the form $ax2+bx+c$ using the 'ac' method
- Factor using substitution

Before you get started, take this readiness quiz.

Find all the factors of 72.
If you missed this problem, review [link].

1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

Find the product: $(3y+4)(2y+5)$.
If you missed this problem, review [link].

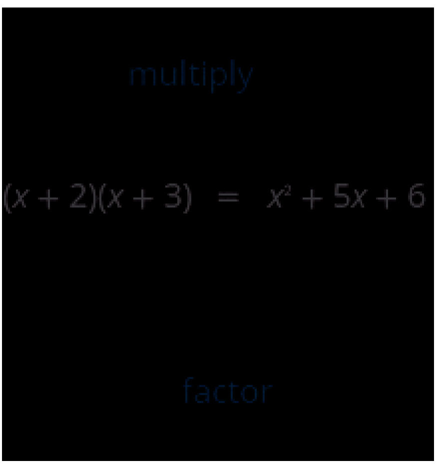
$6y2+23y+20$

Simplify: $-9(6);\ -9(-6)$.
If you missed this problem, review [link].

$-54, 54$

Factor Trinomials of the Form $x2+bx+c$

You have already learned how to multiply binomials using FOIL. Now you’ll need to “undo” this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a trinomial of

the form x2 + bx + c, such as x2 + 5x + 6 and factor it to (x + 2)(x + 3), let’s start with two general binomials of the form (x + m) and (x + n).

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x2 + bx + c.	$\begin{array}{l} x^2 + bx + c \\ x^2+(m+n)x+mn \end{array}$

This tells us that to factor a trinomial of the form x2 + bx + c, we need two factors (x + m) and (x + n) where the two numbers *m* and *n* multiply to *c* and add to *b*.

How to Factor a Trinomial of the form x2 + bx + c

Factor: $x2+11x+24$.

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2+11x+24$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m* and *n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m+n=b$	Find two numbers that multiply to 24 and add to 11.	
Step 3. Use *m* and *n* as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x+3)(x+8)$
Step 4. Check by multiplying the factors.		$(x+3)(x+8)$ $x^2+8x+3x+24$ $x^2+11x+24$ ✓

Factors of 24	Sum of factors
1, 24	$1+24=25$
2, 12	$2+12=14$
3, 8	$3+8=11$*
4, 6	$4+6=10$

Factor: $q2+10q+24$.

$(q+4)(q+6)$

Factor: $t^2+14t+24$.

$(t+2)(t+12)$

Let’s summarize the steps we used to find the factors.

Factor trinomials of the form x^2+bx+c.

Write the factors as two binomials with first terms *x*. $x^2+bx+c(x)(x)$ Find two numbers *m* and *n* that

- multiply to $c, m\cdot n=c$
- add to $b, m+n=b$

Use *m* and *n* as the last terms of the factors. $(x+m)(x+n)$ Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let’s look first at trinomials with only the middle term

negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

Factor: $y2-11y+28$.

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y2-11y+28$
Write the factors as two binomials with first terms y.	$(y)(y)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1, -28$	$-1+(-28)=-29$
$-2, -14$	$-2+(-14)=-16$
$-4, -7$	$-4+(-7)=-11$*

Use $-4, -7$ as the last terms of the binomials.	$(y-4)(y-7)$

Check:
$(y-4)(y-7)y2-7y$
$-4y+28y2-11y$
$+28$✓

Factor: $u2-9u+18$.

$(u-3)(u-6)$

Factor: $y2 - 16y + 63$.

$(y - 7)(y - 9)$

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

Factor: $2x + x2 - 48$.

	$2x + x2 - 48$
First we put the terms in decreasing degree order.	$x2 + 2x - 48$
Factors will be two binomials with first terms x.	$(x)(x)$

Factors of -48	Sum of factors
$-1,48$	$-1 + 48 = 47$
$-2,24$	$-2 + 24 = 22$
$-3,16$	$-3 + 16 = 13$
$-4,12$	$-4 + 12 = 8$
$-6,8$	$-6 + 8 = 2$*

Use $-6,8$as the last terms of the binomials.	$(x - 6)(x + 8)$

Check:
$(x-6)(x+8)x^2-6q$
$+8q-48x^2+2x-48$✓

Factor: $9m+m^2+18$.

$(m+3)(m+6)$

Factor: $-7n+12+n^2$.

$(n-3)(n-4)$

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x\cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the

coefficients *b* and *c*, you use the same process summarized in How To Factor trinomials.

Factor: $r^2 - 8rs - 9s^2$.
We need *r* in the first term of each binomial and *s* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Note that the first terms are r, last terms contain s.	$r^2 - 8rs - 9s^2$ $(r\quad s)(r\quad s)$
Find the numbers that multiply to -9 and add to -8.	

Factors of -9	Sum of factors
$1, -9$	$-1+9=8$
$-1, 9$	$1+(-9)=-8^{*}$
$3, -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of the last terms.	$(r+s)(r-9s)$

Check:

$(r-9s)(r+s)$ $r^2+rs-9rs-9s^2$ $r^2-8rs-9s^2$ ✓

Factor: $a^2-11ab+10b^2$.

$(a-b)(a-10b)$

Factor: $m^2 - 13mn + 12n^2$.
$(m - n)(m - 12n)$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

Factor: $u^2 - 9uv - 12v^2$.
We need *u* in the first term of each binomial and *v* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Note that the first terms are u, last terms contain v.	$u^2 - 9uv - 12v^2$ $(u\ v)(u\ v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1,-12$	$1+(-12)=-11$
$-1,12$	$-1+12=11$
$2,-6$	$2+(-6)=-4$
$-2,6$	$-2+6=4$
$3,-4$	$3+(-4)=-1$
$-3,4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

Factor: x2 − 7xy − 10y2.

prime

Factor: $p^2+15pq+20q^2$.

prime

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c.

Strategy for Factoring Trinomials of the Form x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

x^2+bx+c $(x+m)(x+n)$ When c is positive, m and n have the same sign. b positive b negative m,n positive m,n negative x^2+5x+6 x^2-6x+8 $(x+2)(x+3)$ $(x-4)(x-2)$ same signs same signs When c is negative, m and n have opposite signs. x^2+x-12 $x^2-2x-15$ $(x+4)(x-3)$ $(x-5)(x+3)$ opposite signs opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

Factor completely: $4x^3+16x^2-20x$.

Is there a greatest common factor?	$4x^3+16x^2-20x$
Yes, GCF $=4x$. Factor it.	$4x(x^2+4x-5)$
Binomial, trinomial, or more than three terms?	
It is a trinomial. So	$4x(x)(x)$

“undo FOIL.”

Use a table like the one shown to find two numbers that multiply to -5 and add to 4.

$4x(x-1)(x+5)$

Factors of -5	Sum of factors
$-1,5$	$-1+5=4^{*}$
$1,-5$	$1+(-5)=-4$

Check:

$4x(x-1)(x+5)4x(x2+5x-x-5)4x(x2+4x-5)4x3+16x2-20x$✓

Factor completely: $5x3+15x2-20x$.
$5x(x-1)(x+4)$

Factor completely: $6y3+18y2-60y$.
$6y(y-2)(y+5)$

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let’s factor the trinomial $3x2+5x+2$.

From our earlier work, we expect this will factor into two binomials.
$3x2+5x+2()()$

We know the first terms of the binomial factors will multiply to give us $3x2$. The only factors of $3x2$ are $1x,3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$1x,\ 3x$$
$$(x \quad\ )(3x \quad\ )$$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.

$3x^2 + 5x + 2$		$3x^2 + 5x + 2$
$1x,\ 3x \qquad 1,\ 2$		$1x,\ 3x \qquad 1,\ 2$
$(x + 1)(3x + 2)$	or	$(x + 2)(3x + 1)$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$3x^2 + 5x + 2$		$3x^2 + 5x + 2$
$1x,\ 3x \qquad 1,\ 2$		$1x,\ 3x \qquad 1,\ 2$
$(x + 1)(3x + 2)$	or	$(x + 2)(3x + 1)$
$3x$		$6x$
$2x$		$1x$
$5x$		$7x$

Since the middle term of the trinomial is 5x, the factors in the first case will work. Let’s use FOIL to check.

(x + 1)(3x + 2)3x2 + 2x + 3x + 23x2 + 5x + 2✓

Our result of the factoring is:
3x2 + 5x + 2(x + 1)(3x + 2)

How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error:
3y2 + 22y + 7.

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$

Step 2. Factor any GCF.	There is no GCF.	

Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y$, $3y$.	$3y^2 + 22y + 7$ $1y, 3y$
	Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ $1y, 3y$ $(y\quad)(3y\quad)$

Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ $1y, 3y \quad 1, 7$ $(y\quad)(3y\quad)$

Step 5. Test all the possible combinations of the factors until the correct product is found.

$3y^2 + 22y + 7$

$1y, 3y \quad 1, 7$

$(y + 1)(3y + 7)$

$3y$

$7y$

$10y$ No! We need $22y$

$3y^2 + 22y + 7$

$1y, 3y \quad 1, 7$

$(y + 7)(3y + 1)$

$21y$

$+y$

$22y$

$3y^2 + 22y + 7$	
Possible factors	**Product**
$(y + 1)(3y + 7)$	$3y^2 + 10y + 7$
$(y + 7)(3y + 1)$	$3y^2 + 22y + 7$

Step 6. Check by multiplying.

$(y + 7)(3y + 1)$

$3y^2 + 22y + 7$ ✓

Factor completely using trial and error: 2a2 + 5a + 3.

(a + 1)(2a + 3)

Factor completely using trial and error: 4b2 + 5b + 1.

$(b+1)(4b+1)$

Factor trinomials of the form $ax2+bx+c$ using trial and error.

Write the trinomial in descending order of degrees as needed. Factor any GCF. Find all the factor pairs of the first term. Find all the factor pairs of the third term. Test all the possible combinations of the factors until the correct product is found. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Factor completely using trial and error: $6b2-13b+5$.

The trinomial is already in descending or[redacted]

Find the factors of the first term. [redacted]

Find the factors of the last term. Consider the sig[redacted]m, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.

Consider all the combinations of factors.

$6b^2 - 13b + 5$	
Possible factors	**Product**
$(b-1)(6b-5)$	$6b^2-11b+5$
$(b-5)(6b-1)$	$6b^2-31b+5$

$(2b-1)(3b-5)$	$6b^2-13b+5$*
$(2b-5)(3b-1)$	$6b^2-17b+5$

The correct factors are those whose product is the original trinomial.	$(2b-1)(3b-5)$
Check by multiplying: $(2b-1)(3b-5)$ $6b^2-10b-3b+5$ $6b^2-13b+5$✓	

Factor completely using trial and error: $8x^2-14x+3$.

$(2x-3)(4x-1)$

Factor completely using trial and error: $10y2 - 37y + 7$.
$(2y - 7)(5y - 1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

Factor completely using trial and error: $18x2 - 37xy + 15y2$.

The trinomial is already in descending order. $18x^2 - 37xy + 15y^2$		
Find the factors of the first term.		

$18x^2 - 37xy + 15y^2$
$1x \cdot 18x$
$2x \cdot 9x$
$3x \cdot 6x$

Find the factors of the last term. Consider the sig

$18x^2 - 37xy + 15y^2$

$1x \cdot 18x$	$-1, -15$
$2x \cdot 9x$	$-3, -5$
$3x \cdot 6x$	

Si and the e middle term is negative, we use the negative factors.

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$

Possible factors	Product
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is $(2x - 3y)(9x - 5y)$

the original trinomial.
Check by multiplying:
$(2x-3y)(9x-5y)$
$18x^2-10xy-27xy+15y^2$
$18x^2-37xy+15y^2$ ✓

Factor completely using trial and error $18x^2-3xy-10y^2$.

$(3x+2y)(6x-5y)$

Factor completely using trial and error: $30x^2-53xy-21y^2$.

$(3x+y)(10x-21y)$

Don’t forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

Factor completely using trial and error: $-10y^4-55y^3-60y^2$.

$-10y^4 - 55y^3 - 60y^2$

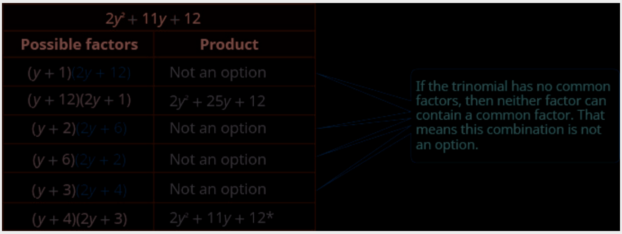
Notice the greatest common factor, so factor it first.

$-5y^2(2y^2+11y+12)$

Factor the trinomial.

$$-5y^2\left(\begin{array}{lr} 2y^2 + 11y + 12 & \\ y \cdot 2y & 1 \cdot 12 \\ & 2 \cdot 6 \\ & 3 \cdot 4 \end{array}\right)$$

Consider all the combinations.



$2y^2 + 11y + 12$	
Possible factors	Product
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

The correct factors are those whose product is the original trinomial. Remember to include the factor $-5y2$.	$-5y2(y+4)(2y+3)$
Check by multiplying: $-5y2(y+4)(2y+3) - 5y2(2y2+8y+3y+12) - 10y4 - 55y3 - 60y2$✓	

Factor completely using trial and error: $15n3 - 85n2 + 100n$.

$5n(n-4)(3n-5)$

Factor completely using trial and error: $56q^3+320q^2-96q$.

$8q(q+6)(7q-2)$

Factor Trinomials of the Form ax^2+bx+c using the "ac" Method

Another way to factor trinomials of the form ax^2+bx+c is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

How to Factor Trinomials using the “ac” Method

Factor using the *‘ac’* method: $6x2+7x+2$.

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers *m* and *n* that: Multiply to *ac*. $m \cdot n = a \cdot c$ Add to *b*. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using *m* and *n*. $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$

Step 6. Check by multiplying the factors.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Factor using the ‘ac’ method: $6x2+13x+2$.

$(x+2)(6x+1)$

Factor using the ‘ac’ method: $4y2+8y+3$.

$(2y+1)(2y+3)$

The “ac” method is summarized here.

Factor trinomials of the form $ax2+bx+c$ using the “ac” method.

Factor any GCF. Find the product *ac*. Find two numbers *m* and *n* that:

Multiply to ac $m\cdot n=a\cdot c$ Add to b $m+n=b$ ax^2+bx+c

Split the middle term using *m* and *n*. $ax^2+mx+nx+c$ Factor by grouping. Check by multiplying the factors.

Don’t forget to look for a common factor!

Factor using the *‘ac’* method: $10y^2-55y+70$.

Is there a greatest common factor?	
Yes. The GCF is 5.	$10y^2 - 55y + 70$
Factor it.	$5(2y^2 - 11y + 14)$

The trinomial inside the pa[redacted] a leading coefficient that is not 1.	
Find the product ac.	$ac = 28$
Find two numbers that multiply to ac	$(-4)(-7) = 28$
and add to *b*.	$-4 + (-7) = -11$
Split the middle term.	
[redacted]	
[redacted]	
Factor the trinomial by grouping.	
$5(y\,(2y-7) - 2(2y-7))$	
[redacted]	
Check by multiplying all three factors.	
$5(y-2)(2y$	

$-7)5(2y^2-7y$
$-4y$
$+14)5(2y^2-11y$
$+14)10y^2-55y$
$+70$ ✓

Factor using the ‘ac’ method: $16x^2-32x+12$.

$4(2x-3)(2x-1)$

Factor using the ‘ac’ method: $18w^2-39w+18$.

$3(3w-2)(2w-3)$

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax2+bx+c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax2+bx+c$ form. This is called factoring by substitution. It is standard to use *u* for the substitution.

In the $ax2+bx+c$, the middle term has a variable, *x*, and its square, $x2$, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Factor by substitution: $x4-4x2-5$.
The variable part of the middle term is $x2$ and its square, $x4$, is the variable part of the first term. (We know $(x2)2=x4$). If we let $u=x2$, we can put our trinomial in the $ax2+bx+c$ form we need to factor it.

[redacted]

Rewrite the trinomial to prepare for the su[redacted]

Let $u = x2$ and substitute.

[redacted]

Factor the trinomial.

[redacted]

Replace u with $x2$.

[redacted]

Check:

$(x2+1)$
$(x2-5)x4-5x2+x2-5x4-4x2-5$✓

Factor by substitution: $h4+4h2-12$.

$(h2-2)(h2+6)$

Factor by substitution: y^4-y^2-20.
$(y^2+4)(y^2-5)$

Sometimes the expression to be substituted is not a monomial.

Factor by substitution: $(x-2)^2+7(x-2)+12$
The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

$(x-2)^2+7(x-2)+12$

Rewrite the trinomial to prepare for the	

su[redacted]

Let u = x − 2 and substitute.

[redacted]

Factor the trinomial.

[redacted]

Replace *u* with x − 2.

[redacted]

Simplify inside the parentheses.

[redacted]

This could also be factored by first multiplying out the (x − 2)2 and the 7(x − 2) and then combining like terms and then factoring. Most students prefer the substitution method.

Factor by substitution: (x − 5)2 + 6(x − 5) + 8.

(x − 3)(x − 1)

Factor by substitution: $(y-4)2+8(y-4)+15$.

$(y-1)(y+1)$

Access this online resource for additional instruction and practice with factoring.

- Factor a trinomial using the AC method

Key Concepts

- **How to factor trinomials of the form $x2+bx+c$.**

  Write the factors as two binomials with first terms *x*. $x2+bx+c(x)(x)$ Find two numbers *m* and *n* that multiply $toc, m·n=cadd tob, m+n=b$ Use *m* and *n* as the last terms of the factors. $(x+m)(x+n)$ Check by multiplying the factors.

- **Strategy for Factoring Trinomials of the Form $x2+bx+c$:** When we factor a trinomial,

we look at the signs of its terms first to determine the signs of the binomial factors.
x2 + bx + c(x + m)(x + n)Whencis positive,mandnhave the same sign.bpositivebnegativem,npositivem,nnegativex2 + 5x + 6x2 − 6x + 8(x + 2)(x + 3)(x − 4)(x − 2)same signssame signsWhencis negative,mandnhave opposite signs.x2 + x − 12x2 − 2x − 15(x + 4)(x − 3)(x − 5)(x + 3)opposite signsopposite signs
Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

- **How to factor trinomials of the form ax2 + bx + c using trial and error.**

  Write the trinomial in descending order of degrees as needed. Factor any GCF. Find all the factor pairs of the first term. Find all the factor pairs of the third term. Test all the possible combinations of the factors until the correct product is found. Check by multiplying.

- **How to factor trinomials of the form ax2 + bx + c using the "ac" method.**

  Factor any GCF. Find the product *ac*. Find two numbers *m* and *n* that:
  Multiply toac.m·n = a·cAdd tob.m + n = bax2 + bx + c Split the middle term using *m* and *n*. ax2 + mx + nx + c Factor by grouping. Check by multiplying the factors.

Practice Makes Perfect

Factor Trinomials of the Form $x2+bx+c$

In the following exercises, factor each trinomial of the form $x2+bx+c$.

$p2+11p+30$

$(p+5)(p+6)$

$w2+10x+21$

$n2+19n+48$

$(n+3)(n+16)$

$b2+14b+48$

$a2+25a+100$

$(a+5)(a+20)$

$u2 + 101u + 100$

$x2 - 8x + 12$

$(x - 2)(x - 6)$

$q2 - 13q + 36$

$y2 - 18y + 45$

$(y - 3)(y - 15)$

$m2 - 13m + 30$

$x2 - 8x + 7$

$(x - 1)(x - 7)$

$y2 - 5y + 6$

$5p - 6 + p2$

$(p-1)(p+6)$

$6n-7+n2$

$8-6x+x2$

$(x-4)(x-2)$

$7x+x2+6$

$x2-12-11x$

$(x-12)(x+1)$

$-11-10x+x2$

In the following exercises, factor each trinomial of the form $x2+bxy+cy2$.

$x2-2xy-80y2$

$(x+8y)(x-10y)$

$p2 - 8pq - 65q2$

$m2 - 64mn - 65n2$

$(m + n)(m - 65n)$

$p2 - 2pq - 35q2$

$a2 + 5ab - 24b2$

$(a + 8b)(a - 3b)$

$r2 + 3rs - 28s2$

$x2 - 3xy - 14y2$

Prime

$u2 - 8uv - 24v2$

$m2 - 5mn + 30n2$

Prime

$c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

$p^3 - 8p^2 - 20p$

$p(p - 10)(p + 2)$

$q^3 - 5q^2 - 24q$

$3m^3 - 21m^2 + 30m$

$3m(m - 5)(m - 2)$

$11n^3 - 55n^2 + 44n$

$5x^4 + 10x^3 - 75x^2$

$5x2(x-3)(x+5)$

$6y4+12y3-48y2$

$2t2+7t+5$

$(2t+5)(t+1)$

$5y2+16y+11$

$11x2+34x+3$

$(11x+1)(x+3)$

$7b2+50b+7$

$4w2-5w+1$

$(4w-1)(w-1)$

$5x2-17x+6$

$4q2-7q-2$

$(4q+1)(q-2)$

$10y2-53y-11$

$6p2-19pq+10q2$

$(2p-5q)(3p-2q)$

$21m2-29mn+10n2$

$4a2+17ab-15b2$

$(4a-3b)(a+5b)$

$6u2+5uv-14v2$

$-16x2-32x-16$

$-16(x+1)(x+1)$

$-81a2+153a+18$

$-30q3-140q2-80q$

$-10q(3q+2)(q+4)$

$-5y3-30y2+35y$

Factor Trinomials of the Form $ax2+bx+c$ using the ‘ac’ Method

In the following exercises, factor using the ‘ac’ method.

$5n2+21n+4$

$(5n+1)(n+4)$

$8w2+25w+3$

$4k2-16k+15$

$(2k-3)(2k-5)$

$5s2 - 9s + 4$

$6y2 + y - 15$

$(3y + 5)(2y - 3)$

$6p2 + p - 22$

$2n2 - 27n - 45$

$(2n + 3)(n - 15)$

$12z2 - 41z - 11$

$60y2 + 290y - 50$

$10(6y - 1)(y + 5)$

$6u2 - 46u - 16$

$48z3 - 102z2 - 45z$

$3z(8z+3)(2z-5)$

$90n^3+42n^2-216n$

$16s^2+40s+24$

$8(2s+3)(s+1)$

$24p^2+160p+96$

$48y^2+12y-36$

$12(4y-3)(y+1)$

$30x^2+105x-60$

Factor Using Substitution

In the following exercises, factor using substitution.

x^4-6x^2-7

$(x^2+1)(x^2-7)$

$x4 + 2x2 - 8$

$x4 - 3x2 - 28$

$(x2 - 7)(x2 + 4)$

$x4 - 13x2 - 30$

$(x - 3)2 - 5(x - 3) - 36$

$(x - 12)(x + 1)$

$(x - 2)2 - 3(x - 2) - 54$

$(3y - 2)2 - (3y - 2) - 2$

$(3y - 4)(3y - 1)$

$(5y - 1)2 - 3(5y - 1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

$u2 - 12u + 36$

$(u - 6)(u - 6)$

$x2 - 14x - 32$

$r2 - 20rs + 64s2$

$(r - 4s)(r - 16s)$

$q2 - 29qr - 96r2$

$12y2 - 29y + 14$

$(4y - 7)(3y - 2)$

$12x2 + 36y - 24z$

$6n2 + 5n - 4$

$(2n - 1)(3n + 4)$

$3q2 + 6q + 2$

$13z2 + 39z - 26$

$13(z2 + 3z - 2)$

$5r2 + 25r + 30$

$3p2 + 21p$

$3p(p + 7)$

$7x2 - 21x$

$6r2 + 30r + 36$

$6(r + 2)(r + 3)$

$18m2 + 15m + 3$

$24n2 + 20n + 4$

$4(2n+1)(3n+1)$

$4a^2+5a+2$

x^4-4x^2-12

$(x^2+2)(x^2-6)$

x^4-7x^2-8

$(x+3)^2-9(x+3)-36$

$(x-9)(x+6)$

$(x+2)^2-25(x+2)-54$

Writing Exercises

Many trinomials of the form x^2+bx+c factor into the product of two binomials $(x+m)(x+n)$. Explain how you find the values of *m* and *n*.

Answers will vary.

Tommy factored $x2-x-20$ as $(x+5)(x-4)$. Sara factored it as $(x+4)(x-5)$. Ernesto factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

List, in order, all the steps you take when using the “ac” method to factor a trinomial of the form $ax2+bx+c$.

Answers will vary.

How is the “ac” method similar to the “undo FOIL” method? How is it different?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
factor trinomials of the form x^2+bx+c			
factor trinomials of the form ax^2+bx+c using trial and error.			
factor trinomials of the form ax^2+bx+c with using the "*ac*" method.			
factor using substitution.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Factor Special Products
By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Before you get started, take this readiness quiz.

1. Simplify: $(3x2)3$.
   If you missed this problem, review [link].
2. Multiply: $(m+4)2$.
   If you missed this problem, review [link].
3. Multiply: $(x-3)(x+3)$.
   If you missed this problem, review [link].

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\overset{(a\ +\ b)^2}{(3x+4)^2}$$

$$\overset{a^2\ \ +\ 2\cdot a\ \cdot\ b\ \ +\ b^2}{(3x)^2+2(3x\cdot 4)+4^2}$$

$$9x^2+24x+16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If *a* and *b* are real numbers

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad a^2 - 2ab + b^2 = (a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

How to Factor Perfect Square Trinomials

Factor: $9x^2 + 12x + 4$.

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \qquad (2)^2$
• Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \qquad (2)^2$ $2(3x)(2)$ $12x$

Step 2. Write the square of the binomial.	$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot \ \ + \ \ ^2$ $(a + b)^2$ $(3x + \ \ )^2$

Step 3. Check.	$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

Factor: $4x2 + 12x + 9$.

$(2x + 3)2$

Factor: $9y2 + 24y + 16$.

$(3y + 4)2$

The sign of the middle term determines which

pattern we will use. When the middle term is negative, we use the pattern $a^2-2ab+b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

Factor perfect square trinomials.

Step 1.Does the trinomial fit the pattern?$a^2+2ab+b^2$ $a^2-2ab+b^2$ Is the first term a perfect square? $(a)^2$ $(a)^2$ Write it as a square. Is the last term a perfect square?$(a)^2(b)^2$ $(a)^2(b)^2$ Write it as a square. Check the middle term. Is it$2ab$? $(a)^2\searrow 2\cdot a\cdot b\swarrow(b)^2$ $(a)^2\searrow 2\cdot a\cdot b\swarrow(b)^2$ Step 2.Write the square of the binomial.$(a+b)^2$ $(a-b)^2$ Step 3.Check by multiplying.

We'll work one now where the middle term is negative.

Factor: $81y^2-72y+16$.
The first and last terms are squares. See if the middle term fits the pattern of a perfect square

trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does it match $(a-b)^2$? Yes.	$a^2 - 2\,a\,b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y-4)^2$
Check by multiplying: $(9y-4)^2$ $(9y)^2 - 2\cdot 9y\cdot 4 + 4^2$ $81y^2 - 72y + 16$ ✓	

Factor: $64y^2 - 80y + 25$.
$(8y - 5)^2$

Factor: $16z^2 - 72z + 81$.
$(4z - 9)^2$

The next example will be a perfect square trinomial with two variables.

Factor: $36x^2 + 84xy + 49y^2$.

[redacted]

Test each term to verify the pa[redacted]

Factor.

[redacted]

Check by multiplying.

$$\begin{array}{l}(6x+7y)^2 \\ (6x)^2+2\cdot 6x\cdot 7y+(7y)^2 \\ 36x^2+84xy+49y^2 \checkmark\end{array}$$

Factor: $49x^2+84xy+36y^2$.

$(7x+6y)^2$

Factor: $64m^2+112mn+49n^2$.

$(8m+7n)2$

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Factor: $100x2y-80xy+16y$.

	$100x^2y-80xy+16y$
Is there a GCF? Yes, 4y, so factor it out.	$4y(25x^2-20x+4)$
Is this a perfect square trinomial?	

Verify the pattern.

$$4y[\underbrace{(5x)^2}_{a^2} - \underbrace{2}_{2} \cdot \underbrace{5x}_{a} \cdot \underbrace{2}_{b} + \underbrace{2^2}_{b^2}]$$

Factor.

$$4y(5x-2)^2$$

Remember: Keep the factor $4y$ in the final product.

Check:

$$4y(5x-2)^2$$
$$4y[(5x)^2 - 2\cdot 5x\cdot 2 + 2^2]$$
$$4y(25x^2 - 20x + 4)$$
$$100x^2y - 80xy + 16y\checkmark$$

Factor: $8x^2y - 24xy + 18y$.
$2y(2x-3)^2$

Factor: $27p^2q + 90pq + 75q$.

$3q(3p+5)2$

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here’s an example:

$$\underset{(a\ -\ b)\ (a\ +\ b)}{(3x-4)(3x+4)}$$

$$\underset{(a)^2\ -\ (b)^2}{(3x)^2-(4)^2}$$

$$9x^2-16$$

A difference of squares factors to a product of conjugates.

Difference of Squares Pattern

If *a* and *b* are real numbers,

$$a^2 - b^2 = (a-b)(a+b) \qquad a^2 \overset{\text{difference}}{-} b^2 = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

squares

Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

How to Factor a Trinomial Using the Difference of Squares

Factor: 64y2 − 1.

Step 1. Does the binomial fit the pattern?		$64y^2 - 1$
• Is this a difference?	Yes	$64y^2 - 1$
• Are the first and last terms perfect squares?	Yes	

Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(8y)^2 - 1^2$

Step 3. Write the product of conjugates. $(a - b)(a + b)$ $(8y - 1)(8y + 1)$

Step 4. Check. $(8y - 1)(8y + 1)$

$64y^2 - 1$ ✓

Factor: $121m2 - 1$.

$(11m - 1)(11m + 1)$

Factor: $81y2 - 1$.

$(9y - 1)(9y + 1)$

Factor differences of squares.

Step 1.Does the binomial fit the pattern?$a2 - b2$Is this a difference?___$-$___Are the first and last

terms perfect squares?Step 2.Write them as squares.$(a)^2-(b)^2$Step 3.Write the product of conjugates.$(a-b)(a+b)$Step 4.Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression a^2+b^2 is prime!

The next example shows variables in both terms.

Factor: $144x^2-49y^2$.

$144x^2-49y^2$Is this a difference of squares? Yes.$(12x)^2-(7y)^2$Factor as the product of conjugates.$(12x-7y)(12x+7y)$Check by multiplying.$(12x-7y)(12x+7y)$$144x^2-49y^2$✓

Factor: $196m2-25n2$.

$(16m-5n)(16m+5n)$

Factor: $121p2-9q2$.

$(11p-3q)(11p+3q)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we’ll factor a difference of squares twice!

Factor: $48x4y2-243y2$.

$48x4y2-243y2$Is there a GCF? Yes,$3y2$—factor it out!$3y2(16x4-81)$Is the binomial a difference of squares? Yes.$3y2((4x2)2-(9)2)$Factor as a product of conjugates.$3y2(4x2-9)(4x2+9)$Notice the first binomial is also a difference of squares! $3y2((2x)2-(3)2)(4x2+9)$Factor it as the product of conjugates.$3y2(2x-3)(2x+3)(4x2+9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:$3y2(2x-3)(2x+3)(4x2+9)3y2(4x2-9)(4x2+9)3y2(16x4-81)48x4y2-243y2$✓

Factor: $2x4y2-32y2$.

$2y2(x-2)(x+2)(x2+4)$

Factor: $7a4c2-7b4c2$.

$7c2(a-b)(a+b)(a2+b2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Factor: $x2-6x+9-y2$.
Notice that the first three terms form a perfect square trinomial.

[redacted]

Factor by grouping the first three terms.

[redacted]

Use the perfect square

tri[redacted]
pattern.

Is this a difference of squares? Yes.

Yes—write them as squares.

[redacted]

Factor as the product of conjugates.

[redacted]

[redacted]

You may want to rewrite the solution as $(x-y-3)(x+y-3)$.

Factor: $x^2-10x+25-y^2$.

$(x-5-y)(x-5+y)$

Factor: $x^2+6x+9-4y^2$.

$(x+3-2y)(x+3+2y)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$a3+b3=(a+b)(a2-ab+b2)a3-b3=(a-b)(a2+ab+b2)$

We'll check the first pattern and leave the second to you.

[redacted]

Distribute.

[redacted]

Multiply.

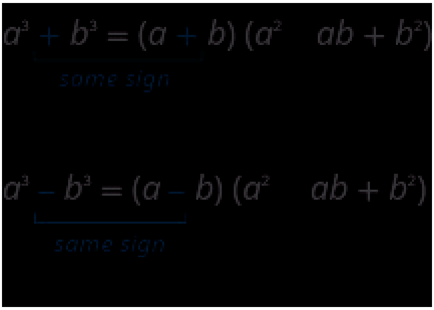
$$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$$

Combine like terms.

$$a^3 + b^3$$

Sum and Difference of Cubes Pattern
a3 + b3 = (a + b)(a2 − ab + b2)a3 − b3 = (a − b)(a2 + ab + b2)

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.



The trinomial factor in the sum and difference of

cubes pattern cannot be factored.

It will be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in [link].

n	1	2	3	4	5	6	7	8	9	10
n3	1	8	27	64	125	216	343	512	729	1000

How to Factor the Sum or Difference of Cubes

Factor: x3 + 64.

Step 1. Does the binomial fit the sum or difference of cubes pattern?		$x^3 + 64$
• Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$

Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$a^3 + b^3$ $x^3 + 4^3$

Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\underset{(x+4)}{(a+b)}\underset{(x^2-4x+4^2)}{(a^2-ab+b^2)}$
Step 4. Simplify inside the parentheses.	It is already simplified.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2-4x+16 \\ x+4 \\ \hline 4x^2-16x+64 \ ✓ \\ x^3-4x^2+16x \quad\quad \\ \hline x^3 \quad\quad\quad\quad +64 \end{array}$

Factor: x3 + 27.

(x + 3)(x2 − 3x + 9)

Factor: y3 + 8.

(y + 2)(y2 − 2y + 4)

Factor the sum or difference of cubes.

Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes? Write them as cubes. Use either the sum or difference of cubes pattern. Simplify inside the parentheses.
Check by multiplying the factors.

Factor: 27u3 − 125v3.

[redacted]

This binomial is a difference. The first and last
terms are perfect cubes.
Write the terms as cubes.

[redacted]

Use the difference of cubes pattern.

[redacted]

Simplify.

[redacted]

Check by multiplying. We'll leave the check to you.

Factor: 8x3 − 27y3.

(2x − 3y)(4x2 + 6xy + 9y2)

Factor: 1000m3 − 125n3.

(10m − 5n)(100m2 + 50mn + 25n2)

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Factor: 6x3y + 48y4.

[redacted]

Factor the common factor.

[redacted]

This binomial is a sum The first and last terms are perfect cubes.

Write the terms as cubes.

[redacted]

Use the sum of cubes pattern.

[redacted]

Simplify.

$6y(x+2y)(x^2-2xy+4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 6y. We’ll leave the multiplication for you.

Factor: 500p3 + 4q3.
4(5p + q)(25p2 − 5pq + q2)

Factor: 432c3 + 686d3.
2(6c + 7d)(36c2 − 42cd + 49d2)

The first term in the next example is a binomial cubed.

Factor: $(x+5)3-64x3$.

[redacted]

This binomial is a difference. The first and last terms are perfect cubes.

Write the terms as cubes.

[redacted]

Use the difference of cubes pattern.

[redacted]

Simplify.

[redacted]

[redacted]

Check by multiplying.

We'll leave the check to you.

Factor: $(y+1)3-27y3$.

$(-2y+1)(13y2+5y+1)$

Factor: $(n+3)3-125n3$.

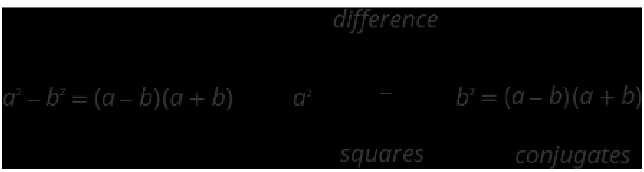
$(-4n+3)(31n2+21n+9)$

Access this online resource for additional instruction and practice with factoring special products.

- Factoring Binomials-Cubes #2

Key Concepts

- **Perfect Square Trinomials Pattern:** If a and b are real numbers,
  a2 + 2ab + b2 = (a + b)2a2 − 2ab + b2 = (a − b)2
- **How to factor perfect square trinomials.**
  Step 1.Does the trinomial fit the pattern? a2 + 2ab + b2a2 − 2ab + b2 Is the first term a perfect square?(a)2(a)2 Write it as a square. Is the last term a perfect square?(a)2(b)2(a)2(b)2 Write it as a square. Check the middle term. Is it2ab?(a)2↘2·a·b↙(b)2(a)2↘2·a·b↙(b)2 Step 2.Write the square of the binomial.(a + b)2(a − b)2 Step 3.Check by multiplying.
- **Difference of Squares Pattern:** If a,b are real numbers,



- **How to factor differences of squares.**
  Step 1.Does the binomial fit the pattern? a2 − b2Is this a difference?___ − ___Are the first and last terms perfect squares?Step 2.Write them as squares.(a)2 − (b)2Step 3.Write the product of conjugates.(a − b)(a + b)Step 4.Check by multiplying.
- **Sum and Difference of Cubes Pattern**
  a3 + b3 = (a + b)(a2 − ab + b2)a3 − b3 = (a − b)

$(a^2 + ab + b^2)$

- **How to factor the sum or difference of cubes.**

  Does the binomial fit the sum or difference of cubes pattern?
  Is it a sum or difference?
  Are the first and last terms perfect cubes? Write them as cubes. Use either the sum or difference of cubes pattern. Simplify inside the parentheses Check by multiplying the factors.

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

$16y^2 + 24y + 9$

$(4y + 3)^2$

$25v^2 + 20v + 4$

$36s^2 + 84s + 49$

$(6s + 7)2$

$49s2 + 154s + 121$

$100x2 - 20x + 1$

$(10x - 1)2$

$64z2 - 16z + 1$

$25n2 - 120n + 144$

$(5n - 12)2$

$4p2 - 52p + 169$

$49x2 + 28xy + 4y2$

$(7x + 2y)2$

$25r2 + 60rs + 36s2$

$100y^2 - 20y + 1$

$(10y - 1)^2$

$64m^2 - 16m + 1$

$10jk^2 + 80jk + 160j$

$10j(k + 4)^2$

$64x^2y - 96xy + 36y$

$75u^4 - 30u^3v + 3u^2v^2$

$3u^2(5u - v)^2$

$90p^4 + 300p^3q + 250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

$25v^2 - 1$

$(5v-1)(5v+1)$

$169q2-1$

$4-49x2$

$(2-7x)(2+7x)$

$121-25s2$

$6p2q2-54p2$

$6p2(q-3)(q+3)$

$98r3-72r$

$24p2+54$

$6(4p2+9)$

$20b2+140$

$121x2 - 144y2$

$(11x - 12y)(11x + 12y)$

$49x2 - 81y2$

$169c2 - 36d2$

$(13c - 6d)(13c + 6d)$

$36p2 - 49q2$

$16z4 - 1$

$(2z - 1)(2z + 1)(4z2 + 1)$

$m4 - n4$

$162a4b2 - 32b2$

$2b2(3a - 2)(3a + 2)(9a2 + 4)$

$48m4n2-243n2$

$x2-16x+64-y2$

$(x-8-y)(x-8+y)$

$p2+14p+49-q2$

$a2+6a+9-9b2$

$(a+3-3b)(a+3+3b)$

$m2-6m+9-16n2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

$x3+125$

$(x+5)(x2-5x+25)$

$n6+512$

$z6-27$

$(z2-3)(z4+3z2+9)$

$v3-216$

$8-343t3$

$(2-7t)(4+14t+49t2)$

$125-27w3$

$8y3-125z3$

$(2y-5z)(4y2+10yz+25z2)$

$27x3-64y3$

$216a3+125b3$

$(6a+5b)(36a2-30ab+25b2)$

$27y3+8z3$

$7k3+56$

$7(k+2)(k2-2k+4)$

$6x3-48y3$

$2x2-16x2y3$

$2x2(1-2y)(1+2y+4y2)$

$-2x3y2-16y5$

$(x+3)3+8x3$

$9(x+1)(x2+3)$

$(x+4)3-27x3$

$(y-5)^3-64y^3$

$-(3y+5)(21y^2-30y+25)$

$(y-5)^3+125y^3$

Mixed Practice

In the following exercises, factor completely.

$64a^2-25$

$(8a-5)(8a+5)$

$121x^2-144$

$27q^2-3$

$3(3q-1)(3q+1)$

$4p^2-100$

$16x^2-72x+81$

$(4x - 9)2$

$36y2 + 12y + 1$

$8p2 + 2$

$2(4p2 + 1)$

$81x2 + 169$

$125 - 8y3$

$(5 - 2y)(25 + 10y + 4y2)$

$27u3 + 1000$

$45n2 + 60n + 20$

$5(3n + 2)2$

$48q3 - 24q2 + 3q$

$x^2 - 10x + 25 - y^2$

$(x + y - 5)(x - y - 5)$

$x^2 + 12x + 36 - y^2$

$(x + 1)^3 + 8x^3$

$(3x + 1)(3x^2 + 1)$

$(y - 3)^3 - 64y^3$

Writing Exercises

Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Answers will vary.

How do you recognize the binomial squares pattern?

Explain why $n^2+25 \neq (n+5)^2$. Use algebra, words, or pictures.

Answers will vary.

Maribel factored $y^2-30y+81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

General Strategy for Factoring Polynomials

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

GCF

Binomial

- **Difference of Squares**
  $a^2 - b^2 = (a - b)(a + b)$
- **Sum of Squares**
  Sums of squares do not factor.
- **Sum of Cubes**
  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- **Difference of Cubes**
  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Trinomial

- $x^2 + bx + c$
  $(x \quad)(x \quad)$
- $ax^2 + bx + c$
  - **'a' and 'c' squares**
    $(a + b)^2 = a^2 + 2ab + b^2$
    $(a - b)^2 = a^2 - 2ab + b^2$
  - **'ac' method**

More than 3 terms

- grouping

Use a general strategy for factoring polynomials.

Is there a greatest common factor?
Factor it out. Is the polynomial a binomial, trinomial, or are there more than three terms?
If it is a binomial: If it is a trinomial: If it has more than three terms:

- Is it a sum?
  Of squares? Sums of squares do not factor.
  Of cubes? Use the sum of cubes pattern.
- Is it a difference?
  Of squares? Factor as the product of conjugates.
  Of cubes? Use the difference of cubes pattern.

- Is it of the form x2 + bx + c? Undo FOIL.
- Is it of the form ax2 + bx + c?
  If *a* and *c* are squares, check if it fits the trinomial square pattern.
  Use the trial and error or “ac” method.

- Use the grouping method.

Check.
Is it factored completely?
Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

Factor completely: $7x^3 - 21x^2 - 70x$.

	$7x^3 - 21x^2 - 70x$
Is there a GCF? Yes, $7x$.	
Factor out the GCF.	$7x(x^2 - 3x - 10)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	
Trinomial with leading	

coefficient 1.	
“Undo” FOIL.	$7x(x)(x)$ $7x(x+2)(x-5)$
Is the expression factored completely? Yes.	
Neither binomial can be factored.	
Check your answer.	
Multiply.	
$7x(x+2)(x-5)$	
$7x(x^2-5x+2x-10)$	
$7x(x^2-3x-10)$	
$7x^3-21x^2-70x$ ✓	

Factor completely: $8y^3+16y^2-24y$.

$8y(y-1)(y+3)$

Factor completely: $5y^3-15y^2-270y$.

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

Factor completely: $24y^2-150$.

	$24y^2-150$
Is there a GCF? Yes, 6. Factor out the GCF.	$6(4y^2-25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.	
Is it a sum? No.	
Is it a difference? Of squares or cubes? Yes, squares.	$6((2y)^2-(5)^2)$
Write as a product of conjugates.	$6(2y-5)(2y+5)$

Is the expression factored completely?
Neither binomial can be factored.
Check:
Multiply.
$6(2y-5)(2y+5)$
$6(4y2-25)$
$24y2-150$✓

Factor completely: $16x3-36x$.
$4x(2x-3)(2x+3)$

Factor completely: $27y2-48$.
$3(3y-4)(3y+4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

Factor completely: $4a^2-12ab+9b^2$.

	$4a^2-12ab+9b^2$
Is there a GCF? No.	
Is it a binomial, trinomial, or are there more terms? Trinomial with $a\neq1$. But the first term is a perfect square.	
Is the last term a perfect square? Yes.	$(2a)^2-12ab+(3b)^2$
Does it fit the pattern, $a^2-2ab+b^2$? Yes.	$(2a)^2\searrow-12ab+-2(2a)(3b)\swarrow(3b)^2$
Write it as a square.	$(2a-3b)^2$
Is the expression factored completely? Yes.	

The binomial cannot be factored.
Check your answer.
Multiply.
$(2a-3b)^2$
$(2a)^2-2\cdot 2a\cdot 3b+(3b)^2$
$4a^2-12ab+9b^2$ ✓

Factor completely: $4x^2+20xy+25y^2$.

$(2x+5y)^2$

Factor completely: $9x^2-24xy+16y^2$.

$(3x-4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

Factor completely 12x3y2 + 75xy2.

	12x3y2 + 75xy2
Is there a GCF? Yes, 3xy2.	
Factor out the GCF.	3xy2(4x2 + 25)
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.	
Is it a sum? Of squares? Yes.	Sums of squares are prime.
Is the expression factored completely? Yes.	
Check:	
Multiply.	
3xy2(4x2 + 25)	
12x3y2 + 75xy2✓	

Factor completely: $50x3y+72xy$.

$2xy(25x2+36)$

Factor completely: $27xy3+48xy$.

$3xy(9y2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

Factor completely: $24x3+81y3$.

Is there a GCF? Yes, 3.

[redacted]

Factor it out.

[redacted]

In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.

Is it a sum or difference? Sum.

Of squares or cubes? Sum of cubes.

[redacted]

Write it using the sum of cubes pattern.

[redacted]

Is the expression factored completely? Yes.

[redacted]

Check by multiplying.

Factor completely: $250m3 + 432n3$.

$2(5m + 6n)(25m2 - 30mn + 36n2)$

Factor completely: $2p^3+54q^3$.

$2(p+3q)(p^2-3pq+9q^2)$

Factor completely: $3x^5y-48xy$.

	$3x^5y-48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4-16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy((x^2)^2-(4)^2)$
Factor it as a product of conjugates	$3xy(x^2-4)(x^2+4)$
The first binomial is again a difference of squares.	$3xy((x)^2-(2)^2)(x^2+4)$

Factor it as a product of conjugates.	$3xy(x-2)(x+2)(x^2+4)$
Is the expression factored completely? Yes.	
Check your answer. Multiply.	
$3xy(x-2)(x+2)(x^2+4)$ $3xy(x^2-4)(x^2+4)$ $3xy(x^4-16)$ $3x^5y-48xy$✓	

Factor completely: $4a^5b-64ab$.
$4ab(a^2+4)(a-2)(a+2)$

Factor completely: $7xy^5-7xy$.
$7xy(y^2+1)(y-1)(y+1)$

Factor completely: $4x2+8bx-4ax-8ab$.

	$4x2+8bx-4ax-8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x2+2bx-ax-2ab)$
There are four terms. Use grouping.	$4[x(x+2b)-a(x+2b)]$ $4(x+2b)(x-a)$
Is the expression factored completely? Yes.	
Check your answer. Multiply. $4(x+2b)(x-a)$ $4(x2-ax+2bx-2ab)$ $4x2+8bx-4ax-8ab$ ✓	

Factor completely: $6x2-12xc+6bx-12bc$.

$6(x+b)(x-2c)$

Factor completely: $16x2+24xy-4x-6y$.
$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

Factor completely: $40x2y+44xy-24y$.

	$40x2y+44xy-24y$
Is there a GCF? Factor out the GCF, 4y.	$4y(10x2+11x-6)$
Factor the trinomial	$4y(10x2+11x-6)$

with $a \neq 1$.

$4y(5x-2)(2x+3)$

Is the expression factored completely?

Yes.

Check your answer.

Multiply.

$4y(5x-2)(2x+3)$

$4y(10x^2+11x-6)$

$40x^2y+44xy-24y$ ✓

Factor completely: $4p^2q-16pq+12q$.

$4q(p-3)(p-1)$

Factor completely: $6pq^2-9pq-6p$.

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Factor completely: $9x^2 - 12xy + 4y^2 - 49$.

	$9x^2 - 12xy + 4y^2 - 49$
Is there a GCF? No. With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping first 3 terms.	$9x^2 - 12xy + 4y^2 - 49$
Factor the trinomial with $a \neq 1$. But the first term is a perfect square.	
Is the last term of the trinomial a perfect square? Yes.	$(3x)^2 - 12xy + (2y)^2 - 49$
Does the trinomial fit	$(3x)^2 \searrow - 12xy +$

the pattern, $a^2-2ab+b^2$? Yes.	$-2(3x)(2y)\checkmark(2y)^2-49$
Write the trinomial as a square.	$(3x-2y)^2-49$
Is this binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$(3x-2y)^2-7^2$
Write it as a product of conjugates.	$((3x-2y)-7)((3x-2y)+7)$ $(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$(3x-2y-7)(3x-2y+7)$	
$9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49$	
$9x^2-12xy+4y^2-49\checkmark$	

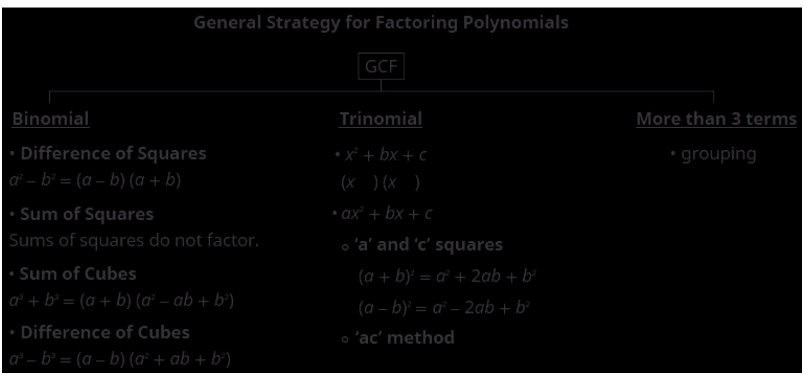
Factor completely: $4x^2-12xy+9y^2-25$.

$(2x - 3y - 5)(2x - 3y + 5)$

Factor completely: $16x2 - 24xy + 9y2 - 64$.

$(4x - 3y - 8)(4x - 3y + 8)$

Key Concepts



- **How to use a general strategy for factoring polynomials.**

  Is there a greatest common factor?

Factor it out. Is the polynomial a binomial, trinomial, or are there more than three terms?
If it is a binomial:
Is it a sum?
Of squares? Sums of squares do not factor.
Of cubes? Use the sum of cubes pattern.
Is it a difference?
Of squares? Factor as the product of conjugates.
Of cubes? Use the difference of cubes pattern.
If it is a trinomial:
Is it of the form $x2+bx+c$? Undo FOIL.
Is it of the form $ax2+bx+c$?
If *a* and *c* are squares, check if it fits the trinomial square pattern.
Use the trial and error or "ac" method.
If it has more than three terms:
Use the grouping method. Check.
Is it factored completely?
Do the factors multiply back to the original polynomial?

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

$2n2+13n-7$

$(2n-1)(n+7)$

$8x2-9x-3$

$a5+9a3$

$a3(a2+9)$

$75m3+12m$

$121r2-s2$

$(11r-s)(11r+s)$

$49b2-36a2$

$8m2-32$

$8(m-2)(m+2)$

$36q2-100$

$25w2-60w+36$

$(5w-6)2$

$49b2-112b+64$

$m2+14mn+49n2$

$(m+7n)2$

$64x2+16xy+y2$

$7b2+7b-42$

$7(b+3)(b-2)$

$30n2+30n+72$

$3x4y-81xy$

$3xy(x-3)(x2+3x+9)$

$4x5y - 32x2y$

$k4 - 16$

$(k - 2)(k + 2)(k2 + 4)$

$m4 - 81$

$5x5y2 - 80xy2$

$5xy2(x2 + 4)(x + 2)(x - 2)$

$48x5y2 - 243xy2$

$15pq - 15p + 12q - 12$

$3(5p + 4)(q - 1)$

$12ab - 6a + 10b - 5$

$4x2 + 40x + 84$

$4(x+3)(x+7)$

$5q2-15q-90$

$4u5+4u2v3$

$4u2(u+v)(u2-uv+v2)$

$5m4n+320mn4$

$4c2+20cd+81d2$

---

prime

$25x2+35xy+49y2$

$10m4-6250$

$10(m-5)(m+5)(m2+25)$

$3v4-768$

$36x2y + 15xy - 6y$

$3y(3x + 2)(4x - 1)$

$60x2y - 75xy + 30y$

$8x3 - 27y3$

$(2x - 3y)(4x2 + 6xy + 9y2)$

$64x3 + 125y3$

$y6 - 1$

$(y + 1)(y - 1)(y2 - y + 1)(y2 + y + 1)$

$y6 + 1$

$9x2 - 6xy + y2 - 49$

$(3x - y + 7)(3x - y - 7)$

$16x^2 - 24xy + 9y^2 - 64$

$(3x + 1)^2 - 6(3x + 1) + 9$

$(3x - 2)^2$

$(4x - 5)^2 - 7(4x - 5) + 12$

Writing Exercises

Explain what it mean to factor a polynomial completely.

Answers will vary.

The difference of squares $y^4 - 625$ can be factored as $(y^2 - 25)(y^2 + 25)$. But it is not completely factored. What more must be done to completely factor.

Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, ‘ac’ method, special products) which is the easiest for you? Which is the hardest? Explain your

answers.

Answers will vary.

Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Polynomial Equations

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Before you get started, take this readiness quiz.

Solve: $5y-3=0$. If you missed this problem, review [link].
$y=35$

Factor completely: $n3-9n2-22n$. If you missed this problem, review [link].
$n(n-11)(n+2)$

If $f(x) = 8x - 16$, find $f(3)$ and solve $f(x) = 0$.
If you missed this problem, review [link].

$8; x = 2$

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

Polynomial Equation

A **polynomial equation** is an equation that contains a polynomial expression.
The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of

degree one. Polynomial equations of degree one are linear equations are of the form $ax+b=c$.

We are now going to solve polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \quad 3y^2+4y=10 \quad 64u^2-81=0 \quad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a \neq 0$. (If $a=0$, then $0 \cdot x^2=0$ and we are left with no quadratic term.)

Quadratic Equation
An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.
$a, b,$ and c are real numbers and $a \neq 0$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property
If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

How to Solve a Quadratic Equation Using the Zero Product Property

Solve: $(5n - 2)(6n - 1) = 0$.

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$

Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $\quad n=\frac{1}{6}$

Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0$ ✓ $n=$ $(5n-2)(6n-1)=0$ $\left(5\cdot\quad-2\right)\left(6\cdot\quad-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0$ ✓

Solve: $(3m-2)(2m+1)=0.$

$m=23, m=-12$

Solve: $(4p+3)(4p-3)=0$.

$p=-34, p=34$

Use the Zero Product Property.

Set each factor equal to zero. Solve the linear equations. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in standard form, $ax2+bx+c=0$. Then we factor the expression on the left.

How to Solve a Quadratic Equation by Factoring

Solve: $2y2 = 13y + 45$.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	Write the equation in standard form.	$2y^2 = 13y + 45$ $2y^2 - 13y - 45 = 0$
Step 2. Factor the quadratic expression.	Factor $2y^2 - 13y + 45$ $(2y + 5)(y - 9)$	$(2y + 5)(y - 9) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y + 5 = 0 \quad y - 9 = 0$
Step 4. Solve the linear equations.		$y = -\frac{5}{2} \qquad y = 9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y =$ $2y^2 = 13y + 45$ $2(\ )^2 \stackrel{?}{=} 13(\ ) + 45$ $2(81) \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓

Solve: $3c2=10c-8$.

$c=2,c=43$

Solve: $2d2-5d=3$.

$d=3,d=-12$

Solve a quadratic equation by factoring.

Write the quadratic equation in standard form, $ax2+bx+c=0$. Factor the quadratic expression. Use the Zero Product Property. Solve the linear equations. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the quadratic equation is in standard form.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Solve: $169q^2=49$.

	$169x^2=49$
Write the quadratic equation in standard form.	$169x^2-49=0$
Factor. It is a difference of squares.	$(13x-7)(13x+7)=0$
Use the Zero Product Property to set each factor to 0. Solve each equation.	$13x-7=0 \quad 13x+7=0$ $13x=7 \quad 13x=-7$ $x=\frac{7}{13} \quad x=-\frac{7}{13}$

Check:

We leave the check up to you.

Solve: $25p2=49$.
$p=75,p=-75$

Solve: $36x2=121$.
$x=116,x=-116$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Solve: $(3x-8)(x-1)=3x$.

	$(3x-8)(x-1)=3x$
Multiply the binomials.	$3x2-11x+8=3x$
Write the quadratic equation in standard form.	$3x2-14x+8=0$
Factor the trinomial.	$(3x-2)(x-4)=0$
Use the Zero Product Property to set each factor to 0.	$3x-2=0x$ $-4=03x=2x=4$
Solve each equation.	$x=23$
Check your answers.	The check is left to you.

Solve: $(2m+1)(m+3)=12m$.

$m=1, m=32$

Solve: $(k+1)(k-1)=8$.

$k=3, k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Solve: $3x^2=12x+63$.

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$

Check your answers.	The check is left to you.

Solve: $18a2-30=-33a$.

$a=-52, a=23$

Solve: $123b=-6-60b2$.

$b=-2, b=-120$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

Solve: $9m3+100m=60m2$.

	$9m3+100m=60m2$
Bring all the terms to one side so that the other side is zero.	$9m3-60m2+100m=0$
Factor the greatest common factor first.	$m(9m2-60m+100)=0$
Factor the trinomial.	$m(3m-10)(3m-10)=0$
Use the Zero Product Property to set each factor to 0. Solve each equation.	$m=03m-10=03m-10=0m=0m=103m=103$
Check your answers.	The check is left to you.

Solve: $8x3=24x2-18x$.

$x = 0, x = 32$

Solve: $16y2 = 32y3 + 2y$.

$y = 0, y = 14$

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the Zero Product Property will be help us find these answers.

For the function $f(x) = x2 + 2x - 2$,

ⓐ find *x* when $f(x) = 6$ ⓑ find two points that

lie on the graph of the function.

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for f(x).	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0$ or $x-2=0$ $x=-4$ or $x=2$
Solve.	
Check:	

$f(x)=x^2+2x-2$ $f(x)=x^2+2x-2$ $f(-4)=(-4)^2+2(-4)-2$ $f(2)=2^2+2\cdot2-2$ $f(-4)=$

ⓑ Since $f(-4)=6$ and $f(2)=6$, the points $(-4,6)$ and $(2,6)$ lie on the graph of the function.

For the function $f(x) = x2 - 2x - 8$,

ⓐ find *x* when $f(x) = 7$ ⓑ Find two points that lie on the graph of the function.

ⓐ $x = -3$ or $x = 5$
ⓑ $(-3,7)$ $(5,7)$

For the function $f(x) = x2 - 8x + 3$,

ⓐ find *x* when $f(x) = -4$ ⓑ Find two points that lie on the graph of the function.

ⓐ $x = 1$ or $x = 7$
ⓑ $(1,-4)$ $(7,-4)$

The Zero Product Property also helps us determine where the function is zero. A value of *x* where the function is 0, is called a **zero of the function**.

Zero of a Function

For any function f, if $f(x)=0$, then x is a **zero of the function**.

When $f(x)=0$, the point $(x,0)$ is a point on the graph. This point is an x-intercept of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

For the function $f(x)=3x2+10x-8$, find

ⓐ the zeros of the function, ⓑ any x-intercepts of the graph of the function, ⓒ any y-intercepts of the graph of the function

ⓐ To find the zeros of the function, we need to find when the function value is 0.

		$f(x)=3x2+10x-8$
Substitute 0 for f(x).		$0=3x2+10x-8$

Factor the trinomial.	$(x+4)(3x-2)=0$
Use the zero product property.	$x+4=0 \text{ or } 3x-2=0$
Solve.	$x=-4 \text{ or } x=23$

ⓑ An *x*-intercept occurs when $y=0$. Since $f(-4)=0$ and $f(23)=0$, the points $(-4,0)$ and $(23,0)$ lie on the graph. These points are *x*-intercepts of the function.

ⓒ A *y*-intercept occurs when $x=0$. To find the *y*-intercepts we need to find $f(0)$.

	$f(x)=3x2+10x-8$
Find $f(0)$ by substituting 0 for x.	$f(0)=3\cdot02+10\cdot0-8$
Simplify.	$f(0)=-8$

Since $f(0)=-8$, the point $(0,-8)$ lies on the graph. This point is the *y*-intercept of the function.

For the function $f(x)=2x2-7x+5$, find

ⓐ the zeros of the function, ⓑ any *x*-intercepts of the graph of the function, ⓒ any *y*-intercepts of the graph of the function.

ⓐ $x=1$ or $x=52$
ⓑ $(1,0)$, $(52,0)$ ⓒ $(0,5)$

For the function $f(x)=6x2+13x-15$, find

ⓐ the zeros of the function, ⓑ any *x*-intercepts of the graph of the function, ⓒ any *y*-intercepts of the graph of the function.

ⓐ $x=-3$ or $x=56$
ⓑ $(-3,0)$, $(56,0)$ ⓒ $(0,-15)$

Solve Applications Modeled by

Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

Use a problem solving strategy to solve word problems.

Read the problem. Make sure all the words and ideas are understood. **Identify** what we are looking for. **Name** what we are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation. **Solve** the equation using appropriate algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

The product of two consecutive odd integers is 323. Find the integers.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the first integer.
	$n + 2 =$ next consecutive odd integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive odd integers is 323.
	$n(n + 2) = 323$
Step 5. Solve the equation.	$n2 + 2n = 323$
Bring all the terms to one side.	$n2 + 2n - 323 = 0$
Factor the trinomial.	$(n - 17)(n + 19) = 0$
Use the Zero Product Property.	$n - 17 = 0n + 19 = 0n = 17n = -19$

Solve the equations. There are two values for *n* that are solutions to this problem. So there are two sets of consecutive odd integers that will work.	
If the first integer is $n=17$ then the next odd integer is $n+2$ $17+2$ 19 $17,19$	If the first integer is $n=-19$ then the next odd integer is $n+2$ $-19+2$ -17 $-17,-19$
Step 6. Check the answer. The results are consecutive odd integers $17,19 \text{and} -19,-17.$ $17\cdot19=323\checkmark$ $-19(-17)=323\checkmark$ Both pairs of consecutive integers are solutions.	
Step 7. Answer the question	The consecutive integers are 17, 19 and $-19,-17$.

The product of two consecutive odd integers is 255. Find the integers.

$-15, -17$ and $15, 17$

The product of two consecutive odd integers is 483 Find the integers.

$-23, -21$ and $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Step 1. Read the problem. In problems inv ge ske vis the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the	$A = l \cdot w$

area of a rectangle.	
Substitute in the variables.	$117=(w+4)w$
Step 5. Solve the equation Distribute first.	$117=w2+4w$
Get zero on one side.	$117=w2+4w$
Factor the trinomial.	$0=w2+4w-117$
Use the Zero Product Property.	$0=(w2+13)(w-9)$
Solve each equation.	$0=w+130=w-9$
Since *w* is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for *w*.	$-13=w9=w$
	$w=9$ Width is 9 feet.
Find the value of the length.	$w+4$ $9+4$ 13 Length is 13 feet.
Step 6. Check the answer. Does the answer make sense?	

Yes, this makes sense.

Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

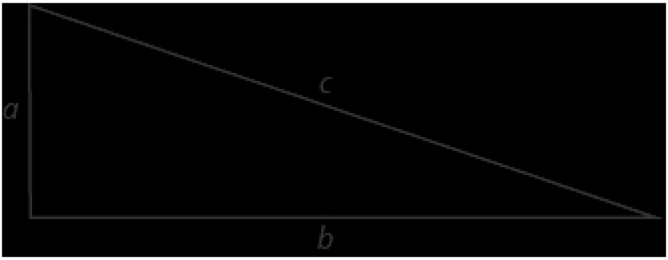
A rectangular sign has an area of 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

The width is 5 feet and length is 6 feet.

A rectangular patio has an area of 180 square feet. The width of the patio is three feet less than the length. Find the length and width of

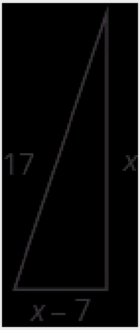
the patio.
The width of the patio is 12 feet and the length is 15 feet.

In the next example, we will use the Pythagorean Theorem (a2 + b2 = c2). This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.

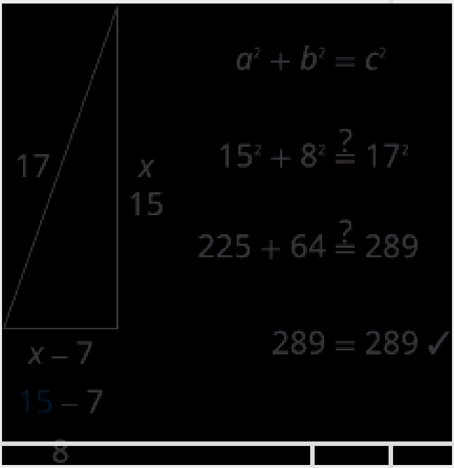


Step 1. Read the problem	
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a2 + b2 = c2$
Substitute in the variables.	$x2 + (x - 7)2 = 172$

Step 5. Solve the equation Simplify.	$x2+x2-14x+49=289$
	$2x2-14x+49=289$
It is a quadratic equation, so get zero on one side.	$2x2-14x-240=0$
Factor the greatest common factor.	$2(x2-7x-120)=0$
Factor the trinomial.	$2(x-15)(x+8)=0$
Use the Zero Product Property.	$2\neq 0x-15=0x+8=0$
Solve.	$2\neq 0x=15x=-8$
Since *x* is a side of the triangle, $x=-8$ does not make sense.	$2\neq 0x=15x=-8$
Find the length of the other side.	
If the length of one side is [redacted] then the length of the other side is [redacted] [redacted]	8 is the length of the

other side.

Step 6. Check the answer in the problem Do these numbers make sense?



Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the

deck.

5 feet and 12 feet

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

The other leg is 24 feet and the hypotenuse is 25 feet.

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80

feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground, ⓑ when the ball will be 80 feet above the ground, ⓒ the height of the ball at $t = 2$ seconds.

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t - 5)(t + 1) = 0$
Use the Zero Product Property.	$t - 5 = 0 \quad t + 1 = 0$
Solve.	$t = 5 \quad t = -1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result $t = -1$ is

discarded.

ⓑ The ball will be 80 feet above the ground when $h(t)=80$.

	$h(t)=80$
Substitute in the polynomial for $h(t)$.	$-16t2+64t+80=80$
Subtract 80 from both sides.	$-16t2+64t=0$
Factor the GCF, $-16t$.	$-16t(t-4)=0$
Use the Zero Product Property.	$-16t=0t$
Solve.	$-4=0t=0t=4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t=2$ seconds we find $h(2)$.

	$h(t) = -16t2 + 64t + 80$
To find h(2) substitute 2 for t.	$h(2) = -16(2)2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t2 + 48t + 160$ models the height, *h*, of the rock above the ocean as a function of time, *t*. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean, ⓑ when the rock will be 160 feet above the ocean, ⓒ the height of the rock at $t = 1.5$ seconds.

ⓐ 5 seconds; ⓑ 0 and 3 seconds; ⓒ 196 feet

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t2 + 32t + 128$ models the height, *h*, of the penny above the ocean as a function of time, *t*. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean, ⓑ when the penny will be 128 feet above the ocean, ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

ⓐ 4 seconds; ⓑ 0 and 2 seconds; ⓒ 144 feet

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax2+bx+c=0$ is called a quadratic equation. a,b,care real numbers $anda\neq 0$
- **Zero Product Property:** If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.
- **How to use the Zero Product Property**

  Set each factor equal to zero. Solve the linear equations. Check.

- **How to solve a quadratic equation by factoring.**

  Write the quadratic equation in standard form, $ax2+bx+c=0$. Factor the quadratic expression. Use the Zero Product Property. Solve the linear equations. Check. Substitute each solution separately into the original equation.

- **Zero of a Function:** For any function *f*, if $f(x)=0$, then *x* is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**

  Read the problem. Make sure all the words and ideas are understood. **Identify** what we are looking for. **Name** what we are looking for.

Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation. **Solve** the equation using appropriate algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

Section Exercises

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

$(3a-10)(2a-7)=0$

$a=10/3, a=7/2$

$(5b+1)(6b+1)=0$

$6m(12m-5)=0$

$m=0, m=5/12$

$2x(6x-3)=0$

$(2x-1)2=0$

$x=1/2$

$(3y+5)2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$5a2-26a=24$

$a=-45, a=6$

$4b2+7b=-3$

$4m2=17m-15$

$m=5/4, m=3$

$n2 = 5n - 6$

$7a2 + 14a = 7a$

$a = -1, a = 0$

$12b2 - 15b = -9b$

$49m2 = 144$

$m = 12/7, m = -12/7$

$625 = x2$

$16y2 = 81$

$y = -9/4, y = 9/4$

$64p2 = 225$

$121n2 = 36$

$n=-6/11, n=6/11$

$100y2=9$

$(x+6)(x-3)=-8$

$x=2, x=-5$

$(p-5)(p+3)=-7$

$(2x+1)(x-3)=-4x$

$x=3/2, x=-1$

$(y-3)(y+2)=4y$

$(3x-2)(x+4)=12x$

$x=2, x=-4/3$

$(2y-3)(3y-1)=8y$

$20x2-60x=-45$

$x=3/2$

$3y2-18y=-27$

$15x2-10x=40$

$x=2,x=-4/3$

$14y2-77y=-35$

$18x2-9=-21x$

$x=-3/2,x=1/3$

$16y2+12=-32y$

$16p3=24p2–9p$

$p=0,p=¾$

$m3 - 2m2 = -m$

$2x3 + 72x = 24x2$

$x = 0, x = 6$

$3y3 + 48y = 24y2$

$36x3 + 24x2 = -4x$

$x = 0, x = -1/3$

$2y3 + 2y2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

For the function, $f(x) = x2 - 8x + 8$, ⓐ find when $f(x) = -4$ ⓑ Use this information to find two points that lie on the graph of the function.

ⓐ $x = 2$ or $x = 6$ ⓑ $(2, -4)$ $(6, -4)$

For the function, $f(x)=x2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

For the function, $f(x)=8x2-18x+5$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

ⓐ $x=32$ or $x=34$
ⓑ $(32,-4)$ $(34,-4)$

For the function, $f(x)=18x2+15x-10$, ⓐ find when $f(x)=15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

$f(x)=9x2-4$

ⓐ $x=23$ or $x=-23$
ⓑ $(23,0)$, $(-23,0)$ ⓒ $(0,-4)$

$f(x)=25x2-49$

$f(x)=6x2-7x-5$

ⓐ $x=53$ or $x=-12$
ⓑ $(53,0)$, $(-12,0)$ ⓒ $(0,-5)$

$f(x)=12x2-11x+2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

The product of two consecutive odd integers is 143. Find the integers.

$-13,-11$ and 11, 13

The product of two consecutive odd integers is 195. Find the integers.

The product of two consecutive even integers is 168. Find the integers.

$-14,-12$ and 12, 14

The product of two consecutive even integers is 288. Find the integers.

The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Width: 4 feet; Length: 7 feet.

A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

The area of a bulletin board is 55 square feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

Width: 5 feet; Length: 11 feet.

A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

The sides are 6 feet and 8 feet.

A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

The building side is 8 feet, the hypotenuse is 17 feet, and the third side is 15 feet.

A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The

hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t2 + 32t$ models the height, *h*, of the rocket above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function, which tell us when the rocket will be on the ground. ⓑ the time the rocket will be 16 feet above the ground.

ⓐ 0 seconds and 2 seconds ⓑ 1 second

Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t2 + 32t + 48$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Answers will vary.

Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

$12a^2b^3, 15ab^2$

$3ab^2$

$12m^2n^3, 42m^5n^3$

$15y^3, 21y^2, 30y$

$3y$

$45x^3y^2, 15x^4y, 10x^5y^3$

Factor the Greatest Common Factor from a

Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

$35y+84$

$7(5y+12)$

$6y2+12y-6$

$18x3-15x$

$3x(6x2-5)$

$15m4+6m2n$

$4x3-12x2+16x$

$4x(x2-3x+4)$

$-3x+24$

$-3x3+27x2-12x$

$-3x(x^2-9x+4)$

$3x(x-1)+5(x-1)$

Factor by Grouping

In the following exercises, factor by grouping.

$ax-ay+bx-by$

$(a+b)(x-y)$

$x^2y-xy^2+2x-2y$

$x^2+7x-3x-21$

$(x-3)(x+7)$

$4x^2-16x+3x-12$

m^3+m^2+m+1

$(m^2+1)(m+1)$

$5x - 5y - y + x$

Factor Trinomials

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

$a^2 + 14a + 33$

$(a + 3)(a + 11)$

$k^2 - 16k + 60$

$m^2 + 3m - 54$

$(m + 9)(m - 6)$

$x^2 - 3x - 10$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

$x^2 + 12xy + 35y^2$

$(x+5y)(x+7y)$

$r2+3rs-28s2$

$a2+4ab-21b2$

$(a+7b)(a-3b)$

$p2-5pq-36q2$

$m2-5mn+30n2$

Prime

Factor Trinomials of the Form $ax2+bx+c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

$x3+5x2-24x$

$3y3-21y2+30y$

$3y(y-5)(y-2)$

$5x4+10x3-75x2$

$5y2+14y+9$

$(5y+9)(y+1)$

$8x2+25x+3$

$10y2-53y-11$

$(5y+1)(2y-11)$

$6p2-19pq+10q2$

$-81a2+153a+18$

$-9(9a+1)(a-2)$

Factor Trinomials of the Form $ax2+bx+c$ using the ‘ac’ Method

In the following exercises, factor.

$2x2+9x+4$

$18a2-9a+1$

$(3a-1)(6a-1)$

$15p2+2p-8$

$15x2+6x-2$

Prime

$8a2+32a+24$

$3x2+3x-36$

$3(x+4)(x-3)$

$48y2+12y-36$

$18a2-57a-21$

$3(2a-7)(3a+1)$

$3n^4-12n^3-96n^2$

Factor using substitution

In the following exercises, factor using substitution.

x^4-13x^2-30

$(x^2-15)(x^2+2)$

$(x-3)^2-5(x-3)-36$

Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

$25x^2+30x+9$

$(5x+3)^2$

$36a^2 - 84ab + 49b^2$

$40x^2 + 360x + 810$

$10(2x + 9)^2$

$5k^3 - 70k^2 + 245k$

$75u^4 - 30u^3v + 3u^2v^2$

$3u^2(5u - v)^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

$81r^2 - 25$

$169m^2 - n^2$

$(13m + n)(13m - n)$

$25p^2 - 1$

$9-121y2$

$(3+11y)(3-11y)$

$20x2-125$

$169n3-n$

$n(13n+1)(13n-1)$

$6p2q2-54p2$

$24p2+54$

$6(4p2+9)$

$49x2-81y2$

$16z4-1$

$(2z-1)(2z+1)(4z2+1)$

$48m4n2-243n2$

$a2+6a+9-9b2$

$(a+3-3b)(a+3+3b)$

$x2-16x+64-y2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

$a3-125$

$(a-5)(a2+5a+25)$

$b3-216$

$2m3+54$

$2(m+3)(m2-3m+9)$

$81m3+3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

$24x3+44x2$

$4x2(6x+11)$

$24a4-9a3$

$16n2-56mn+49m2$

$(4n-7m)2$

$6a2-25a-9$

$5u4-45u2$

$5u2(u+3)(u-3)$

$n4 - 81$

$64j2 + 225$

---

prime

$5x2 + 5x - 60$

$b3 - 64$

$(b - 4)(b2 + 4b + 16)$

$m3 + 125$

$2b2 - 2bc + 5cb - 5c2$

$(2b + 5c)(b - c)$

$48x5y2 - 243xy2$

$5q2 - 15q - 90$

$5(q+3)(q-6)$

$4u^5v+4u^2v^3$

$10m^4-6250$

$10(m-5)(m+5)(m^2+25)$

$60x^2y-75xy+30y$

$16x^2-24xy+9y^2-64$

$(4x-3y+8)(4x-3y-8)$

Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

$(a-3)(a+7)=0$

$(5b+1)(6b+1)=0$

$b = -1/5, b = -1/6$

$6m(12m-5)=0$

$(2x-1)2=0$

$x=1/2$

$3m(2m-5)(m+6)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

$x2+9x+20=0$

$x=-4, x=-5$

$y2-y-72=0$

$2p2-11p=40$

$p=-52, p=8$

$q3+3q2+2q=0$

$144m2-25=0$

$m=512, m=-512$

$4n2=36$

$(x+6)(x-3)=-8$

$x=2, x=-5$

$(3x-2)(x+4)=12x$

$16p3=24p2-9p$

$p=0, p=¾$

$2y3+2y2=12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

For the function, $f(x)=x2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

ⓐ $x=-7$ or $x=-4$
ⓑ $(-7,-8)$ $(-4,-8)$

For the function, $f(x)=9x2-18x+5$, ⓐ find when $f(x)=-3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

$f(x)=64x2-49$

ⓐ $x=78$ or $x=-78$
ⓑ $(78,0)$, $(-78,0)$ ⓒ $(0,-49)$

$f(x)=6x2-13x-5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

The product of two consecutive odd numbers is 399. Find the numbers.

The numbers are -21 and -19 or 19 and 21.

The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

The lengths are 8, 15, and 17 ft.

Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us

when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t=2$ seconds which is when the ball will be at its highest point.

Chapter Practice Test

In the following exercises, factor completely.

$80a2+120a3$

$40a2(2+3a)$

$5m(m-1)+3(m-1)$

$x2+13x+36$

$(x+4)(x+9)$

$p2+pq-12q2$

$xy-8y+7x-56$

$(x - 8)(y + 7)$

$40r2 + 810$

$9s2 - 12s + 4$

$(3s - 2)2$

$6x2 - 11x - 10$

$3x2 - 75y2$

$3(x + 5y)(x - 5y)$

$6u2 + 3u - 18$

$x3 + 125$

$(x + 5)(x2 - 5x + 25)$

$32x5y2 - 162xy2$

$6x4-19x2+15$

$(3x2-5)(2x2-3)$

$3x3-36x2+108x$

In the following exercises, solve

$5a2+26a=24$

$a=4/5, a=-6$

The product of two consecutive integers is 156. Find the integers.

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

The width is 12 inches and the length is 14 inches.

Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80

feet above the ground, the function $h(t) = -16t2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t = 4$ seconds.

For the function, $f(x) = x2 - 7x + 5$, ⓐ find when $f(x) = -7$ ⓑ Use this information to find two points that lie on the graph of the function.

ⓐ $x = 3$ or $x = 4$ ⓑ $(3, -7)$ $(4, -7)$

For the function $f(x) = 25x2 - 81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

Glossary

degree of the polynomial equation
: The degree of the polynomial equation is the degree of the polynomial.

polynomial equation
: A polynomial equation is an equation that contains a polynomial expression.

quadratic equation
: Polynomial equations of degree two are called quadratic equations.

zero of the function
: A value of x where the function is 0, is called a zero of the function.

Zero Product Property
: The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

Multiply and Divide Rational Expressions

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Before you get started, take this readiness quiz.

1. Simplify: 90y15y2.
   If you missed this problem, review [link].
2. Multiply: 1415·635.
   If you missed this problem, review [link].
3. Divide: 1210÷825.
   If you missed this problem, review [link].

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Rational Expression
A rational expression is an expression of the form pq, where *p* and *q* are polynomials and $q \neq 0$.

Here are some examples of rational expressions:
$-24565x12y4x+1x2-94x2+3x-12x-8$

Notice that the first rational expression listed above, -2456, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the

number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Determine the values for which a rational expression is undefined.

Set the denominator equal to zero. Solve the equation.

Determine the value for which each rational expression is undefined:

ⓐ $8a2b3c$ ⓑ $4b-32b+5$ ⓒ $x+4x2+5x+6$.

The expression will be undefined when the denominator is zero.

ⓐ

8a2b3c Set the denominator equal to zero and solvefor the variable.3c=0 c=0 8a2b3cis undefined forc=0.

ⓑ

4b−32b+5 Set the denominator equal to zero and solvefor the variable.2b+5=02b=−5b= −52 4b−32b+5is undefined forb=−52.

ⓒ

x+4x2+5x+6 Set the denominator equal to zero and solvefor the variable.x2+5x+6=0(x +2)(x+3)=0x+2=0orx+3=0x=−2orx= −3 x+4x2+5x+6is undefined forx= −2orx=−3.

Determine the value for which each rational expression is undefined.

ⓐ 3y28x ⓑ 8n−53n+1 ⓒ a+10a2+4a+3

ⓐ x=0 ⓑ n=−13
ⓒ a=−1,a=−3

Determine the value for which each rational expression is undefined.

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

ⓐ $q=0$ ⓑ $y=-\frac{2}{3}$
ⓒ $m=2, m=-3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression
A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,
$x+2x+3$is simplified because there are no common factors of$x+2$and$x+3$. $2x3x$is not simplified because$x$is a common factor of$2x$and$3x$.

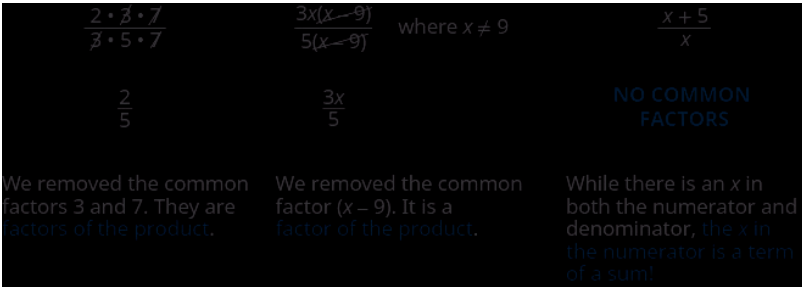
We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property
If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$,
then$ab = a \cdot cb \cdot c$and$a \cdot cb \cdot c = ab$.

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

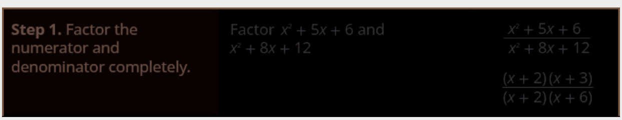
Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

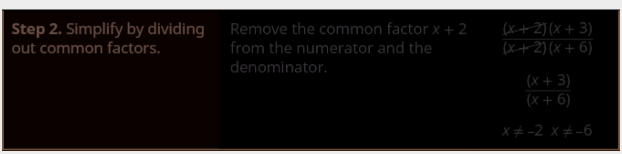


Removing the *x*’s from x + 5x would be like cancelling the 2’s in the fraction 2 + 52!

How to Simplify a Rational Expression

Simplify: x2 + 5x + 6x2 + 8x + 12.





Simplify: $x2-x-2x2-3x+2$.
$x+1x-1, x\neq 2, x\neq 1$

Simplify: $x2-3x-10x2+x-2$.
$x-5x-1, x\neq -2, x\neq 1$

We now summarize the steps you should follow to simplify rational expressions.

Simplify a rational expression. Factor the numerator and denominator completely. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We’ll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Simplify: 3a2 − 12ab + 12b26a2 − 24b2.

3a2 − 12ab + 12b26a2 − 24b2 Factor the numerator and denominator,first factoring out the GCF.3(a2 − 4ab + 4b2)6(a2 − 4b2) 3(a − 2b)(a − 2b)6(a + 2b)(a − 2b) Remove the common factors ofa − 2band3.3(a − 2b)(a − 2b)3·2(a + 2b)(a − 2b) a − 2b2(a + 2b)

Simplify: 2x2 − 12xy + 18y23x2 − 27y2.

2(x − 3y)3(x + 3y)

Simplify: $5x2-30xy+25y22x2-50y2$.

$5(x-y)2(x+5y)$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of *a* is $-a$ and $-a=-1\cdot a$.

The numerical fraction, say $7-7$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $a-a$, whose numerator and denominator are opposites also simplifies to -1. Let’s look at the expressionb $-a$.b $-a$ Rewrite. $-a+b$ Factor out–1. $-1(a-b)$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $a-bb-a$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a}=-1 \quad a\neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a\neq -b$, then $\frac{a+b}{b+a}=1$.

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$

Recognize the factors that are opposites.

$$(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$$

Simplify.

$$-\frac{x+4}{x+8}$$

Simplify: $x2-4x-525-x2$.

$-x+1x+5$

Simplify: $x2+x-21-x2$.

$-x+2x+1$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, s \neq 0$, then

$$pq \cdot rs = prqs$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0, x \neq 3$, and $x \neq 4$.

How to Multiply Rational Expressions

Simplify: $2xx2-7x+12 \cdot x2-96x2$.

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\frac{2x}{x^2-7x+12}\cdot\frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)}\cdot\frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\,\cancel{x}\,\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot \cancel{x}\cdot x\,\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

Simplify: 5xx2 + 5x + 6·x2 − 410x.

x − 22(x + 3)

Simplify: 9x2x2 + 11x + 30·x2 − 363x2.

3(x − 6)x + 5

Multiply rational expressions.

Factor each numerator and denominator completely. Multiply the numerators and denominators. Simplify by dividing out common factors.

Multiply: 3a2 − 8a − 3a2 − 25·a2 + 10a + 253a2 − 14a − 5.

3a2 − 8a − 3a2 − 25·a2 + 10a + 253a2 − 14a − 5 Factor the numerators and denominatorsand then multiply.(3a + 1)(a − 3)(a + 5)(a + 5)(a − 5)(a + 5)(3a + 1)(a − 5) Simplify by dividing outcommon factors.(3a + 1)(a − 3)(a + 5)(a + 5)(a − 5)(a + 5)(3a + 1)(a − 5) Simplify.(a − 3)(a + 5)(a − 5)(a − 5) Rewrite(a − 5)(a − 5)using an exponent.(a − 3)(a + 5)(a − 5)2

Simplify: $2x2+5x-12x2-16 \cdot x2-8x +162x2-13x+15$.

$x-4x-5$

Simplify: $4b2+7b-21-b2 \cdot b2-2b +14b2+15b-4$.

$-(b+2)(b-1)(1+b)(b+4)$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where

$q \neq 0, r \neq 0, s \neq 0,$ then
$pq \div rs = pq \cdot sr$
To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

How to Divide Rational Expressions

Divide: $p3 + q32p2 + 2pq + 2q2 \div p2 - q26.$

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$

Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$

Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$

Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2}\cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Simplify: x3 − 83x2 − 6x + 12 ÷ x2 − 46.

2(x2 + 2x + 4)(x + 2)(x2 − 2x + 4)

Simplify: 2z2z2 − 1 ÷ z3 − z2 + zz3 + 1.

2zz − 1

Divide rational expressions.

Rewrite the division as the product of the first rational expression and the reciprocal of the second. Factor the numerators and denominators

completely. Multiply the numerators and denominators together. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Divide: $6x2-7x+24x-82x2-7x+3x2-5x+6$.

$6x2-7x+24x-82x2-7x+3x2-5x+6$ Rewrite with a division sign. $6x2-7x+24x-8\div 2x2-7x+3x2-5x+6$ Rewrite as product of first times reciprocalof second. $6x2-7x+24x-8\cdot x2-5x+62x2-7x+3$ Factor the numerators and thedenominators, and then multiply. $(2x-1)(3x-2)(x-2)(x-3)4(x-2)(2x-1)(x-3)$ Simplify by dividing out common factors. $(2x-1)(3x-2)(x-2)(x-3)4(x-2)(2x-1)(x-3)$ Simplify. $3x-24$

Simplify: $3x2+7x+24x+243x2-14x-5x2+x-30$.

$x+24$

Simplify: $y2-362y2+11y-62y2-2y-608y-4$.

$2y+5$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

Perform the indicated operations: $3x-64x-4\cdot x2+2x-3x2-3x-10\div 2x+128x+16$.

$$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\div\frac{2x+12}{8x+16}$$

Rewrite the division as multiplication by the reciprocal.

$$\frac{3x-6}{4x-4}\cdot\frac{x^2+2x-3}{x^2-3x-10}\cdot\frac{8x+16}{2x+12}$$

Factor the numerators and the denominators.

$$\frac{3(x-2)}{4(x-1)}\cdot\frac{(x+3)(x-1)}{(x+2)(x-5)}\cdot\frac{8(x+2)}{2(x+6)}$$

Multiply the fractions. Bringing the constants to the front will help when removing common factors.

Simplify by dividing out common factors.

$$\frac{3\cdot\cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4}\cdot\cancel{2}\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$$

Simplify.

$$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$$

Perform the indicated operations: 4m+43m −15·m2−3m−10m2−4m−32÷12m−366m

$-48.$

$2(m+1)(m+2)3(m+4)(m-3)$

Perform the indicated operations: $2n2+10nn-1 \div n2+10n+24n2+8n-9 \cdot n+48n2+12n.$

$(n+5)(n+9)2(n+6)(2n+3)$

Multiply and Divide Rational Functions

We started this section stating that a rational expression is an expression of the form pq, where *p* and *q* are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x)=p(x)q(x)$ where p(x) and q(x) are polynomial functions and q(x) is not zero.

Rational Function

A rational function is a function of the form

$R(x)=p(x)q(x)$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x)=0$.

Determine the domain of a rational function.

Set the denominator equal to zero. Solve the equation. The domain is all real numbers excluding the values found in Step 2.

Find the domain of $R(x)=2x2-14x4x2-16x-48$.

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero ,

solve that equation, and then exclude those values from the domain.

Set the denominator to zero.$4x2-16x-48=0$Factor, first factor out the GCF.$4(x2-4x-12)=04(x-6)(x+2)=0$Use the Zero Product Property.$4\neq 0x-6=0x+2=0$Solve.$x=6x=-2$The domain of$R(x)$is all real numberswhere$x\neq 6$and$x\neq -2$.

Find the domain of $R(x)=2x2-10x4x2-16x-20$.
The domain of $R(x)$ is all real numbers where $x\neq 5$ and $x\neq -1$.

Find the domain of $R(x)=4x2-16x8x2-16x-64$.
The domain of $R(x)$ is all real numbers where $x\neq 4$ and $x\neq -2$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

Find $R(x)=f(x)\cdot g(x)$ where $f(x)=2x-6x2-8x+15$ and $g(x)=x2-252x+10$.
$R(x)=f(x)\cdot g(x)$ $R(x)=2x-6x2-8x+15\cdot x2-252x+10$ Factor each numerator and denominator. $R(x)=2(x-3)(x-3)(x-5)\cdot(x-5)(x+5)2(x+5)$ Multiply the numerators and denominators. $R(x)=2(x-3)(x-5)(x+5)2(x-3)(x-5)(x+5)$ Remove common factors. $R(x)=2(x-3)(x-5)(x+5)2(x-3)(x-5)(x+5)$ Simplify. $R(x)=1$

Find $R(x)=f(x)\cdot g(x)$ where $f(x)=3x-21x2-9x+14$ and $g(x)=2x2-83x+6$.
$R(x)=2$

Find R(x) = f(x)·g(x) where f(x) = x2 − x3x2 + 27x − 30 and g(x) = x2 − 100x2 − 10x.

R(x) = 13

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

Find R(x) = f(x)g(x) where f(x) = 3x2x2 − 4x and g(x) = 9x2 − 45xx2 − 7x + 10.

R(x) = f(x)g(x) Substitute in the functionsf(x),g(x).R(x) = 3x2x2 − 4x9x2 − 45xx2 − 7x + 10 Rewrite the division as the product off(x)and the reciprocal ofg(x).R(x) = 3x2x2 − 4x·x2 − 7x + 109x2 − 45x Factor the numerators and denominatorsand then multiply.R(x) = 3·x·x·(x − 5)(x − 2)x(x − 4)·3·3·x·(x − 5) Simplify by dividing out

common factors.$R(x)=3\cdot x\cdot x(x-5)(x-2)x(x-4)\cdot 3\cdot 3\cdot x(x-5)$ $R(x)=x-23(x-4)$

Find $R(x)=f(x)g(x)$ where $f(x)=2x2x2-8x$ and $g(x)=8x2+24xx2+x-6$.

$R(x)=x-24(x-8)$

Find $R(x)=f(x)g(x)$ where $f(x)=15x23x2+33x$ and $g(x)=5x-5x2+9x-22$.

$R(x)=x(x-2)x-1$

Key Concepts

- **Determine the values for which a rational**

expression is undefined.

Set the denominator equal to zero. Solve the equation.

- **Equivalent Fractions Property**
  If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0$, then $ab = a \cdot cb \cdot c$ and $a \cdot cb \cdot c = ab$.
- **How to simplify a rational expression.**

  Factor the numerator and denominator completely. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.
  $a - bb - a = -1 a \neq b$
  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, s \neq 0$, then
  $pq \cdot rs = prqs$
- **How to multiply rational expressions.**

  Factor each numerator and denominator completely. Multiply the numerators and denominators. Simplify by dividing out common factors.

- **Division of Rational Expressions**
  If *p*, *q*, *r*, and *s* are polynomials where

$q \neq 0, r \neq 0, s \neq 0$, then
$pq \div rs = pq \cdot sr$

- **How to divide rational expressions.**

  Rewrite the division as the product of the first rational expression and the reciprocal of the second. Factor the numerators and denominators completely. Multiply the numerators and denominators together. Simplify by dividing out common factors.

- **How to determine the domain of a rational function.**

  Set the denominator equal to zero. Solve the equation. The domain is all real numbers excluding the values found in Step 2.

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

ⓐ $2x2z$, ⓑ $4p-16p-5$, ⓒ $n-3n2+2n-8$

ⓐ z = 0 ⓑ p = 56
ⓒ n = −4, n = 2

ⓐ 10m11n, ⓑ 6y + 134y − 9, ⓒ b − 8b2 − 36

ⓐ 4x2y3y, ⓑ 3x − 22x + 1, ⓒ u − 1u2 − 3u − 28

ⓐ y = 0, ⓑ x = −12, ⓒ u = −4, u = 7

ⓐ 5pq29q, ⓑ 7a − 43a + 5, ⓒ 1x2 − 4

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

−4455

−45

5663

8m3n12mn2

2m23n 36v3w227vw3 8n − 963n − 36
83 12p − 2405p − 100 x2 + 4x − 5x2 − 2x + 1
x + 5x − 1 y2 + 3y − 4y2 − 6y + 5 a2 − 4a2 + 6a − 16
a + 2a + 8 y2 − 2y − 3y2 − 9

$p3+3p2+4p+12p2+p-6$

$p2+4p-2$

$x3-2x2-25x+50x2-25$

$8b2-32b2b2-6b-80$

$4b(b-4)(b+5)(b-8)$

$-5c2-10c-10c2+30c+100$

$3m2+30mn+75n24m2-100n2$

$3(m+5n)4(m-5n)$

$5r2+30rs-35s2r2-49s2$

$a-55-a$

-1

$5-dd-5$

$20-5yy2-16$

$-5y+4$

$4v-3264-v2$

$w3+216w2-36$

$w2-6w+36w-6$

$v3+125v2-25$

$z2-9z+2016-z2$

$-z-54+z$

$a2-5a-3681-a2$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

$1216\cdot410$

310

$325\cdot1624$

$5x2y412xy3\cdot6x220y2$

$x38y$

$12a3bb2\cdot2ab29b3$

$5p2p2-5p-36\cdot p2-1610p$

$p(p-4)2(p-9)$

$3q2q2+q-6\cdot q2-99q$

$2y2-10yy2+10y+25\cdot y+56y$

$y-53(y+5)$

$z2+3zz2-3z-4\cdot z-4z2$

$28-4b3b-3\cdot b2+8b-9b2-49$

$-4(b+9)3(b+7)$

$72m-12m28m+32\cdot m2+10m+24m2-36$

$c2-10c+25c2-25\cdot c2+10c+253c2-14c-5$

$c+53c+1$

$2d2+d-3d2-16\cdot d2-8d+162d2-9d-18$

$2m2-3m-22m2+7m+3\cdot 3m2-14m$
$+153m2+17m-20$

$(m-3)(m-2)(m+4)(m+3)$

$2n2-3n-1425-n2\cdot n2-10n+252n2-13n$
$+21$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

v − 511 − v ÷ v2 − 25v − 11

− 1v + 5

10 + ww − 8 ÷ 100 − w28 − w

3s2s2 − 16 ÷ s3 + 4s2 + 16ss3 − 64

3ss + 4

r2 − 915 ÷ r3 − 275r2 + 15r + 45

p3 + q33p2 + 3pq + 3q2 ÷ p2 − q212

4(p2 − pq + q2)(p − q)(p2 + pq + q2)

v3 − 8w32v2 + 4vw + 8w2 ÷ v2 − 4w24

x2 + 3x − 104x ÷ (2x2 + 20x + 50)

$x - 28x(x + 5)$

$2y2 - 10yz - 48z22y - 1 \div (4y2 - 32yz)$

$2a2 - a - 215a + 20a2 + 7a + 12a2 + 8a + 16$

$2a - 75$

$3b2 + 2b - 812b + 183b2 + 2b - 82b2 - 7b - 15$

$12c2 - 122c2 - 3c + 14c + 46c2 - 13c + 5$

$3(3c - 5)$

$4d2 + 7d - 235d + 10d2 - 47d2 - 12d - 4$

For the following exercises, perform the indicated operations.

$10m2 + 80m3m - 9 \cdot m2 + 4m - 21m2 - 9m$
$+ 20 \div 5m2 + 10m2m - 10$

$4(m + 8)(m + 7)3(m - 4)(m + 2)$

$4n2+32n3n+2\cdot3n2-n-2n2+n$
$-30\div108n2-24nn+6$

$12p2+3pp+3\div p2+2p-63p2-p-12\cdot p$
$-79p3-9p2$

$(4p+1)(p-4)3p(p+9)(p-1)$

$6q+39q2-9q\div q2+14q+33q2+4q$
$-5\cdot4q2+12q12q+6$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

$R(x)=x3-2x2-25x+50x2-25$

$x\neq5$ and $x\neq-5$

$R(x)=x3+3x2-4x-12x2-4$

$R(x)=3x2+15x6x2+6x-36$

$x\neq2$ and $x\neq-3$

$R(x)=8x2-32x2x2-6x-80$

For the following exercises, find $R(x)=f(x)\cdot g(x)$ where $f(x)$ and $g(x)$ are given.

$f(x)=6x2-12xx2+7x-18$
$g(x)=x2-813x2-27x$

$R(x)=2$

$f(x)=x2-2xx2+6x-16$
$g(x)=x2-64x2-8x$

$f(x)=4xx2-3x-10$
$g(x)=x2-258x2$

$R(x)=x+52x(x+2)$

$f(x)=2x2+8xx2-9x+20$
$g(x)=x-5x2$

For the following exercises, find $R(x)=f(x)g(x)$ where $f(x)$ and $g(x)$ are given.

$f(x) = 27x23x - 21$
$g(x) = 3x2 + 18xx2 + 13x + 42$

$R(x) = 3x(x + 7)x - 7$

$f(x) = 24x22x - 8$
$g(x) = 4x3 + 28x2x2 + 11x + 28$

$f(x) = 16x24x + 36$
$g(x) = 4x2 - 24xx2 + 4x - 45$

$R(x) = x(x - 5)x - 6$

$f(x) = 24x22x - 4$
$g(x) = 12x2 + 36xx2 - 11x + 18$

Writing Exercises

Explain how you find the values of *x* for which the rational expression $x2 - x - 20x2 - 4$ is undefined.

Answers will vary.

Explain all the steps you take to simplify the rational expression p2+4p−219−p2.

ⓐ Multiply 74·910 and explain all your steps. ⓑ Multiply nn−3·9n+3 and explain all your steps. ⓒ Evaluate your answer to part ⓑ when n=7. Did you get the same answer you got in part ⓐ? Why or why not?

Answers will vary.

ⓐ Divide 245÷6 and explain all your steps. ⓑ Divide x2−1x÷(x+1) and explain all your steps. ⓒ Evaluate your answer to part ⓑ when x=5. Did you get the same answer you got in part ⓐ? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

rational expression

A rational expression is an expression of the form pq, where *p* and *q* are polynomials and $q \neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

rational function

A rational function is a function of the form $R(x) = p(x)q(x)$ where p(x) and q(x) are polynomial functions and q(x) is not zero.

Add and Subtract Rational Expressions
By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Before you get started, take this readiness quiz.

1. Add: $710+815$.
   If you missed this problem, review [link].
2. Subtract: $3x4-89$.
   If you missed this problem, review [link].
3. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review [link].

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If *p*, *q*, and *r* are polynomials where $r \neq 0$, then

$pr+qr=p+qr$ and $pr-qr=p-qr$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would

make the denominator zero. What value of x should be excluded in the next example?

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.
Since the denominator is $x+4$, we must exclude the value $x=-4$. $\frac{11x+28}{x+4}+\frac{x^2}{x+4}, x\neq -4$ The fractions have a common denominator,so add the numerators and place the sumover the common denominator.$\frac{11x+28+x^2}{x+4}$ Write the degrees in descending order.$\frac{x^2+11x+28}{x+4}$ Factor the numerator.$\frac{(x+4)(x+7)}{x+4}$ Simplify by removing common factors.$\frac{(x+4)(x+7)}{x+4}$ Simplify.$x+7$ The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

$x+2$

Simplify: $x2+8xx+5+15x+5$.

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

Subtract: $5x2-7x+3x2-3x+18-4x2+x-9x2-3x+18$.

$5x2-7x+3x2-3x+18-4x2+x-9x2-3x+18$ Subtract the numerators and place thedifference over the common denominator.$5x2-7x+3-(4x2+x-9)x2-3x$

$+18$ Distribute the sign in the numerator.$5x2-7x+3-4x2-x+9x2-3x-18$ Combine like terms.$x2-8x+12x2-3x-18$ Factor the numerator and the denominator.$(x-2)(x-6)(x+3)(x-6)$ Simplify by removing common factors.$(x-2)(x-6)(x+3)(x-6)$ $(x-2)(x+3)$

Subtract: $4x2-11x+8x2-3x+2-3x2+x-3x2-3x+2$.

$x-11x-2$

Subtract: $6x2-x+20x2-81-5x2+11x-7x2-81$.

$x-3x+9$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $-1-1$.

Let’s see how this works.

	$\frac{7}{a}+\frac{5}{-a}$
Multiply the second fraction by $-1-1$.	$\frac{7}{a}+\frac{(-1)5}{(-1)(-a)}$
The denominators are the same.	$\frac{7}{a}+\frac{-5}{a}$
Simplify.	$\frac{2}{a}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

Subtract: m2 − 6mm2 − 1 − 3m + 21 − m2.

$$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$$

The denominators are opposites, so multiply the second fraction by −1 −1.

$$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$$

Simplify the second fraction.

$$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$$

The denominators are the same. Subtract the nu

$$\frac{m^2-6m-(-3m-2)}{m^2-1}$$

Distribute.

$$\frac{m^2-6m+3m+2}{m^2-1}$$

Combine like terms.

$$\frac{m^2-3m+2}{m^2-1}$$

Factor the numerator

an [redacted]

Simplify by removing common factors.

[redacted]

Simplify.

[redacted]

Subtract: $y2-5yy2-4-6y-64-y2$.

$y+3y+2$

Subtract: $2n2+8n-1n2-1-n2-7n$
$-11-n2$.

$3n-2n-1$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let’s look at this example: $712+518$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we “brought down” one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12} + \frac{5}{18}$$

$$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$$

$$\frac{21}{36} + \frac{10}{36}$$

$$\begin{array}{rl} 12 = & 2 \cdot 2 \cdot 3 \\ 18 = & 2 \cdot \phantom{2 \cdot{}} 3 \cdot 3 \\ \hline \text{LCD} = & 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCD} = & 36 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Find the least common denominator of rational expressions.

Factor each denominator completely. List the factors of each denominator. Match factors vertically when possible. Bring down the columns by including all factors, but do not include common factors twice. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

ⓐ Find the LCD for the expressions 8x2 − 2x − 3,3xx2 + 4x + 3 and ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

ⓐ

Find the LCD for 8x2 − 2x − 3,3xx2 + 4x + 3.

Factor each denominator

co

co

$$x^2 - 2x - 3 = (x+1)(x-3)$$
$$x^2 + 4x + 3 = (x+1)\qquad(x+3)$$
$$\text{LCD} \qquad = (x+1)(x-3)(x+3)$$

Bring down the columns.

Write the LCD as the product of the factors.

The LCD is $(x+1)(x-3)(x+3)$.

ⓑ

[redacted]

Factor each denominator.

[redacted]

Multiply each denominator by the ‘m[redacted]

LCD factor and multiply each numerator by the same factor.

Simplify the numerators.

[redacted]

ⓐ Find the LCD for the expressions 2x2 − x − 12,1x2 − 16 ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

ⓐ (x − 4)(x + 3)(x + 4)
ⓑ 2x + 8(x − 4)(x + 3)(x + 4), x + 3(x − 4)(x + 3)(x + 4)

ⓐ Find the LCD for the expressions $3xx2-3x–10,5x2+3x+2$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

ⓐ $(x+2)(x-5)(x+1)$
ⓑ $3x2+3x(x+2)(x-5)(x+1)$,
$5x-25(x+2)(x-5)(x+1)$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

How to Add Rational Expressions with Unlike Denominators

Add: $3x-3+2x-2$.

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3:(x-3)$ $x-2:(x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$

Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$

Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Add: $2x-2+5x+3$.

$7x-4(x-2)(x+3)$

Add:$4m+3+3m+4$.

$7m+25(m+3)(m+4)$

The steps used to add rational expressions are summarized here.

Add or subtract rational expressions.

Determine if the expressions have a common denominator.
 	
• **Yes** – go to step 2.
• **No –** Rewrite each rational expression with the LCD.
    ○ Find the LCD.
    ○ Rewrite each rational expression as an equivalent rational expression with the LCD.
 	
Add or subtract the rational expressions. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational

expression as $3x-6(x-3)(x-2)$ to be able to add it to $2x-6(x-2)(x-3)$. Simplify *only* after you have combined the numerators.

Add: $8x2-2x-3+3xx2+4x+3$.

[redacted]

Do the expressions have a common denominator? No.

Rewrite each expression with the LCD.

Find the LCD.$x2-2x-3=(x+1)(x-3)x2+4x+3=(x+1)(x+3)$ LCD$=(x+1)(x-3)(x+3)$

Rewrite each rational

ex[redacted] equivalent rational expression with the LCD.

Simplify the numerators.

[redacted]

Add the rational expressions.

[redacted]

Simplify the numerator.

[redacted]

The numerator is prime, so there are no common factors.

Add: 1m2 − m − 2 + 5mm2 + 3m + 2.

5m2 − 9m + 2(m + 1)(m − 2)(m + 2)

Add: $2nn2-3n-10+6n2+5n+6$.
$2n2+12n-30(n+2)(n-5)(n+3)$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Subtract: $8yy2-16-4y-4$.

[redacted]

Do the expressions have a common denominator?	No.
Rewrite each	

expression with the LCD.

Find the LCD. $y^2 - 16 = (y - 4)(y + 4)$ $y - 4 = y - 4$ ________ $\text{LCD} = (y - 4)(y + 4)$

Rewrite each rational expression as an eq[redacted]expression with the LCD.

Simplify the numerators.

[redacted]

Subtract the rational expressions.

[redacted]

Simplify the numerator.

[redacted]

Factor the numerator to look for common fac[redacted]

Remove common factors

[redacted]

Simplify.

[redacted]

Subtract: $2xx2 - 4 - 1x + 2$.

$1x - 2$

Subtract: $3z + 3 - 6zz2 - 9$.

$-3z - 3$

There are lots of negative signs in the next example. Be extra careful.

Subtract: $-3n - 9n2 + n - 6 - n + 32 - n$.

$$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$$

Factor the denominator.

$$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$$

Since n − 2 and 2 − n are opposites, we will mu

$$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$$

rational expression by −1 −1.

$$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(n-2)}$$

Simplify. Remember, a − (− b) = a + b.

$$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$$

Do the rational expressions have a common denominator? No.

Find the LCD.n2 + n − 6 = (n − 2)(n + 3)n − 2 = (n − 2)______________LCD = (n − 2)(n + 3)

Rewrite each rational expression as an eq

$$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$$

expression with the LCD.

Simplify the numerators.

[redacted]

Add the rational expressions.

[redacted]

Simplify the numerator.

[redacted]

Factor the numerator to look for common fac[redacted]

Simplify.

[redacted]

Subtract :$3x-1x2-5x-6-26-x$.

$5x+1(x-6)(x+1)$

Subtract: $-2y-2y2+2y-8-y-12-y$.

$y+3y+4$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Subtract: $4a2+6a+5-3a2+7a+10$.

[redacted]

Factor the denominators.

[redacted]

Do the rational expressions have a

common denominator?

No.

Find the LCD.a2 + 6a + 5 = (a + 1)(a + 5)a2 + 7a + 10 = (a + 5)(a + 2)________________LCD = (a + 1)(a + 5)(a + 2)

Rewrite each rational expression as an eq[redacted]

expression with the LCD.

Simplify the numerators.

[redacted]

Subtract the rational expressions.

[redacted]

Simplify the numerator.

[redacted]

Look for common factors.

[redacted]

Simplify.

[redacted]

Subtract: $3b2-4b-5-2b2-6b+5$.

$1(b+1)(b-1)$

Subtract: $4x2-4-3x2-x-2$.

$1(x+2)(x+1)$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

Simplify: $2uu-1+1u-2u-1u2-u$.

$$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$$

Do the expressions have a common denominator? No. Rewrite each expression with the LCD.

Find the LCD. $u-1=(u-1)$ $u=u$ $u^2-u=u(u-1)$___________ $\text{LCD}=u(u-1)$

Rewrite each rational expression as an eq

$$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$$

expression with the LCD.

$$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$$

Write as one rational expression.

$$\frac{2u^2+u-1-2u+1}{u(u-1)}$$

Simplify.

$$\frac{2u^2-u}{u(u-1)}$$

Factor the numerator, and remove common factors.

[redacted]

Simplify.

[redacted]

Simplify: vv + 1 + 3v − 1 − 6v2 − 1.

v + 3v + 1

Simplify: 3ww + 2 + 2w + 7 − 17w + 4w2 + 9w + 14.

3ww + 7

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

Find $R(x)=f(x)-g(x)$ where $f(x)=x+5x-2$ and $g(x)=5x+18x2-4$.

[redacted]

Substitute in the functions $f(x), g(x)$.

[redacted]

Factor the denominators.

[redacted]

Do the expressions have a common denominator? No. Rewrite each

expression with the LCD.

Find the LCD. x − 2 = (x − 2) x2 − 4 = (x − 2)(x + 2)

LCD = (x − 2)(x + 2)

Rewrite each rational expression as an eq

$$R(x) = \frac{(x+5)(x+2)}{(x-2)(x+2)} - \frac{5x+18}{(x-2)(x+2)}$$

expression with the LCD.

Write as one rational expression.

$$R(x) = \frac{(x+5)(x+2) - (5x+18)}{(x-2)(x+2)}$$

Simplify.

$$R(x) = \frac{x^2 + 7x + 10 - 5x - 18}{(x-2)(x+2)}$$

$$R(x) = \frac{x^2 + 2x - 8}{(x-2)(x+2)}$$

Factor the numerator, and remove co

$$R(x) = \frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$$

Simplify.

$$R(x) = \frac{(x+4)}{(x+2)}$$

Find $R(x) = f(x) - g(x)$ where $f(x) = x + 1x + 3$ and $g(x) = x + 17x2 - x - 12$.

$x - 7x - 4$

Find $R(x) = f(x) + g(x)$ where $f(x) = x - 4x + 3$ and $g(x) = 4x + 6x2 - 9$.

$x2 - 3x + 18(x + 3)(x - 3)$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If *p*, *q*, and *r* are polynomials where $r \neq 0$, then
  $pr + qr = p + qr$ and $pr - qr = p - qr$
- **How to find the least common denominator of rational expressions.**

  Factor each expression completely. List the factors of each expression. Match factors vertically when possible. Bring down the columns. Write the LCD as the product of the factors.

- **How to add or subtract rational expressions.**

  Determine if the expressions have a common denominator.

  - Yes – go to step 2.
  - No – Rewrite each rational expression with the LCD.

    - Find the LCD.
    - Rewrite each rational expression as an equivalent rational expression with the LCD.

  Add or subtract the rational expressions. Simplify, if possible.

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

$215+715$

35

$724+1124$

$3c4c-5+54c-5$

$3c+54c-5$

$7m2m+n+42m+n$

$2r22r-1+15r-82r-1$

$r+8$

$3s23s-2+13s-103s-2$

$2w2w2 - 16 + 8ww2 - 16$

$2ww - 4$

$7x2x2 - 9 + 21xx2 - 9$

In the following exercises, subtract.

$9a23a - 7 - 493a - 7$

$3a + 7$

$25b25b - 6 - 365b - 6$

$3m26m - 30 - 21m - 306m - 30$

$m - 22$

$2n24n - 32 - 18n - 164n - 32$

$6p2 + 3p + 4p2 + 4p - 5 - 5p2 + p + 7p2 + 4p - 5$

$p + 3p + 5$

$5q2+3q-9q2+6q+8-4q2+9q+7q2+6q+8$

$5r2+7r-33r2-49-4r2+5r+30r2-49$

$r+9r+7$

$7t2-t-4t2-25-6t2+12t-44t2-25$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

$10v2v-1+2v+41-2v$

4

$20w5w-2+5w+62-5w$

$10x2+16x-78x-3+2x2+3x-13-8x$

$x+2$

$6y2+2y-113y-7+3y2-3y+177-3y$

$z2+6zz2-25-3z+2025-z2$

$z+4z-5$

$a2+3aa2-9-3a-279-a2$

$2b2+30b-13b2-49-2b2-5b-849-b2$

$4b-3b-7$

$c2+5c-10c2-16-c2-8c-1016-c2$

Find the Least Common Denominator of Rational Expressions

In the following exercises, ⓐ find the LCD for the given rational expressions ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

$5x2-2x-8,2xx2-x-12$

ⓐ $(x+2)(x-4)(x+3)$
ⓑ $5x+15(x+2)(x-4)(x+3)$,
$2x2+4x(x+2)(x-4)(x+3)$

$8y2+12y+35,3yy2+y-42$

$9z2+2z-8,4zz2-4$

ⓐ $(z-2)(z+4)(z-4)$
ⓑ $9z-36(z-2)(z+4)(z-4),$
$4z2-8z(z-2)(z+4)(z-4)$

$6a2+14a+45,5aa2-81$

$4b2+6b+9,2bb2-2b-15$

ⓐ $(b+3)(b+3)(b-5)$
ⓑ $4b-20(b+3)(b+3)(b-5),$
$2b2+6b(b+3)(b+3)(b-5)$

$5c2-4c+4,3cc2-7c+10$

$23d2+14d-5,5d3d2-19d+6$

ⓐ $(d+5)(3d-1)(d-6)$
ⓑ $2d-12(d+5)(3d-1)(d-6),$
$5d2+25d(d+5)(3d-1)(d-6)$

$35m2-3m-2,6m5m2+17m+6$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

$710x2y+415xy2$

$21y+8x30x2y2$

$112a3b2+59a2b3$

$3r+4+2r-5$

$5r-7(r+4)(r-5)$

$4s-7+5s+3$

$53w-2+2w+1$

$11w+1(3w-2)(w+1)$

$42x+5+2x-1$

$2yy+3+3y-1$

$2y2+y+9(y+3)(y-1)$

$3zz-2+1z+5$

$5ba2b-2a2+2bb2-4$

$b(5b+10+2a2)a2(b-2)(b+2)$

$4cd+3c+1d2-9$

$-3m3m-3+5mm2+3m-4$

$-mm+4$

$84n+4+6n2-n-2$

$3rr2+7r+6+9r2+4r+3$

$3(r2+6r+18)(r+1)(r+6)(r+3)$

$2ss2+2s-8+4s2+3s-10$

$tt-6-t-2t+6$

$2(7t-6)(t-6)(t+6)$

$x-3x+6-xx+3$

$5aa+3-a+2a+6$

$4a2+25a-6(a+3)(a+6)$

$3bb-2-b-6b-8$

$6m+6-12mm2-36$

$-6m-6$

$4n+4-8nn2-16$

$-9p-17p2-4p-21-p+17-p$

$p+2p+3$

$-13q\text{-}8q2+2q-24-q+24-q$

$-2r-16r2+6r-16-52-r$

$3r-2$

$2t-30t2+6t-27-23-t$

$2x+710x-1+3$

$4(8x+1)10x-1$

$8y-45y+2-6$

$3x2-3x-4-2x2-5x+4$

$x-5(x-4)(x+1)(x-1)$

$4x2 - 6x + 5 - 3x2 - 7x + 10$

$5x2 + 8x - 9 - 4x2 + 10x + 9$

$1(x - 1)(x + 1)$

$32x2 + 5x + 2 - 12x2 + 3x + 1$

$5aa - 2 + 9a - 2a + 18a2 - 2a$

$5a2 + 7a - 36a(a - 2)$

$2bb - 5 + 32b - 2b - 152b2 - 10b$

$cc + 5 + 5c - 2 - 10cc2 - 4$

$c - 5c + 2$

$6dd - 5 + 1d + 4 - 7d - 5d2 - d - 20$

$3dd + 2 + 4d - d + 8d2 + 2d$

$3(d+1)d+2$

$2qq+5+3q-3-13q+15q2+2q-15$

Add and Subtract Rational Functions

In the following exercises, find ⓐ $R(x)=f(x)+g(x)$ ⓑ $R(x)=f(x)-g(x)$.

$f(x)=-5x-5x2+x-6$ and
$g(x)=x+12-x$

ⓐ $R(x)=-(x+8)(x+1)(x-2)(x+3)$ ⓑ
$R(x)=x+1x+3$

$f(x)=-4x-24x2+x-30$ and
$g(x)=x+75-x$

$f(x)=6xx2-64$ and
$g(x)=3x-8$

ⓐ $3(3x+8)(x-8)(x+8)$
ⓑ $R(x)=3x+8$

$f(x)=5x+7$ and
$g(x)=10xx2-49$

Writing Exercises

Donald thinks that $3x+4x$ is $72x$. Is Donald correct? Explain.

Answers will vary.

Explain how you find the Least Common Denominator of $x2+5x+4$ and $x2-16$.

Felipe thinks $1x+1y$ is $2x+y$. ⓐ Choose numerical values for *x* and *y* and evaluate $1x+1y$. ⓑ Evaluate $2x+y$ for the same values of *x* and *y* you used in part ⓐ. ⓒ Explain why Felipe is wrong. ⓓ Find the correct expression for $1x+1y$.

ⓐ Answers will vary.
ⓑ Answers will vary.
ⓒ Answers will vary.
ⓓ $x+yxy$

Simplify the expression $4n2+6n+9-1n2-9$ and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Simplify Complex Rational Expressions

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Before you get started, take this readiness quiz.

Simplify: 35910.
If you missed this problem, review [link].

23

Simplify: $1-1342+4\cdot5$.
If you missed this problem, review [link].

154

Solve: $12x+14=18$.
If you missed this problem, review [link].

$x=-14$

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:
3458x2xy6

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression
A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:
$4y - 38y2 - 91x + 1yxy - yx2x + 64x - 6 - 4x2 - 36$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.
$6x2 - 7x + 24x - 82x2 - 8x + 3x2 - 5x + 6$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:
$(6x2 - 7x + 24x - 8) \div (2x2 - 8x + 3x2 - 5x + 6)$.

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Simplify the complex rational expression by

writing it as division: 6x − 43x2 − 16.

	6x − 43x2 − 16
Rewrite the complex fraction as division.	6x − 4 ÷ 3x2 − 16
Rewrite as the product of first times the reciprocal of the second.	6x − 4·x2 − 163
Factor.	3·2x − 4·(x − 4)(x + 4)3
Multiply.	3·2(x − 4)(x + 4)3(x − 4)
Remove common factors.	3·2(x − 4)(x + 4)3(x − 4)
Simplify.	2(x + 4)

Are there any value(s) of *x* that should not be allowed? The original complex rational expression had denominators of x − 4 and x2 − 16. This expression would be undefined if x = 4 or x = − 4.

Simplify the complex rational expression by writing it as division: $2x2-13x+1$.

$23(x-1)$

Simplify the complex rational expression by writing it as division: $1x2-7x+122x-4$.

$12(x-3)$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Simplify the complex rational expression by writing it as division: $13+1612-13$.

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$	
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator.	$$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$
Find the LCD and subtract the fractions in the denominator.	
Simplify the numerator and denominator.	$$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$
Rewrite the complex rational expression as a division problem.	$$\frac{3}{6}\div\frac{1}{6}$$
Multiply the first by the reciprocal of the second.	$$\frac{3}{6}\cdot\frac{6}{1}$$
Simplify.	3

Simplify the complex rational expression by writing it as division: $12+2356+112$.

1411

Simplify the complex rational expression by writing it as division: $34-1318+56$.

1023

We follow the same procedure when the complex rational expression contains variables.

How to Simplify a Complex Rational Expression using Division

Simplify the complex rational expression by writing it as division: $1x+1yxy-yx$.

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
		$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Find a common denominator and subtract the fractions in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	

Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$

Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\frac{1}{x-y}$

Simplify the complex rational expression by writing it as division: 1x+1y1x−1y.

y + xy − x

Simplify the complex rational expression by writing it as division: 1a + 1b1a2 − 1b2.

abb − a

We summarize the steps here.

Simplify a complex rational expression by writing it as division.

Simplify the numerator and denominator. Rewrite the complex rational expression as a division problem. Divide the expressions.

Simplify the complex rational expression by writing it as division: n − 4nn + 51n + 5 + 1n

$-5.$

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$$

Simplify the numerator and denominator.

Fi
de
nu
denominator.

$$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$$

Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$$

Subtract the rational expressions in the
nu
ad
de

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

Simplify. (We now have one rational expression over

on [redacted]
ex

Rewrite as fraction division.

[redacted]

Multiply the first times the reciprocal of the sec [redacted]

Factor any expressions if possible.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Simplify the complex rational expression by writing it as division: b − 3bb + 52b + 5 + 1b − 5.

$b(b+2)(b-5)3b-5$

Simplify the complex rational expression by writing it as division: $1-3c+41c+4+c3$.
$3c+3$

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in [link]. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by LCDLCD we are

multiplying by 1, so the value stays the same.

Simplify the complex rational expression by using the LCD: 13+1612−13.

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$

The LCD of all the fractions in the whole expression is 6.

Clear the fractions by multiplying the nu d de by that LC

$$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$$

Distribute.

$$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$$

Simplify.

$$\frac{2+1}{3-2}$$

$$\frac{3}{1}$$

$$3$$

Simplify the complex rational expression by using the LCD: 12+15110+15.

73

Simplify the complex rational expression by using the LCD: 14+3812−516.

103

We will use the same example as in [link]. Decide which method works better for you.

How to Simplify a Complex Rational Expressing using the LCD

Simplify the complex rational expression by using the LCD: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is *xy*.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$

Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by *xy*.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$

Step 3. Simplify the expression.	Distribute.	$\dfrac{xy \cdot \frac{1}{x} + xy \cdot \frac{1}{y}}{xy \cdot \frac{x}{y} - xy \cdot \frac{y}{x}}$
		$\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$\dfrac{1}{x-y}$

Simplify the complex rational expression by using the LCD: 1a + 1bab + ba.

b + aa2 + b2

Simplify the complex rational expression by using the LCD: 1x2 − 1y21x + 1y.

y − xxy

Simplify a complex rational expression by using the LCD.

Find the LCD of all fractions in the complex rational expression. Multiply the numerator and denominator by the LCD. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Simplify the complex rational expression by using the LCD: 2x + 64x − 6 − 4x2 − 36.

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Find the LCD of all fractions in the complex rational expression. The LCD is x2 − 36 = (x + 6)(x − 6). Multiply the numerator

an[redacted]
the

Simplify the expression.

Distribute in the denominator.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

To simplify the denominator, dis[redacted] and combine like terms.

Factor the denominator.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Notice that there are no more factors

common to the numerator and denominator.

Simplify the complex rational expression by using the LCD: $3x+25x-2-3x2-4$.
$3(x-2)5x+7$

Simplify the complex rational expression by using the LCD: $2x-7-1x+76x+7-1x2-49$.
$x+216x-43$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Simplify the complex rational expression by using the LCD: 4m2 − 7m + 123m − 3 − 2m − 4.

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Find the LCD of all fractions in the complex rational expression.

The LCD is (m − 3)(m − 4).

Multiply the numerator and de
LC

$$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$$

Simplify.

$$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$$

Simplify.

[redacted]

Distribute.

[redacted]

Combine like terms.

[redacted]

Simplify the complex rational expression by using the LCD: 3x2 + 7x + 104x + 2 + 1x + 5.

35x + 22

Simplify the complex rational expression by using the LCD: 4yy + 5 + 2y + 63yy2 + 11y + 30.

2(2y2 + 13y + 5)3y

Simplify the complex rational expression by using the LCD: yy+11+1y−1.

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Find the LCD of all fractions in the complex rational expression.

The LCD is (y+1)(y−1).

Multiply the numerator and denominator by the

$$\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$$

Distribute in the denominator and sin

$$\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$$

Simplify.

[redacted]

Simplify the denominator and leave the [redacted] numerator factored.

[redacted]

Factor the denominator and remove factors co[redacted] numerator.

Simplify.

[redacted]

Simplify the complex rational expression by using the LCD: xx + 31 + 1x + 3.

xx + 4

Simplify the complex rational expression by using the LCD: $1+1x-13x+1$.

$x(x+1)3(x-1)$

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**

  Simplify the numerator and denominator. Rewrite the complex rational expression as a division problem. Divide the expressions.

- **How to simplify a complex rational expression by using the LCD.**

  Find the LCD of all fractions in the complex

rational expression. Multiply the numerator and denominator by the LCD. Simplify the expression.

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

$2aa+44a2a2-16$

$a-42a$

$3bb-5b2b2-25$

$5c2+5c-1410c+7$

$12(c-2)$

$8d2+9d+1812d+6$

$12+5623+79$

1213

$12+3435+710$

$23-1934+56$

2057

$12-1623+34$

$nm+1n1n-nm$

$n2+mm-n2$

$1p+pqqp-1q$

$1r+1t1r2-1t2$

$rtt-r$

$2v+2w1v2-1w2$

$x-2xx+31x+3+1x-3$

$(x+1)(x-3)2$

$y-2yy-42y-4+2y+4$

$2-2a+31a+3+a2$

$4a+1$

$4+4b-51b-5+b4$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

$13+1814+112$

118

$14 + 1916 + 112$

$56 + 29718 - 13$

19

$16 + 41535 - 12$

$cd + 1d1d - dc$

$c2 + cc - d2$

$1m + mnnm - 1n$

$1p + 1q1p2 - 1q2$

$pqq - p$

$2r + 2t1r2 - 1t2$

$2x + 53x - 5 + 1x2 - 25$

$2x - 103x + 16$

$5y - 43y + 4 + 2y2 - 16$

$5z2 - 64 + 3z + 81z + 8 + 2z - 8$

$3z - 193z + 8$

$3s + 6 + 5s - 61s2 - 36 + 4s + 6$

$4a2 - 2a - 151a - 5 + 2a + 3$

$43a - 7$

$5b2 - 6b - 273b - 9 + 1b + 3$

$5c + 2 - 3c + 75cc2 + 9c + 14$

$2c + 295c$

$6d - 4 - 2d + 72dd2 + 3d - 28$

$2+1p-35p-3$

$2p-55$

$nn-23+5n-2$

$mm+54+1m-5$

$m(m-5)(4m-19)(m+5)$

$7+2q-21q+2$

In the following exercises, simplify each complex rational expression using either method.

$34-2712+514$

1324

$vw+1v1v-vw$

$2a+41a2-16$

2(a − 4)

3b2 − 3b − 405b + 5 − 2b − 8

3m + 3n1m2 − 1n2

3mnn − m

2r − 91r + 9 + 3r2 − 81

x − 3xx + 23x + 2 + 3x − 2

(x − 1)(x − 2)6

yy + 32 + 1y − 3

Writing Exercises

In this section, you learned to simplify the complex fraction 3x + 2xx2 − 4 two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Answers will vary.

Efraim wants to start simplifying the complex fraction $1a+1b1a-1b$ by cancelling the variables from the numerator and denominator, $1a+1b1a-1b$. Explain what is wrong with Efraim's plan.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

complex rational expression
: A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

Solve Rational Equations

By the end of this section, you will be able to:

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Before you get started, take this readiness quiz.

1. Solve: $16x+12=13$.
   If you missed this problem, review [link].
2. Solve: $n2-5n-36=0$.
   If you missed this problem, review [link].
3. Solve the formula $5x+2y=10$ for y.
   If you missed this problem, review [link].

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will *solve* a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational ExpressionRational Equation 18x+12 y
+6y2−36 1n−3+1n+4 18x+12=14 y
+6y2−36=y+1 1n−3+1n+4=15n2+n−12

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we

examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, *c*, by writing $x \neq c$ next to the equation.

How to Solve a Rational Equation

Solve: $1x+13=56$.

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6},\ x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x \cdot \left(\frac{1}{x} + \frac{1}{3}\right) = 6x \cdot \left(\frac{5}{6}\right)$ $6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left(\frac{5}{6}\right)$ $6 + 2x = 5x$
Step 4. Solve the resulting equation.	Simplify.	$6 = 3x$ $2 = x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x = 2$ into the original equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$ $\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6} = \frac{5}{6}$ ✓ The solution is $x = 2$.

Solve: $1y+23=15$.

$y=-715$

Solve: $23+15=1x$.

$x=1315$

The steps of this method are shown.

Solve equations with rational expressions.

Note any value of the variable that would make any denominator zero. Find the least common denominator of *all* denominators in the equation. Clear the fractions by multiplying both sides of the equation by the LCD. Solve the resulting equation. Check:

- If any values found in Step 1 are algebraic solutions, discard them.

- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

How to Solve a Rational Equation using the Zero Product Property

Solve: 1 − 5y = − 6y2.

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2},\ y\neq 0$$

Find the least common denominator of all denominators in

the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of [redacted] the equation by the LCD.

Distribute.

[redacted]

Multiply.

[redacted]

Solve the resulting equation. First wr[redacted] equation in standard form.

Factor.

[redacted]

Use the Zero Product Property.

[redacted]

Solve.

[redacted]

Check.

We did not get 0 as an algebraic solution.

[redacted]

The solution is $y = 2$, $y = 3$.

Solve: $1 - 2x = 15x2$.

$x = -3, x = 5$

Solve: $1 - 4y = 12y2$.

$y = -2, y = 6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Solve: $2x+2+4x-2=x-1x2-4$.

[redacted]

Note any value of the variable
tha[redacted]
denominator
zero.

Find the least common denominator of all denominators

in the equation.
The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying bo[redacted] equation by the LCD.

Distribute.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Distribute.

[redacted]

Solve.

[redacted]

[redacted]

[redacted]

Check:

We did not get 2 or -2 as algebraic solutions.

Check $x = -1$ in the original equation.

$$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$$

$$\frac{2}{(-1)+2} + \frac{4}{(-1)-2} \stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4}$$

$$\frac{2}{1} + \frac{4}{-3} \stackrel{?}{=} \frac{-2}{-3}$$

$$\frac{6}{3} - \frac{4}{3} \stackrel{?}{=} \frac{2}{3}$$

$$\frac{2}{3} = \frac{2}{3} \checkmark$$

The solution is $x = -1$.

Solve: $2x+1+1x-1=1x2-1$.

$x=23$

Solve: 5y+3+2y−3=5y2−9.

y=2

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Solve: m+11m2−5m+4=5m−4−3m−1.

[redacted]

Note any value of the variable that [redacted] denominator zero. Use the factored form of the quadratic

denominator.

Find the least common denominator of all denominators in the equation.

The LCD is $(m-4)(m-1)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

[redacted]

Distribute.

[redacted]

Remove common factors.

[redacted]

Simplify.

[redacted]

Solve the resulting equation.

[redacted]

[redacted]

Check.

The only algebraic solution was 4, but we said that 4 would make

a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Solve: $x+13x2-7x+10=6x-5-4x-2$.
There is no solution.

Solve: $y-6y2+3y-4=2y+4+7y-1$.
There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to

the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

Solve: $yy+6=72y2-36+4$.

[redacted]

Factor all the denominators, so [redacted] value of the variable that would make any denominator zero.

Find the least common denominator.

The LCD is $(y-6)(y+6)$.

Clear the fractions.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Solve the resulting equation.

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

Check.

[redacted]

The solution is $y=4$.

Solve: $\frac{x}{x+4}=\frac{32}{x^2-16}+5$.

$x=3$

Solve: $\frac{y}{y+8}=\frac{128}{y^2-64}+9$.

$y=7$

In some cases, all the algebraic solutions are extraneous.

Solve: x2x − 2 − 23x + 3 = 5x2 − 2x +912x2 − 12.

[redacted]

We will start by factoring all de[redacted] it easier to identify extraneous solutions and the LCD.

Note any value of the variable tha[redacted] denominator zero.

Find the least common denominator.

The LCD is 12(x − 1)(x

+ 1).

Clear the fractions.

[redacted]

Simplify.

[redacted]

Simplify.

[redacted]

Solve the resulting equation.

[redacted]

[redacted]

[redacted]

[redacted]

Check.

$x = 1$ and $x = -1$ are extraneous solutions.

The equation has no solution.

Solve: $y5y-10-53y+6=2y2-19y+5415y2-60$.

There is no solution.

Solve: $z2z+8-34z-8=3z2-16z-168z2+2z-64$.

There is no solution.

Solve: $43x2-10x+3+33x2+2x-1=2x2-2x-3$.

[redacted]

Factor all the denominators, so we ca[redacted] the variable that would make any denominator zero.

$x \neq -1, x \neq 13, x \neq 3$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

[redacted]

Simplify.

[redacted]

Distribute.

[redacted]

Simplify.

[redacted]

[redacted]

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic

solution is an extraneous solution.

There is no solution to this equation.

Solve: $15x2+x-6-3x-2=2x+3$.

There is no solution.

Solve: $5x2+2x-3-3x2+x-2=1x2+5x+6$.

There is no solution.

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

For rational function, $f(x)=2x-6x2-8x+15$, ⓐ find the domain of the function, ⓑ solve $f(x)=1$, and ⓒ find the points on the graph at this function value.

ⓐ The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$x2-8x+15=0$Factor the trinomial.$(x-3)(x-5)=0$Use the Zero Product Property.$x-3=0x-5=0$Solve.$x=3x=5$The domain is all real numbers except$x\neq 3, x\neq 5$..

ⓑ

[redacted]

Substitute in the rational expression.

[redacted]

Factor the denominator.

[redacted]

Multiply both sides by the LCD, (x [redacted]

Simplify.

[redacted]

Solve.

[redacted]

Factor.

[redacted]

Use the Zero Product Property.

[redacted]

Solve.

[redacted]

© The value of the function is 1 when x=7,x=3. So the points on the graph of this function when f(x)=1, will be (7,1),(3,1).

For rational function, $f(x)=8-xx2-7x+12$, ⓐ find the domain of the function ⓑ solve $f(x)=3$ ⓒ find the points on the graph at this function value.

ⓐ The domain is all real numbers except $x\neq 3$ and $x\neq 4$. ⓑ $x=2, x=143$
ⓒ $(2,3),(143,3)$

For rational function, $f(x)=x-1x2-6x+5$, ⓐ find the domain of the function ⓑ solve $f(x)=4$ ⓒ find the points on the graph at this function value.

ⓐ The domain is all real numbers except $x\neq 1$ and $x\neq 5$. ⓑ $x=214$ ⓒ $(214,4)$

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$m=y-y1x-x1$ Multiply both sides of the equation by$x-x1$.$m(x-x1)=(y-y1x-x1)(x-x1)$ Simplify.$m(x-x1)=y-y1$ Rewrite the equation with the$y$terms on the left.$y-y1=m(x-x1)$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point (2,3). We will add one more step to solve for *y*.

Solve:$m=y-2x-3$ for y.

[redacted]

Note any value of the variable that would ma [redacted] zero.

Clear the fractions by multiplying both sides of [redacted] the equation by the LCD, $x - 3$.

Simplify.

[redacted]

Isolate the term with *y*.

[redacted]

Solve: $m = y - 5x - 4$ for y.

$y = mx - 4m + 5$

Solve: $m = y - 1x + 5$ for y.
$y = mx + 5m + 1$

Remember to multiply both sides by the LCD in the next example.

Solve: $1c + 1m = 1$ for c.

[redacted]

Note any value of the variable that would ma[redacted]
any denominator zero.
Clear the fractions by

mu[redacted]
of
the equations by the LCD, cm.

Distribute.

[redacted]

Simplify.

[redacted]

Collect the terms with c to the right.

[redacted]

Factor the expression on the right.

[redacted]

To isolate c, divide both sides by $m-1$.

[redacted]

Simplify by removing common factors.

[redacted]

Notice that even though we excluded $c=0, m=0$ from the original equation, we must also now state that $m \neq 1$.

Solve: $1a+1b=c$ for a.

$a=bcb-1$

Solve: $2x+13=1y$ for y.

$y=3xx+6$

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions

Key Concepts

- **How to solve equations with rational**

expressions.

Note any value of the variable that would make any denominator zero. Find the least common denominator of all denominators in the equation. Clear the fractions by multiplying both sides of the equation by the LCD. Solve the resulting equation. Check:

- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve each rational equation.

$1a+25=12$

$a=10$

$63-2d=49$

$45+14=2v$

$v=4021$

$38+2y=14$

$1-2m=8m2$

$m=-2,m=4$

$1+4n=21n2$

$1+9p=-20p2$

$p=-5,p=-4$

$1-7q=-6q2$

$53v-2=74v$

$v=14$

$82w+1=3w$

$3x+4+7x-4=8x2-16$

$x=-45$

$5y-9+1y+9=18y2-81$

$8z-10-7z+10=5z2-100$

$z=-145$

$9a+11-6a-11=6a2-121$

$-10q-2-7q+4=1$

$q=-18,q=-1$

$2s+7-3s-3=1$

$v-10v2-5v+4=3v-1-6v-4$

---

no solution

$w+8w2-11w+28=5w-7+2w-4$

$x-10x2+8x+12=3x+2+4x+6$

---

no solution

$y-5y2-4y-5=1y+1+1y-5$

$b+33b+b24=1b$

$b=-8$

$c+312c+c36=14c$

$dd+3=18d2-9+4$

$d=2$

$mm+5=50m2-25+6$

$nn+2-3=8n2-4$

$m=1$

$pp+7-8=98p2-49$

$q3q-9-34q+12=7q2+6q+6324q2-216$

---

no solution

$r3r-15-14r+20=3r2+17r+4012r2-300$

$s2s+6-25s+5=5s2-3s-710s2+40s+30$

$s=54$

$t6t-12-52t+10=t2-23t+7012t2+36t-120$

$2x2+2x-8-1x2+9x+20=4x2+3x-10$

$x=-43$

$5x2+4x+3+2x2+x-6=3x2-x-2$

$3x2-5x-6+3x2-7x+6=6x2-1$

---

no solution

$2x2+2x-3+3x2+4x+3=6x2-1$

Solve Rational Equations that Involve Functions

For rational function, $f(x)=x-2x2+6x+8$, ⓐ find the domain of the function ⓑ solve $f(x)=5$ ⓒ find the points on the graph at this function value.

> ⓐ The domain is all real numbers except $x\neq-2$ and $x\neq-4$. ⓑ $x=-3, x=-145$ ⓒ $(-3,5)$, $(-145,5)$

For rational function, $f(x)=x+1x2-2x-3$, ⓐ find the domain of the function ⓑ solve $f(x)=1$ ⓒ find the points on the graph at this function value.

For rational function, $f(x)=2-xx2-7x+10$, ⓐ find the domain of the function ⓑ solve $f(x)=2$

ⓒ find the points on the graph at this function value.

ⓐ The domain is all real numbers except $x \neq 2$ and $x \neq 5$. ⓑ $x = 92$, ⓒ $(92,2)$

For rational function, $f(x) = 5 - xx2 + 5x + 6$,
ⓐ find the domain of the function
ⓑ solve $f(x) = 3$
ⓒ the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

$Cr = 2\pi$ for r.

$r = C2\pi$

$Ir = P$ for r.

$v + 3w - 1 = 12$ for w.

$w = 2v + 7$

$x + 52 - y = 43$ for y.

$a = b + 3c - 2$ for c.

$c = b + 3 + 2aa$

$m = n2 - n$ for n.

$1p + 2q = 4$ for p.

$p = q4q - 2$

$3s + 1t = 2$ for s.

$2v + 15 = 3w$ for w.

$w = 15v10 + v$

$6x + 23 = 1y$ for y.

$m + 3n - 2 = 45$ for n.

$n = 5m + 234$

$r = s3 - t$ for t.

$Ec = m2$ for c.

$c = Em2$

$RT = W$ for T.

$3x - 5y = 14$ for y.

$y = 20x12 - x$

$c = 2a + b5$ for a.

Writing Exercises

Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Answers will vary.

Alek thinks the equation $yy+6=72y2-36+4$ has two solutions, $y=-6$ and $y=4$. Explain why Alek is wrong.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations involving functions.			
solve rational equations for a specific variable.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

extraneous solution to a rational equation
: An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

rational equation

A rational equation is an equation that contains a rational expression.

Solve Applications with Rational Equations

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

Before you get started, take this readiness quiz.

1. Solve: $2(n-1)-4=-10$.
   If you missed this problem, review [link].
2. An express train and a charter bus leave Chicago to travel to Champaign. The express train can make the trip in two hours and the bus takes five hours for the trip. The speed of the express train is 42 miles per hour faster than the speed of the bus. Find the speed of the bus.
   If you missed this problem, review [link].
3. Solve $13x+14x=56$.
   If you missed this problem, review [link].

Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

> Proportion
>
> A **proportion** is an equation of the form $ab=cd$, where $b\neq 0, d\neq 0$.
>
> The proportion is read "*a* is to *b* as *c* is to *d*."

The equation $12=48$ is a proportion because the two fractions are equal. The proportion $12=48$ is read "1 is to 2 as 4 is to 8."

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

> Solve: $nn+14=57$.

[redacted]

Multiply both sides by LCD.

[redacted]

Remove common fa[redacted] side.

Simplify.

[redacted]

Solve for *n*.

[redacted]

[redacted]

Check.

[redacted]

[redacted]

Simplify.

[redacted]

Show common

fa[redacted]

Simplify.

[redacted]

Solve the proportion: $yy+55=38$.

$y=33$

Solve the proportion: $zz-84=-15$.

$z=14$

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\begin{array}{cc} \dfrac{n}{n+14}=\dfrac{5}{7} & \dfrac{n}{n+14}=\dfrac{5}{7} \\[2ex] 7(n+14)\left(\dfrac{n}{n+14}\right)=7(n+14)\left(\dfrac{5}{7}\right) & \dfrac{n}{n+14}=\dfrac{5}{7} \\[2ex] 7n=5(n+14) & 7n=5(n+14) \end{array}$$

For any proportion, $ab=cd$, we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\begin{array}{cc} \dfrac{a}{b}=\dfrac{c}{d} & \dfrac{a}{b}=\dfrac{c}{d} \\[2ex] bd\left(\dfrac{a}{b}=\dfrac{c}{d}\right)bd & \dfrac{a}{b}=\dfrac{c}{d} \\[2ex] ad=bc & ad=bc \end{array}$$

To solve applications with proportions, we will follow our usual strategy for solving applications But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how

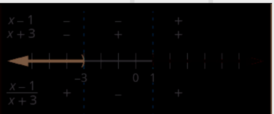
many milliliters of acetaminophen will her doctor prescribe?

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let $a =$ ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion—be careful of the units.	

Multiply both sides by the LCD, 400.	

Step 4. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.	Use values in each interval to determine the value of each factor in the interval. In the interval (–3, 1), zero is a good value to test. For example, when $x = 0$ then $x - 1 = -1$ and $x + 3 = 3$ The factor $x - 1$ is marked negative and $x + 3$ marked positive. Since a negative divided by a positive is negative, the quotient is marked negative in that interval.

Remove common factors on each side.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.	 We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty, -3)$ and $(1, \infty)$ are solutions. Since 3 must be excluded since it makes the rational expression 0, we cannot include it in the solution. We can include 1 in our solution. $(-\infty, -3) \cup [1, \infty)$

Simplify, but don't multiply on the left. Notice what the next step will be.

$$16 \cdot 5 = 5a$$

Solve for *a*.

$$\frac{16 \cdot 5}{5} = \frac{5a}{5}$$

$$16 = a$$

Check.
Is the answer reasonable?

Step 2. Determine the critical points-the points where the rational expression will be zero or undefined.	The rational expression will be zero when the numerator is zero. Since $x - 1 = 0$ when $x = 1$, then 1 is a critical point. The rational expression will be undefined when the denominator is zero. Since $x + 3 = 0$ when $x = -3$, then -3 is a critical point.

Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

The pediatrician will prescribe 12 ml of acetaminophen to Emilia.

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

The pediatrician will prescribe 180 mg of fever reducer to Isabella.

Solve similar figure applications

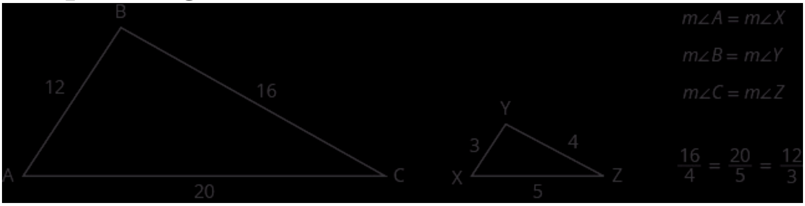
When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

For example, the two triangles in [link] are similar. Each side of ΔABC is four times the length of the

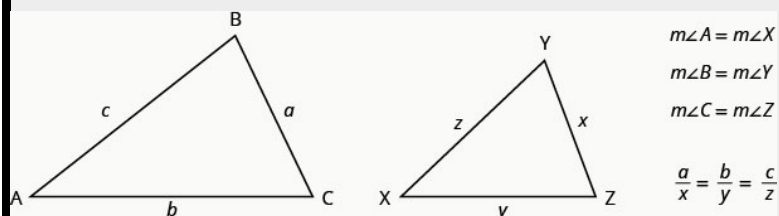
corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

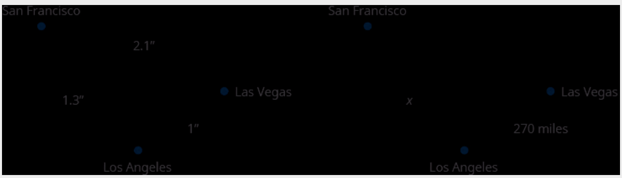
Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. If the actual distance from Los Angeles to Las Vegas is 270 miles, find the distance from Los Angeles to San Francisco.



Since the triangles are similar, the corresponding sides are proportional.

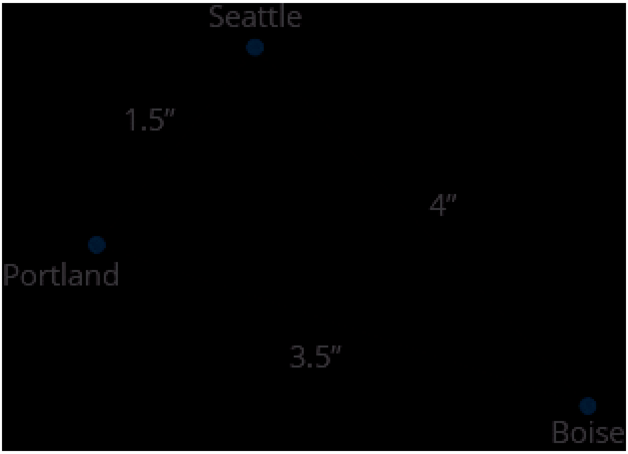
Read the problem. Draw the figures and label it with the given information.	The figures are shown above.
Identify what we are looking for.	the actual distance from Los Angeles to San Francisco

Name the variables.	Let $x =$ distance from Los Angeles to San Francisco.
Translate into an equation. Si[redacted] similar, the corresponding sides are proportional. We’ll make the numerators “miles” and the denominators “inches”.	
Solve the equation.	
[redacted]	
[redacted]	
Check. On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.	

[redacted]

Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. The actual distance from Seattle to Boise is 400 miles.



Find the actual distance from Seattle to Portland.
The distance is 150 miles.

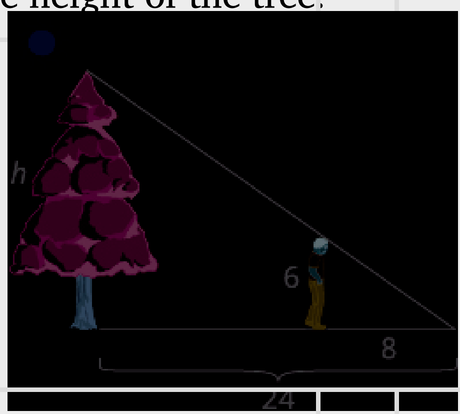
Find the actual distance from Portland to Boise.
The distance is 350 miles.

We can use similar figures to find heights that we cannot directly measure.

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Read the problem and draw a figure.

We are looking for h, the height of the tree.



We will use similar

triangles to write an equation.

The small triangle is similar to the large tri[redacted]

Solve the proportion.

[redacted]

Simplify.

Check.

[redacted]

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a

shadow that is 10 feet long. How tall is the telephone pole?

The telephone pole is 40 feet tall.

A pine tree casts a shadow of 80 feet next to a 30 foot tall building which casts a 40 feet shadow. How tall is the pine tree?

The pine tree is 60 feet tall.

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

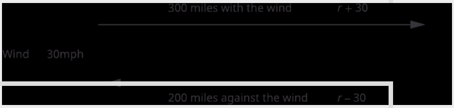
	Rate	• Time	= Distance

The formula $D = rt$ assumes we know *r* and *t* and use them to find *D*. If we know *D* and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t = Dr$.

We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.

Let $r =$ the speed of the airplane.

When the plane flies with the wind,
the wind increases its speed and so the rate is $r + 30$.

When the plane flies against the wind,
the wind decreases its speed and the rate is $r - 30$.

Write in the rates.
Write in the distances.

	Rate ·	Time	= Distance
Headwind	$r-30$	$\frac{200}{r-30}$	200
Tailwind	$r+30$	$\frac{300}{r+30}$	300

Since D = r t, we solve for t and get t=Dr.

We divide the distance by the rate in each row, and place the expression in the time column.

We know the times are equal and so we write our equation.	$200r-30=300r+30$
We multiply both sides by the LCD.	$(r+30)(r-30)(200r-30)=(r+30)(r-30)(300r+30)$
Simplify.	$(r+30)(200)=(r-30)300$
	$200r+6000=300r-9000$
Solve.	$15000=100r$

Check.
Is 150 mph a reasonable speed for an airplane? Yes.
If the plane is traveling 150 mph and the wind is 30 mph,
$\text{Tailwind} \; 150+30=180 \text{mph} \; 300180=53 \text{hours} \; \text{Headwind}$

The times are equal, so it checks.	The plane was traveling 150 mph.

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Link's biking speed is 15 mph.

Danica can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Danica's boat without a wind?

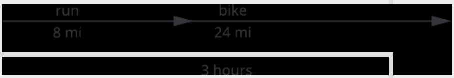
The speed of Danica's boat is 17 mph.

In the next example, we will know the total time resulting from travelling different distances at different speeds.

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

This is a uniform motion situation. A diagram

will help us visualize the situation.



We fill in the chart to organize the information.	
We are looking for Jazmine's running speed.	Let $r =$ Jazmine's running speed.
Her biking speed is 4 miles faster than her running speed.	$r + 4 =$ her biking speed
The distances are given, enter them into the	

	Rate	· Time	= Distance
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

Si[illegible] for[illegible]

We divide the distance by the rate in each row, and place the expression in the time column.

Write a word sentence.	Her time plus the time biking is 3 hours.
Translate the sentence to get the equation.	8r + 24r + 4 = 3
Solve.	r(r + 4)(8r + 24r + 4) = 3·r(r + 4)8(r + 4) + 24r = 3r(r + 4)8r + 32 + 24r = 3r2 + 12r32 + 32r = − 320 = (3r + 4)(r − 8)
	(3r + 4) = 0(r − 8) = 0
	r = − 43r = 8
Check. A negative speed does not make sense in this problem, so r = 8 is the solution. Is 8 mph a reasonable running speed? Yes. If Jazmine's running rate is 4, then her biking rate, r + 4, which is 8 + 4 = 12. Run8mph8miles8mph = 1hour Bike12mph24miles12mph = 2hours Total 3 hours.	Jazmine's running speed is 8 mph.

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis’ speed going uphill and his speed going downhill?

Dennis’s uphill speed was 10 mph and his downhill speed was 5 mph.

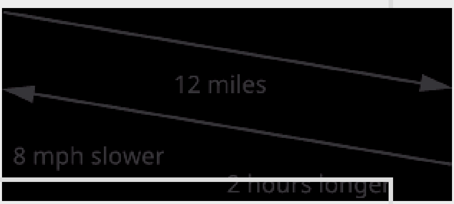
Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Joon’s rate on the country roads is 50 mph.

Once again, we will use the uniform motion formula solved for the variable t.

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.	Let $h =$ Hamilton's downhill speed.
His uphill speed is 8	$h - 8 =$ Hamilton's

miles per hour slower. Enter the rates into the chart.	uphill speed
The distance is the same in both di 12	

	Rate	Time	Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

Since $D=r\cdot t$, we solve for t and get $t=Dr$.

We divide the distance by the rate in each row, and place the expression in the time column.

Write a word sentence about the line.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation.	$12h-8=12h+2$
Solve.	$h(h-8)(12h-8)=h(h-8)(12h+2)12h=12(h-8)+2h(h-8)12h=12h-96+2h2-16h0=2h2-16h-960=2(h2-8h-48)0=2(h-12)(h$

$+4)h-12=0h$
$+4=0h=12h=-4$

Check.
Is 12 mph a reasonable speed for biking downhill? Yes.
Downhill12 mph12 miles12 mph $=1$ hourUphill12 $-8=4$ mph12 miles4 mph $=3$ hours.

The uphill time is 2 hours more that the downhill time.

Hamilton’s downhill speed is 12 mph.

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla’s biking speed. Find Kayla’s biking speed.

Kayla’s biking speed was 15 mph.

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 20 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Victoria jogged 6 mph on the flat trail.

Solve Work Applications

The weekly gossip magazine has a big story about the Princess’ baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running

together?

This is a typical ‘work’ application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete 16 of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete 112 of the job.

We will let *t* be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed 1t of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for t.

A chart will help us organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

Let $t =$ the number of hours needed to complete the job together.

Enter the hours per job for Press #1, Pr... wo...

	Number of hours to complete the job.	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

If a... 6 h... of the job is completed.

Similarly find the part of the job completed/hours for Press #2 and when thet both together.

Write a word sentence.

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

Translate into an equation.

Work completed by

Press #1 + Press #2 = Together

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Solve.

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

Mutiply by the LCD, $12t$

$$12t\left(\frac{1}{6}+\frac{1}{12}\right)=12t\left(\frac{1}{t}\right)$$

Simplify.

$$2t+t=12$$

$$3t=12$$

$$t=4$$

When both presses are running it takes 4 hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it alone.

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint 110 of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint 18 of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a 'work' application. A chart will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did 110 of the job. Alicia did 18 of the job. And together they did 1t of the job.

Let t be the number of hours needed to paint the room together.

Enter the hours per job for Pete, Alicia, and wh

tog

	Number of complete hours to the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

In 1 hour working together, they have completed 1t of the job.

Similarly, find the part of the job completed/hour by Pete and then by Alicia.

Write a word sentence. The work completed by Pete plus the work completed by Alicia equals the total work completed.

Work completed by:

[redacted]

[redacted]

Multiply by the LCD, 40*t*.

[redacted]

Distribute.

[redacted]

Simplify and solve.

[redacted]

We'll write as a mixed number so [redacted] it to hours and minutes.

Remember, 1 hour = 60 minutes.

[redacted]

Multiply, and then round to the [redacted]

It would take Pete and Alica about 4 hours and 27

minutes to paint the room.

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

When the two gardeners work together it takes 2 hours and 24 minutes.

Daria can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

When Daria and her mother work together it takes 2 hours and 6 minutes.

Ra’shon can clean the house in 7 hours. When his sister helps him it takes 3 hours. How long does it take his sister when she cleans the house alone?

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take Ra’shon’s sister to complete the job by herself.

Let s be the number of hours Ra’shon’s sister takes to clean the house alone.

Enter the hours per job for Ra’shon, his

sis

wo

If I

	Number of hours to clean the house	Part of job completed/hour
Ra’shon	7	$\frac{1}{7}$
His sister	s	$\frac{1}{s}$
Together	3	$\frac{1}{3}$

hours, then in 1 hour
17
of the job is completed.

If Ra’shon’s sister takes s hours, then in 1 hour $\frac{1}{s}$ of the job is completed.

Write a word sentence. The part completed by Ra’shon plus the part by his sister equals the amount completed together.

Translate to an equation.

Work completed by Ra’shon	+ His sister	= Together
$\frac{1}{7}$	$+ \quad \frac{1}{s}$	$= \quad \frac{1}{3}$

Solve.

$$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$$

Multiply by the LCD, 21s.

$$21s\left(\frac{1}{7} + \frac{1}{s}\right) = \left(\frac{1}{3}\right)21s$$

$$3s + 21 = 7s$$

Simplify.

$$-4s = -21$$

$$s = \frac{-21}{-4} = \frac{21}{4}$$

Write as a mixed number to convert it to hours and minutes.

$$s = 5\tfrac{1}{4}\text{ hours}$$

There are 60 minutes in 1 hour.

$$s = 5\text{ hours} + \tfrac{1}{4}(60\text{ minutes})$$

$$s = 5\text{ hours} + 15\text{ minutes}$$

It would take Ra’shon’s sister 5 hours and 15 minutes to clean the

house alone.

Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

Kristina can paint the room in 12 hours.

Tracy can lay a slab of concrete in 3 hours, with Jordan's help they can do it in 2 hours. If Jordan works alone, how long will it take?

It will take Jordan 6 hours.

Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Lindsay gets paid $15 per hour at her job. If we let *s* be her salary and *h* be the number of hours she has worked, we could model this situation with the equation

$$s = 15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables *x* and *y*, *y* varies directly with *x* if

$$y = kx, \text{where} k \neq 0$$

The constant *k* is called the constant of variation.

In applications using direct variation, generally we

will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values of y for other values of x.

We'll list the steps here.

Solve direct variation problems.

Write the formula for direct variation. Substitute the given values for the variables. Solve for the constant of variation. Write the equation that relates x and y using the constant of variation.

Now we'll solve an application of direct variation.

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates c and m. ⓑ How many calories would he burn if he ran on

the treadmill for 25 minutes?

ⓐ

The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.

Write the formula for direct variation. [redacted]

We will use c in place of y and m in place of x. [redacted]

Substitute the given values for the variables. [redacted]

Solve for the constant of variation. [redacted]

[redacted]

Write the equation that relates c and m.

[redacted]

Substitute in the constant of variation.

[redacted]

ⓑ

Find c when $m = 25$.

Write the equation that relates c and m.

[redacted]

Substitute the given value for m.

[redacted]

Simplify.

[redacted]

Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

The number of calories, *c*, burned varies directly with the amount of time, *t*, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates *c* and *t*. ⓑ How many calories would he burn if he exercises for 90 minutes?

ⓐ $c = 4.8t$ ⓑ He would burn 432 calories.

The distance a moving body travels, *d*, varies directly with time, *t*, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates *d* and *t*. ⓑ How many miles would it travel in 5 hours?

ⓐ $d = 50t$ ⓑ It would travel 250 miles.

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is y=kx.

Inverse Variation

For any two variables *x* and *y*, *y* varies inversely with *x* if

y=kx,wherek≠0

The constant *k* is called the constant of variation.

The word 'inverse' in inverse variation refers to the multiplicative inverse. The multiplicative inverse of *x* is 1x.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change 'direct' to 'inverse'.

Solve inverse variation problems.

Write the formula for inverse variation. Substitute

the given values for the variables. Solve for the constant of variation. Write the equation that relates x and y using the constant of variation.

The frequency of a guitar string varies inversely with its length. A 26 in.-long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation. ⓑ How many vibrations per second will there be if the string's length is reduced to 20 inches by putting a finger on a fret?

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let f = frequency. L = length
Write the formula for inverse variation.	

[redacted]

We will use f in place of y and L in place of x.

[redacted]

Substitute the given values for the va[redacted]

[redacted]

Solve for the constant of variation

[redacted]

[redacted]

Write the equation that relates f and L.

[redacted]

Substitute the constant of variation

[redacted]

ⓑ

FindfwhenL=20. Write the equation that relatesfandL.f=11,440L Substitute the given value forL.f=11,44020 Simplify.f=572 A20″-guitar string has frequency 572vibrations per

second.

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

ⓐ Write the equation of variation. ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

ⓐ $h = 130t$ ⓑ 123 hours

Xander's new business found that the daily demand for its product was inversely proportional to the price, p. When the price is $5, the demand is 700 units.

ⓐ Write the equation of variation. ⓑ What is the demand if the price is raised to $7?

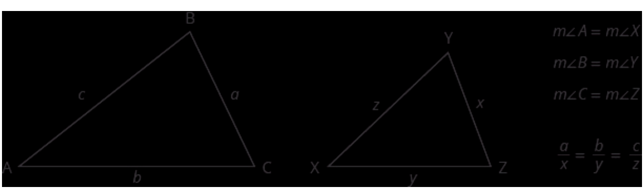
ⓐ $x=3500p$ ⓑ 500 units

Access this online resource for additional instruction and practice with applications of rational expressions

- Applications of Rational Expressions

Key Concepts

- A proportion is an equation of the form $ab=cd$, where $b\neq0, d\neq0$. The proportion is read "*a* is to *b* as *c* is to *d*."
- **Property of Similar Triangles**
  If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



- **Direct Variation**
  - For any two variables *x* and *y*, *y* varies directly with *x* if $y = kx$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve direct variation problems.

    Write the formula for direct variation. Substitute the given values for the variables. Solve for the constant of variation. Write the equation that relates x and y.

- **Inverse Variation**
  - For any two variables *x* and *y*, *y* varies inversely with *x* if $y = kx$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve inverse variation problems.

    Write the formula for inverse variation. Substitute the given values for the variables. Solve for the constant of variation. Write the equation that relates *x* and *y*.

Practice Makes Perfect

Solve Proportions

In the following exercises, solve each proportion.

$x56=78$

$x=49$

$5672=y9$

$98154=-7p$

$p=-11$

$72156=-6q$

$aa+12=47$

$a=16$

$bb-16=119$

$m+9025=m+3015$

$m=60$

$n+104=40-n6$

$2p+48=p+186$

$p=30$

$q-22=2q-718$

In the following exercises, solve.

Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this? ⓑ Has Kevin met his target heart rate?

ⓐ 162 beats per minute ⓑ yes

Jesse’s car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then

home? ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

---

9 ml

A veterinarian prescribed Sunny, a 65-pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

---

159 calories

One 12-ounce can of soda has 150 calories. If Josiah drinks the big 32-ounce size from the local mini-mart, how many calories does he

get?

Kyra is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.3 Canadian. How many Canadian dollars will she get for her trip?

---

325 Canadian dollars

Maurice is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

---

3 cups

Sonya drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

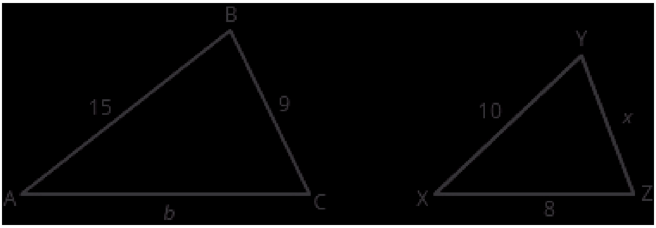
---

4 bags

An oatmeal cookie recipe calls for 12 cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

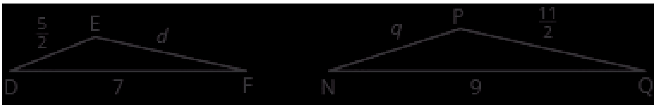
Solve Similar Figure Applications

In the following exercises, the triangles are similar. Find the length of the indicated side.



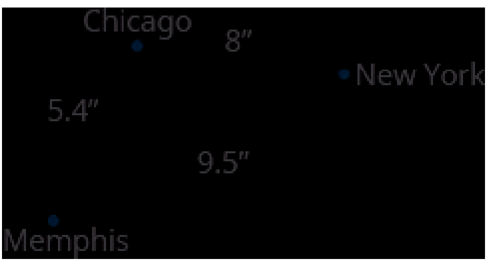
ⓐ side x ⓑ side b

ⓐ 6 ⓑ 12



ⓐ side *d* ⓑ side *q*

In the following exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle. The actual distance from New York to Chicago is 800 miles.



Find the actual distance from New York to Memphis.

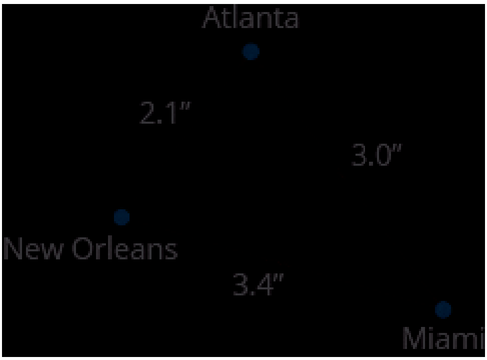
---

950 miles

Find the actual distance from Chicago to

Memphis.

In the following exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle. The actual distance from Atlanta to New Orleans is 420 miles.



Find the actual distance from New Orleans to Miami.

---

680 miles

Find the actual distance from Atlanta to Miami.

In the following exercises, answer each question.

A 2-foot-tall dog casts a 3-foot shadow at the same time a cat casts a one foot shadow. How tall is the cat ?

23 foot (8 in.)

Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven-foot shadow. How long is the windmill's shadow?

247.3 feet

The height of the Statue of Liberty is 305 feet. Nikia, who is standing next to the statue, casts a 6-foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Solve Uniform Motion Applications

In the following exercises, solve the application problem provided.

Mary takes a sightseeing tour on a helicopter

that can fly 450 miles against a 35-mph headwind in the same amount of time it can travel 702 miles with a 35-mph tailwind. Find the speed of the helicopter.

---

160 mph

A private jet can fly 1,210 miles against a 25-mph headwind in the same amount of time it can fly 1694 miles with a 25-mph tailwind. Find the speed of the jet.

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

---

29 mph

Darrin can skateboard 2 miles against a 4-mph wind in the same amount of time he skateboards 6 miles with a 4-mph wind. Find the speed Darrin skateboards with no wind.

Jane spent 2 hours exploring a mountain with a dirt bike. First, she rode 40 miles uphill. After

she reached the peak she rode for 12 miles along the summit. While going uphill, she went 5 mph slower than when she was on the summit. What was her rate along the summit?

---

30 mph

Laney wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Byron wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

---

20 mph

Nancy took a 3-hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had

driven when the weather was good. What was her speed driving in the storm?

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, what was his uphill rate?

---

4 mph

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

---

60 mph

Nathan walked on an asphalt pathway for 12

miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel.

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

---

650 mph

Jim's speedboat can travel 20 miles upstream against a 3-mph current in the same amount of time it travels 22 miles downstream with a 3-mph current speed . Find the speed of the Jim's boat.

Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

---

50 mph

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

When driving the 9-hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

---

50 mph

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Dana enjoys taking her dog for a walk, but sometimes her dog gets away, and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed.

Find her walking speed.

4.2 mph

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Solve Work Applications

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

---

2 hours

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Mia can clean her apartment in 6 hours while her roommate can clean the apartment in 5

hours. If they work together, how long would it take them to clean the apartment?

---

2 hours and 44 minutes

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Josephine can correct her students test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

---

7 hours and 30 minutes

Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take Levi's dad to wash the car by himself?

At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

10 min

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solve Direct Variation Problems

In the following exercises, solve.

If y varies directly as x and $y=14$, when $x=3$. find the equation that relates x and y.

$y=143x$

If a varies directly as b and $a=16$, when $b=4$. find the equation that relates a and b.

If p varies directly as q and $p=9.6$, when $q=3$. find the equation that relates p and q.

$p=3.2q$

If v varies directly as w and $v=8$, when $w=12$.

find the equation that relates v and w.

The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him $50 to buy 20 gallons of gas.

ⓐ Write the equation that relates P and g. ⓑ How much would 33 gallons cost Eric?

ⓐ $P = 2.5g$ ⓑ $82.50

Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates d and v. ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume. ⓑ What is the volume of this liquid if its mass is 128 kilograms?

ⓐ $m = 8v$ ⓑ 16 liters

The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight. ⓑ What weight of watermelon would stretch the spring 6 inches?

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section. ⓑ What load will a beam with a 10-inch diagonal support?

ⓐ $L = 3d2$ ⓑ 300 pounds

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to

the radius. ⓑ What is the area of a personal pizza with a radius 4 inches?

Solve Inverse Variation Problems

In the following exercises, solve.

If y varies inversely with x and $y=5$ when $x=4$, find the equation that relates x and y.

$y=20x$

If p varies inversely with q and $p=2$ when $q=1$, find the equation that relates p and q.

If v varies inversely with w and $v=6$ when $w=12$, find the equation that relates v and w.

$v=3w$

If a varies inversely with b and $a=12$ when $b=13$, find the equation that relates a and b.

In the following exercises, write an inverse variation equation to solve the following problems.

The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds getting 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car's weight. ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

ⓐ $g = 92{,}400w$ ⓑ 16.8 mpg

A car's value varies inversely with its age. Jackie bought a 10-year-old car for $2,400.

ⓐ Write the equation that relates the car's value to its age. ⓑ What will be the value of Jackie's car when it is 15 years old?

The time required to empty a tank varies inversely as the rate of pumping. It took Ada 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute).

ⓐ Write the equation that relates the number of hours to the pump rate. ⓑ How long would it take Ada to pump her basement if she used a pump rated at 400 gpm?

ⓐ $t = 1000r$ ⓑ 2.5 hours

On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency. ⓑ What is the frequency of a 10 inch string?

Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had four cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing. ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

ⓐ $c = 2t$ ⓑ 1 cavity

Boyle's law states that if the temperature of a gas stays constant, then the pressure varies

inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume. ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

The cost of a ride service varies directly with the distance traveled. It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, *c*, with the number of miles, m. ⓑ What would it cost to travel 22 miles with this service?

ⓐ $c = 2.5m$ ⓑ $55

The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, h, with the speed, s. ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

Marisol solves the proportion $144a=94$ by 'cross multiplying,' so her first step looks like $4\cdot144=9\cdot a$. Explain how this differs from the method of solution shown in [link].

Answers will vary.

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $13+14=1t$ can be used to find *t*, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

In your own words, explain the difference between direct variation and inverse variation.

Answers will vary.

Make up an example from your life experience of inverse variation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			
solve uniform motion applications.			
solve work applications.			
solve direct variation problems.			
solve inverse variation problems.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

proportion
: When two rational expressions are equal, the equation relating them is called a proportion.

similar figures
: Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

Simplify Expressions with Roots
By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)2$ ⓑ -92 ⓒ $(-9)3$.
   If you missed this problem, review [link].
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review [link].
3. Simplify: ⓐ $x3·x3$ ⓑ $y2·y2·y2$ ⓒ $z3·z3·z3·z3$.
   If you missed this problem, review [link].

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write n2 and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

132is read "13 squared"169 is called thesquareof

13, since132=16913 is asquare rootof 169

Square and Square Root of a number
Square
Ifn2=m,thenmis thesquareofn.
Square Root
Ifn2=m,thennis asquare rootofm.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, m, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**. This symbol, as well as other radicals to be introduced later, are **grouping symbols**.

We also use the radical sign for the square root of zero. Because 02=0, 0=0. Notice that zero has only one square root.

Square Root Notation
mis read “the square root ofm”.Ifn2 = m,thenn = m,forn ≥ 0.

We know that every positive number has two square roots and the radical sign indicates the positive one. We write 169 = 13. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, − 169 = − 13.

Simplify: ⓐ 144 ⓑ − 289.

ⓐ
144Since122 = 144.12

ⓑ
− 289Since172 = 289and the negative is infront of the radical sign. − 17

Simplify: ⓐ -64 ⓑ 225.

ⓐ -8 ⓑ 15

Simplify: ⓐ 100 ⓑ -121.

ⓐ 10 ⓑ -11

Can we simplify -49? Is there a number whose square is -49?

$()2=-49$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to -49. The square root of a negative number is not a real number.

Simplify: ⓐ -196 ⓑ -64.

ⓐ
-196There is no real number whose square is -196. $-196$is not a real number.

ⓑ
-64The negative is in front of the radical. -8

Simplify: ⓐ -169 ⓑ -81.

ⓐ not a real number ⓑ -9

Simplify: ⓐ -49 ⓑ -121.

ⓐ -7 ⓑ not a real number

So far we have only talked about squares and square roots. Let’s now extend our work to include higher powers and higher roots.

Let’s review some vocabulary first.

We write:We say:n2nsquaredn3ncubedn4nto the fourth powern5nto the fifth power

The terms ‘squared’ and ‘cubed’ come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See [link].

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We’ll copy the row with the powers of -2 to help you see this.

n		n^3		n^5
–2	4	–8	16	–32

power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

nth Root of a Number
If $b^n = a$, then b is an nth root of a. The principal nth root of a is written $\sqrt[n]{a}$. n is called the index of the radical.

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to [link] to help find higher roots.
$\sqrt[3]{64} = 4 \quad \sqrt[4]{81} = 3 \quad \sqrt[5]{-32} = -2$ $43=6434=81(-2)5=-32643=4814=3-325=-2$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$
When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

We will apply these properties in the next two examples.

Simplify: ⓐ 643 ⓑ 814 ⓒ 325.

ⓐ

643Since43 = 64.4

ⓑ

814Since(3)4 = 81.3

ⓒ

325Since(2)5 = 32.2

Simplify: ⓐ 273 ⓑ 2564 ⓒ 2435.

ⓐ 3 ⓑ 4 ⓒ 3

Simplify: ⓐ 10003 ⓑ 164 ⓒ 2435.

ⓐ 10 ⓑ 2 ⓒ 3

In this example be alert for the negative signs as well as even and odd powers.

Simplify: ⓐ −1253 ⓑ 164 ⓒ −2435.

ⓐ

−1253Since(−5)3=−125.−5

ⓑ

−164Think,(?)4=−16.No real number raisedto the fourth power is negative.Not a real number.

ⓒ

−2435Since(−3)5=−243.−3

Simplify: ⓐ -273 ⓑ -2564 ⓒ -325.

ⓐ -3 ⓑ not real ⓒ -2

Simplify: ⓐ -2163 ⓑ -814 ⓒ -10245.

ⓐ -6 ⓑ not real ⓒ -4

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $4=2$, what is the value of 21 or 503? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

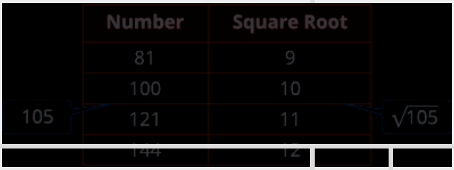
To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of 11, we see 11 is between perfect square numbers 9 and 16, *closer* to

9. Its square root then will be between 3 and 4, but closer to 3.

Similarly, to estimate 913, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

Estimate each root between two consecutive whole numbers: ⓐ 105 ⓑ 433.
ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

[redacted]



Locate 105 between two consecutive perfect squares.

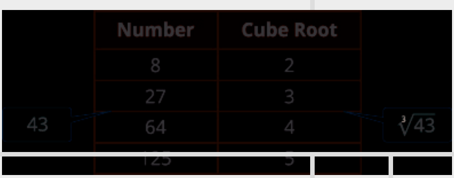
$100 < 105 < 121$

105 is between their square roots.

$10 < \sqrt{105} < 11$

ⓑ Similarly we locate 43 between two perfect cube numbers.

$\sqrt[3]{43}$



Locate 43 between two consecutive perfect cubes.

$27 < 43 < 64$

433 is between their cube roots.

[redacted]

Estimate each root between two consecutive whole numbers:

ⓐ 38 ⓑ 933

ⓐ $6 < 38 < 7$
ⓑ $4 < 933 < 5$

Estimate each root between two consecutive whole numbers:

ⓐ 84 ⓑ 1523

ⓐ $9 < 84 < 10$
ⓑ $5 < 1523 < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the x key on your calculator. To find a cube root, or any root with higher index, you will use the xy key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is ≈ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that
$5 \approx 2.236067978$rounded to two decimal places is$5 \approx 2.24934 \approx 3.105422799$rounded to two decimal places is$934 \approx 3.11$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:
$(2.236067978)2 = 5.000000002(2.24)2 = 5.0176(3.105422$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

Round to two decimal places: ⓐ 17 ⓑ 493 ⓒ 514.

ⓐ
17Use the calculator square root key.4.123105626…Round to two decimal places.4.1217 $\approx$ 4.12

ⓑ
493Use the calculatorxykey.3.659305710…
Round to two decimal places.3.66493 $\approx$ 3.66

ⓒ
514Use the calculatorxykey.2.6723451177…
Round to two decimal places.2.67514 $\approx$ 2.67

Round to two decimal places:

ⓐ 11 ⓑ 713 ⓒ 1274.

ⓐ ≈ 3.32 ⓑ ≈ 4.14
ⓒ ≈ 3.36

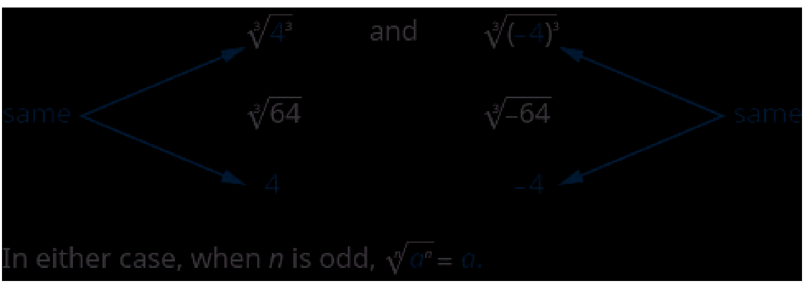
Round to two decimal places:

ⓐ 13 ⓑ 843 ⓒ 984.

ⓐ ≈ 3.61 ⓑ ≈ 4.38
ⓒ ≈ 3.15

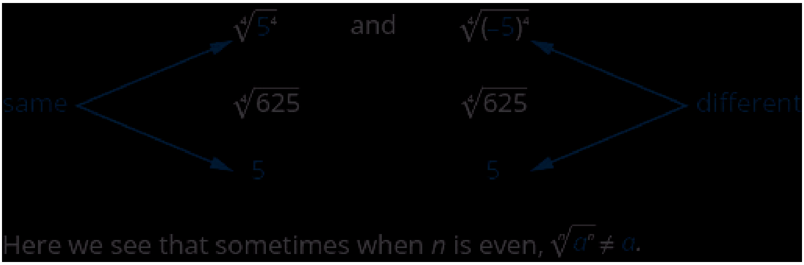
Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,



But what about an even root? We want the principal root, so $6254 = 5$.

But notice,



How can we make sure the fourth root of -5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even ann $= |a|$. This guarantees the principal root is positive.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

when the indexnis oddann $=$ awhen the indexnis evenann $= |a|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Simplify: ⓐ x2 ⓑ n33 ⓒ p44 ⓓ y55.

ⓐ We use the absolute value to be sure to get the positive root.

x2Since the indexnis even,ann = |a|.|x|

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

m33Since the indexnis odd,ann = a.m

ⓒ

p44Since the indexnis evenann = |a|.|p|

ⓓ

y55Since the indexnis odd,ann = a.y

Simplify: ⓐ b2 ⓑ w33 ⓒ m44 ⓓ q55.
ⓐ |b| ⓑ *w* ⓒ |m| ⓓ *q*

Simplify: ⓐ y2 ⓑ p33 ⓒ z44 ⓓ q55.
ⓐ |y| ⓑ *p* ⓒ |z| ⓓ *q*

What about square roots of higher powers of variables? The Power Property of Exponents says (am)n = am·n. So if we square *am*, the exponent will become 2*m*.
(am)2 = a2m

Looking now at the square root,

a2mSince(am)2 = a2m.(am)2Sincenis evenann = |a|.|am|Soa2m = |am|.

We apply this concept in the next example.

Simplify: ⓐ x6 ⓑ y16.

ⓐ
x6Since(x3)2 = x6.(x3)2Since the indexnis evenan = |a|.|x3|

ⓑ
y16Since(y8)2 = y16.(y8)2Since the indexnis evenann = |a|.y8In this case the absolute value sign isnot needed asy8is positive.

Simplify: ⓐ y18 ⓑ z12.

ⓐ |y9| ⓑ z6

Simplify: ⓐ m4 ⓑ b10.

ⓐ m2 ⓑ |b5|

The next example uses the same idea for highter roots.

Simplify: ⓐ y183 ⓑ z84.

ⓐ
y183Since(y6)3=y18.(y6)33Sincenis odd,ann=a.y6

ⓑ
z84Since(z2)4=z8.(z2)44Sincez2is positive, we do not need anz2absolute value sign.

Simplify: ⓐ u124 ⓑ v153.

ⓐ |u3| ⓑ v5

Simplify: ⓐ c205 ⓑ d246

ⓐ c4 ⓑ d4

In the next example, we now have a coefficient in front of the variable. The concept a2m=|am| works in much the same way.
16r22=4|r11|because(4r11)2=16r22.

But notice 25u8=5u4 and no absolute value sign is needed as *u*4 is always positive.

Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.
ⓐ $\sqrt{16n^2}$ Since $(4n)^2 = 16n^2$. $\sqrt{(4n)^2}$ Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$. $4\lvert n \rvert$ ⓑ $-\sqrt{81c^2}$ Since $(9c)^2 = 81c^2$. $-\sqrt{(9c)^2}$ Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$. $-9\lvert c \rvert$

Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.
ⓐ $8\lvert x \rvert$ ⓑ $-10\lvert p \rvert$

Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.
ⓐ $13\lvert y \rvert$ ⓑ $-11\lvert y \rvert$

This example just takes the idea farther as it has roots of higher index.

Simplify: ⓐ 64p63 ⓑ 16q124.

ⓐ
64p63Rewrite64p6as(4p2)3.(4p2)33Take the cube root.4p2

ⓑ
16q124Rewrite the radicand as a fourth power.(2q3)44Take the fourth root.2|q3|

Simplify: ⓐ 27x273 ⓑ 81q284.

ⓐ 3x9 ⓑ 3|q7|

Simplify: ⓐ 125q93 ⓑ 243q255.

ⓐ 5p3 ⓑ 3q5

The next examples have two variables.

Simplify: ⓐ 36x2y2 ⓑ 121a6b8 ⓒ 64p63q93.

ⓐ
36x2y2Since(6xy)2=36x2y2(6xy)2Take the square root.6|xy|

ⓑ
121a6b8Since(11a3b4)2=121a6b8(11a3b4)2Take the square root.11|a3|b4

ⓒ
64p63q93Since(4p21q3)3=64p63q9(4p21q3)33Take the cube root.4p21q3

Simplify: ⓐ 100a2b2 ⓑ 144p12q20 ⓒ 8x30y123

ⓐ 10|ab| ⓑ 12p6q10
ⓒ 2x10y4

Simplify: ⓐ 225m2n2 ⓑ 169x10y14 ⓒ 27w36z153

ⓐ 15|mn| ⓑ 13|x5y7|
ⓒ 3w12z5

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

Key Concepts

- **Square Root Notation**

- m is read ‘the square root of *m*’
- If $n2 = m$, then n = m, for n ≥ 0.

$\sqrt{m}$

- The square root of *m*, m, is a positive number whose square is *m*.

- ***n*th Root of a Number**
  - If bn = a, then *b* is an *nth* root of *a*.
  - The principal *nth* root of *a* is written an.
  - *n* is called the *index* of the radical.

- **Properties of an**
  - When *n* is an even number and
    - a ≥ 0, then an is a real number
    - a < 0, then an is not a real number
  - When *n* is an odd number, an is a real number for all values of *a*.

- **Simplifying Odd and Even Roots**
  - For any integer n ≥ 2,
    - when *n* is odd ann = a
    - when *n* is even ann = |a|

- ○ We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

ⓐ 64 ⓑ -81

ⓐ 8 ⓑ -9

ⓐ 169 ⓑ -100

ⓐ 196 ⓑ -1

ⓐ 14 ⓑ -1

ⓐ 144 ⓑ -121

ⓐ 49 ⓑ -0.01

ⓐ 23 ⓑ -0.1

ⓐ 64121 ⓑ -0.16

ⓐ -121 ⓑ -289

ⓐ not real number ⓑ -17

ⓐ -400 ⓑ -36

ⓐ -225 ⓑ -9

ⓐ -15 ⓑ not real number

ⓐ -49 ⓑ -256

ⓐ 2163 ⓑ 2564

ⓐ 6 ⓑ 4

ⓐ 273 ⓑ 164 ⓒ 2435

ⓐ 5123 ⓑ 814 ⓒ 15

ⓐ 8 ⓑ 3 ⓒ 1

ⓐ 1253 ⓑ 12964 ⓒ 10245

ⓐ −83 ⓑ −814 ⓒ −325

ⓐ −2 ⓑ not real ⓒ −2

ⓐ −643 ⓑ −164 ⓒ −2435

ⓐ −1253 ⓑ −12964 ⓒ −10245

ⓐ −5 ⓑ not real ⓒ −4

ⓐ −5123 ⓑ −814 ⓒ −15

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

ⓐ 70 ⓑ 713

ⓐ $8 < 70 < 9$
ⓑ $4 < 713 < 5$

ⓐ 55 ⓑ 1193

ⓐ 200 ⓑ 1373

ⓐ $14 < 200 < 15$
ⓑ $5 < 1373 < 6$

ⓐ 172 ⓑ 2003

In the following exercises, approximate each root and round to two decimal places.

ⓐ 19 ⓑ 893 ⓒ 974

ⓐ 4.36 ⓑ ≈ 4.46
ⓒ ≈ 3.14

ⓐ 21 ⓑ 933 ⓒ 1014

ⓐ 53 ⓑ 1473 ⓒ 4524

ⓐ 7.28 ⓑ ≈5.28
ⓒ ≈4.61

ⓐ 47 ⓑ 1633 ⓒ 5274

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

ⓐ u55 ⓑ v88

ⓐ *u* ⓑ |v|

ⓐ a33 ⓑ b99

ⓐ y44 ⓑ m77

ⓐ |y| ⓑ m

ⓐ k88 ⓑ p66

ⓐ x6 ⓑ y16

ⓐ $|x3|$ ⓑ $y8$

ⓐ $a14$ ⓑ $w24$

ⓐ $x24$ ⓑ $y22$

ⓐ $x12$ ⓑ $|y11|$

ⓐ $a12$ ⓑ $b26$

ⓐ $x93$ ⓑ $y124$

ⓐ $x3$ ⓑ $|y3|$

ⓐ $a105$ ⓑ $b273$

ⓐ $m84$ ⓑ $n205$

ⓐ $m2$ ⓑ $n4$

ⓐ $r126$ ⓑ $s303$

ⓐ 49x2 ⓑ −81x18

ⓐ 7|x| ⓑ −9|x9|

ⓐ 100y2 ⓑ −100m32

ⓐ 121m20 ⓑ −64a2

ⓐ 11m10 ⓑ −8|a|

ⓐ 81x36 ⓑ −25x2

ⓐ 16x84 ⓑ 64y126

ⓐ 2x2 ⓑ 2y2

ⓐ −8c93 ⓑ 125d153

ⓐ 216a63 ⓑ 32b205

ⓐ 6a2 ⓑ 2b4

ⓐ 128r147 ⓑ 81s244

ⓐ 144x2y2 ⓑ 169w8y10 ⓒ 8a51b63

ⓐ 12|xy| ⓑ 13w4|y5|
ⓒ 2a17b2

ⓐ 196a2b2 ⓑ 81p24q6 ⓒ 27p45q93

ⓐ 121a2b2 ⓑ 9c8d12 ⓒ 64x15y663

ⓐ 11|ab| ⓑ 3c4d6
ⓒ 4x5y22

ⓐ 225x2y2z2 ⓑ 36r6s20 ⓒ 125y18z273

Writing Exercises

Why is there no real number equal to -64?

Answers will vary.

What is the difference between 92 and 9?

Explain what is meant by the nth root of a number.

Answers will vary.

Explain the difference of finding the nth root of a number when the index is even compared to when the index is odd.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

ⓑ If most of your checks were:

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

Simplify Radical Expressions

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Before you get started, take this readiness quiz.

Simplify: x9x4.
If you missed this problem, review [link].

x5

Simplify: y3y11.
If you missed this problem, review [link].

1y8

Simplify: (n2)6.
If you missed this problem, review [link].

n12

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, an, is considered simplified if it has no factors of mn. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Simplified Radical Expression
For real numbers *a* and *m*, and $n \geq 2$,

anis considered simplified ifahas no factors ofmn

For example, 5 is considered simplified because there are no perfect square factors in 5. But 12 is not simplified because 12 has a perfect square factor of 4.

Similarly, 43 is simplified because there are no perfect cube factors in 4. But 243 is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that (ab)n = anbn. The corresponding of **Product Property of Roots** says that abn = an·bn.

Product Property of *n*th Roots
If an and bn are real numbers, and n ≥ 2 is an integer, then
abn = an·bnandan·bn = abn

We use the Product Property of Roots to remove all perfect square factors from a square root.

Simplify Square Roots Using the Product Property of Roots

Simplify: 98.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.		$7\sqrt{2}$

Simplify: 48.

43

Simplify: 45.
35

Notice in the previous example that the simplified form of 98 is 72, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression 72 is very different from 27.

Simplify a radical expression using the Product Property.
Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

Simplify: ⓐ 500 ⓑ 163 ⓒ 2434.

ⓐ

	500
Rewrite the radicand as a product using the largest perfect square factor.	100·5
Rewrite the radical as the product of two radicals.	100·5
Simplify.	105

ⓑ

	163
Rewrite the radicand as a product using the greatest perfect cube factor. 23 = 8	8·23
Rewrite the radical as the product of two radicals.	83·23
Simplify.	223

©

	2434
Rewrite the radicand as a product using the greatest perfect fourth power factor. 34 = 81	81·34
Rewrite the radical as the product of two radicals.	814·34
Simplify.	334

Simplify: ⓐ 288 ⓑ 813 ⓒ 644.

ⓐ 122 ⓑ 333 ⓒ 244

Simplify: ⓐ 432 ⓑ 6253 ⓒ 7294.

ⓐ 123 ⓑ 553 ⓒ 394

The next example is much like the previous examples, but with variables. Don’t forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Simplify: ⓐ x3 ⓑ x43 ⓒ x74.

ⓐ

	x3
Rewrite the radicand as a product using the largest perfect square factor.	x2·x
Rewrite the radical as the product of two radicals.	x2·x
Simplify.	|x|x

ⓑ

	x43
Rewrite the radicand as a product using the largest perfect cube factor.	x3·x3.
Rewrite the radical as the product of two radicals.	x33·x3
Simplify.	xx3

ⓒ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x\rvert \sqrt[4]{x^3}$

Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

ⓐ $b^2\sqrt{b}$ ⓑ $\lvert y\rvert \sqrt[4]{y^2}$ ⓒ $z\sqrt[3]{z^2}$

Simplify: ⓐ p9 ⓑ y85 ⓒ q136

ⓐ p4p ⓑ yy35
ⓒ q2q6

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

Simplify: ⓐ 72n7 ⓑ 24x73 ⓒ 80y144.

ⓐ

	72n7
Rewrite the radicand as a product using the largest perfect square factor.	36n6·2n

Rewrite the radical as the product of two radicals.	36n6·2n
Simplify.	6|n3|2n

ⓑ

	24x73
Rewrite the radicand as a product using perfect cube factors.	8x6·3x3
Rewrite the radical as the product of two radicals.	8x63·3x3
Rewrite the first radicand as (2x2)3.	(2x2)33·3x3
Simplify.	2x23x3

ⓒ

	80y144
Rewrite the radicand as a product using perfect fourth power factors.	16y12·5y24
Rewrite the radical as the product of two radicals.	16y124·5y24
Rewrite the first radicand as (2y3)4.	(2y3)44·5y24
Simplify.	2|y3|5y24

Simplify: ⓐ 32y5 ⓑ 54p103 ⓒ 64q104.

ⓐ 4y22y ⓑ 3p32p3
ⓒ 2q24q24

Simplify: ⓐ 75a9 ⓑ 128m113 ⓒ 162n74.

ⓐ 5a43a ⓑ 4m32m23

ⓒ 3|n|2n34

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Simplify: ⓐ 63u3v5 ⓑ 40x4y53 ⓒ 48x4y74.

ⓐ

	63u3v5
Rewrite the radicand as a product using the largest perfect square factor.	9u2v4·7uv
Rewrite the radical as the product of two radicals.	9u2v4·7uv
Rewrite the first radicand as (3uv2)2.	(3uv2)2·7uv

Simplify.	3|u|v27uv

ⓑ

	40x4y53
Rewrite the radicand as a product using the largest perfect cube factor.	8x3y3·5xy23
Rewrite the radical as the product of two radicals.	8x3y33·5xy23
Rewrite the first radicand as (2xy)3.	(2xy)33·5xy23
Simplify.	2xy5xy23

ⓒ

	48x4y74
Rewrite the radicand as a product using the largest perfect fourth power factor.	16x4y4·3y34
Rewrite the radical as the product of two radicals.	16x4y44·3y34
Rewrite the first radicand as (2xy)4.	(2xy)44·3y34
Simplify.	2|xy|3y34

Simplify: ⓐ 98a7b5 ⓑ 56x5y43 ⓒ 32x5y84.

ⓐ 7|a3|b22ab
ⓑ 2xy7x2y3 ⓒ 2|x|y22x4

Simplify: ⓐ 180m9n11 ⓑ 72x6y53 ⓒ 80x7y44.

ⓐ 6m4|n5|5mn
ⓑ 2x2y9y23 ⓒ 2|xy|5x34

Simplify: ⓐ −273 ⓑ −164.

ⓐ

	−273
Rewrite the radicand as a product using perfect cube factors.	(−3)33
Take the cube root.	−3

ⓑ

There is no real number n where $n4 = -16$.	-164 Not a real number.

Simplify: ⓐ -643 ⓑ -814.

ⓐ -4 ⓑ no real number

Simplify: ⓐ -6253 ⓑ -3244.

ⓐ -553 ⓑ no real number

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one

term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Simplify: ⓐ $3+32$ ⓑ $4-482$.

ⓐ

	$3+32$
Rewrite the radicand as a product using the largest perfect square factor.	$3+16\cdot2$
Rewrite the radical as the product of two radicals.	$3+16\cdot2$
Simplify.	$3+42$

The terms cannot be added as one has a radical and the other does not. Trying to add

an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

	$4-482$
Rewrite the radicand as a product using the largest perfect square factor.	$4-16\cdot 32$
Rewrite the radical as the product of two radicals.	$4-16\cdot 32$
Simplify.	$4-432$
Factor the common factor from the numerator.	$4(1-3)2$
Remove the common factor, 2, from the numerator and denominator.	$2\cdot 2(1-3)2$
Simplify.	$2(1-3)$

Simplify: ⓐ $5+75$ ⓑ $10-755$

ⓐ $5+53$ ⓑ $2-3$

Simplify: ⓐ $2+98$ ⓑ $6-453$

ⓐ $2+72$ ⓑ $2-5$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Simplify: ⓐ 4580 ⓑ 16543 ⓒ 5804.

ⓐ

	4580
Simplify inside the radical first.	
Rewrite showing the common factors of the numerator and denominator.	5·95·16
Simplify the fraction by removing common factors.	916
Simplify. Note (34)2 = 916.	34

ⓑ

Simplify inside the radical first.	16543
Rewrite showing the common factors of the numerator and denominator.	2·82·273
Simplify the fraction by removing common factors.	8273
Simplify. Note (23)3 = 827.	23

©

Simplify inside the radical first.	5804
Rewrite showing the common factors of the numerator and denominator.	5·15·164
Simplify the fraction by removing common	1164

factors.

Simplify. Note 12

(12)4 = 116.

Simplify: ⓐ 7548 ⓑ 542503 ⓒ 321624.

ⓐ 54 ⓑ 35 ⓒ 23

Simplify: ⓐ 98162 ⓑ 243753 ⓒ 43244.

ⓐ 79 ⓑ 25 ⓒ 13

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, aman = am − n,a ≠ 0

Simplify: ⓐ m6m4 ⓑ a8a53 ⓒ a10a24.

ⓐ

	m6m4
Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.	m2
Simplify.	|m|

ⓑ

	a8a53
Use the Quotient Property of exponents	a33

to simplify the fraction under the radical first.	
Simplify.	a

ⓒ

	a10a24
Use the Quotient Property of exponents to simplify the fraction under the radical first.	a84
Rewrite the radicand using perfect fourth power factors.	(a2)44
Simplify.	a2

Simplify: ⓐ a8a6 ⓑ x7x34 ⓒ y17y54.

ⓐ |a| ⓑ |x| ⓒ y3

Simplify: ⓐ x14x10 ⓑ m13m73 ⓒ n12n25.

ⓐ x2 ⓑ m2 ⓒ n2

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

(ab)m=ambm,b≠0

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

Quotient Property of Radical Expressions

If an and bn are real numbers,b≠0, and for any integer n≥2 then,

abn=anbnandanbn=abn

How to Simplify the Quotient of Radical Expressions

Simplify: 27m3196.

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3\lvert m\rvert\sqrt{3m}}{14}$

Simplify: 24p349.

2|p|6p7

Simplify: 48x5100.

2x23x5

Simplify a square root using the Quotient Property.

Simplify the fraction in the radicand, if possible. Use the Quotient Property to rewrite the radical as the quotient of two radicals. Simplify the radicals in the numerator and the denominator.

Simplify: ⓐ 45x5y4 ⓑ 24x7y33 ⓒ 48x10y84.

ⓐ

45x5y4

We cannot simplify the 45x5y4

fraction in the radicand. Rewrite using the Quotient Property.	
Simplify the radicals in the numerator and the denominator.	9x4·5xy2
Simplify.	3x25xy2

ⓑ

	24x7y33
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	24x73y33
Rewrite each radicand as a product using perfect cube factors.	8x6·3x3y33
Rewrite the numerator as the product of two radicals.	(2x2)33·3x3y33
Simplify.	2x23x3y

ⓒ

	48x10y84
The fraction in the radicand cannot be simplified.	48x104y84
Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.	16x8·3x24y84
Rewrite the numerator as the product of two radicals.	(2x2)44·3x24(y2)44
Simplify.	2x23x24y2

Simplify: ⓐ 80m3n6 ⓑ 108c10d63 ⓒ 80x10y44.

ⓐ 4|m|5m|n3| ⓑ 3c34c3d2
ⓒ 2x25x24|y|

Simplify: ⓐ 54u7v8 ⓑ 40r3s63 ⓒ 162m14n124.

ⓐ 3u36uv4 ⓑ 2r53s2
ⓒ 3|m3|2m24|n3|

Be sure to simplify the fraction in the radicand first, if possible.

Simplify: ⓐ 18p5q732pq2 ⓑ 16x5y754x2y23 ⓒ 5a8b680a3b24.

ⓐ

	18p5q732pq2
Simplify the fraction in the radicand, if possible.	9p4q516
Rewrite using the Quotient Property.	9p4q516
Simplify the radicals in the numerator and the denominator.	9p4q4·q4
Simplify.	3p2q2q4

ⓑ

	16x5y754x2y23
Simplify the fraction in the radicand, if possible.	8x3y5273
Rewrite using the Quotient Property.	8x3y53273
Simplify the radicals in the numerator and the denominator.	8x3y33·y23273
Simplify.	2xyy233

ⓒ

	$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[4]{\frac{a^5b^4}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$
Simplify.	$\frac{\lvert ab\rvert\sqrt[4]{a}}{2}$

Simplify: ⓐ $\sqrt{\frac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

ⓐ $\frac{5\lvert y\rvert\sqrt{x}}{6}$ ⓑ $\frac{2xy\sqrt[3]{y^2}}{3}$
ⓒ $\frac{\lvert ab\rvert\sqrt[4]{a}}{2}$

Simplify: ⓐ 48m7n2100m5n8 ⓑ 54x7y5250x2y23 ⓒ 32a9b7162a3b34.

ⓐ 2|m|35|n3| ⓑ 3xyx235
ⓒ 2|ab|a243

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

Simplify: ⓐ 48a73a ⓑ −108323 ⓒ 96x743x24.

ⓐ

	48a73a
The denominator cannot be simplified, so use the Quotient Property to write as one radical.	48a73a
Simplify the fraction under the radical.	16a6
Simplify.	4|a3|

ⓑ

	−108323
The denominator cannot be simplified, so use the Quotient Property to write as one radical.	−10823
Simplify the fraction under the radical.	−543
Rewrite the radicand as a product using perfect cube factors.	(−3)3·23
Rewrite the radical as	(−3)33·23

the product of two radicals.	
Simplify.	-323

©

	96x743x24
The denominator cannot be simplified, so use the Quotient Property to write as one radical.	96x73x24
Simplify the fraction under the radical.	32x54
Rewrite the radicand as a product using perfect fourth power factors.	16x44·2x4
Rewrite the radical as the product of two radicals.	(2x)44·2x4
Simplify.	2|x|2x4

Simplify: ⓐ 98z52z ⓑ −500323 ⓒ 486m1143m54.

ⓐ 7z2 ⓑ −523
ⓒ 3|m|2m24

Simplify: ⓐ 128m92m ⓑ −192333 ⓒ 324n742n34.

ⓐ 8m4 ⓑ −4 ⓒ 3|n|24

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

Key Concepts

- **Simplified Radical Expression**
    - For real numbers *a*, *m* and $n \geq 2$
      an is considered simplified if *a* has no factors of mn
- **Product Property of n^{th} Roots**
    - For any real numbers, an and bn, and for any integer $n \geq 2$
      $abn = an \cdot bn$ and $an \cdot bn = abn$
- **How to simplify a radical expression using the Product Property**

  Find the largest factor in the radicand that is a perfect power of the index.
  Rewrite the radicand as a product of two factors, using that factor. Use the product rule to rewrite the radical as the product of two radicals. Simplify the root of the perfect power.
- **Quotient Property of Radical Expressions**
    - If an and bn are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
      $abn = anbn$ and $anbn = abn$

- **How to simplify a radical expression using the Quotient Property.**

  Simplify the fraction in the radicand, if possible. Use the Quotient Property to rewrite the radical as the quotient of two radicals. Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

27

33

80

125

55

96

147

73

450

800

202

675

ⓐ 324 ⓑ 645

ⓐ 224 ⓑ 225

ⓐ 6253 ⓑ 1286

ⓐ 644 ⓑ 2563

ⓐ 244 ⓑ 443

ⓐ 31254 ⓑ 813

In the following exercises, simplify using absolute value signs as needed.

ⓐ y11 ⓑ r53 ⓒ s104

ⓐ | y5 |y ⓑ rr23 ⓒ s2s24

ⓐ m13 ⓑ u75 ⓒ v116

ⓐ n21 ⓑ q83 ⓒ n108

ⓐ n10n ⓑ q2q23
ⓒ |n|n28

ⓐ r25 ⓑ p85 ⓒ m54

ⓐ 125r13 ⓑ 108x53 ⓒ 48y64

ⓐ 5r65r ⓑ 3x4x23
ⓒ 2|y|3y24

ⓐ 80s15 ⓑ 96a75 ⓒ 128b76

ⓐ 242m23 ⓑ 405m104 ⓒ 160n85

ⓐ 11|m11|2m ⓑ 3m25m24 ⓒ 2n5n35

ⓐ 175n13 ⓑ 512p55 ⓒ 324q74

ⓐ 147m7n11 ⓑ 48x6y73 ⓒ 32x5y44

ⓐ 7|m3n5|3mn ⓑ 2x2y26y3 ⓒ 2|xy|2x4

ⓐ 96r3s3 ⓑ 80x7y63 ⓒ 80x8y94

ⓐ 192q3r7 ⓑ 54m9n103 ⓒ 81a9b84

ⓐ 8|qr3|3qr ⓑ 3m3n32n3 ⓒ 3a2b2a4

ⓐ 150m9n3 ⓑ 81p7q83 ⓒ 162c11d124

ⓐ -8643 ⓑ -2564

ⓐ -643 ⓑ not real

ⓐ -4865 ⓑ -646

ⓐ -325 ⓑ -18

ⓐ -2 ⓑ not real

ⓐ -83 ⓑ -164

ⓐ $5+12$ ⓑ $10-242$

ⓐ $5+23$ ⓑ $5-6$

ⓐ $8+96$ ⓑ $8-804$

ⓐ $1+45$ ⓑ $3+903$

ⓐ $1+35$ ⓑ $1+10$

ⓐ 3+125 ⓑ 15+755

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

ⓐ 4580 ⓑ 8273 ⓒ 1814

ⓐ 34 ⓑ 23 ⓒ 13

ⓐ 7298 ⓑ 24813 ⓒ 6964

ⓐ 10036 ⓑ 813753 ⓒ 12564

ⓐ 53 ⓑ 35 ⓒ 14

ⓐ 12116 ⓑ 162503 ⓒ 321624

ⓐ x10x6 ⓑ p11p23 ⓒ q17q134

ⓐ x2 ⓑ p3 ⓒ |q|

ⓐ p20p10 ⓑ d12d75 ⓒ m12m48

ⓐ y4y8 ⓑ u21u115 ⓒ v30v126

ⓐ 1y2 ⓑ u2 ⓒ |v3|

ⓐ q8q14 ⓑ r14r53 ⓒ c21c94

96x7121

4|x3|6x11

108y449

300m564

5m23m4

125n7169

98r5100

7r22r10

180s10144

28q6225

2|q3|715

150r3256

ⓐ 75r9s8 ⓑ 54a8b33 ⓒ 64c5d44

ⓐ 5r43rs4 ⓑ 3a22a23b
ⓒ 2|c|4c4|d|

ⓐ 72x5y6 ⓑ 96r11s55 ⓒ 128u7v126

ⓐ 28p7q2 ⓑ 81s8t33 ⓒ 64p15q124

ⓐ 2|p3|7p|q| ⓑ 3s23s23t
ⓒ 2|p3|4p34|q3|

ⓐ 45r3s10 ⓑ 625u10v33 ⓒ 729c21d84

ⓐ 32x5y318x3y ⓑ 5x6y940x5y33 ⓒ 5a8b680a3b24

ⓐ 4|xy|3 ⓑ y2x32 ⓒ |ab|a42

ⓐ 75r6s848rs4 ⓑ 24x8y481x2y3 ⓒ 32m9n2162mn24

ⓐ 27p2q108p4q3 ⓑ 16c5d7250c2d23 ⓒ 2m9n7128m3n6

ⓐ 12|pq| ⓑ 2cdd235
ⓒ |mn|2

ⓐ 50r5s2128r2s6 ⓑ 24m9n7375m4n3 ⓒ 81m2n8256m1n24

ⓐ 45p95q2 ⓑ 64424 ⓒ 128x852x25

ⓐ 3p4p|q| ⓑ 224
ⓒ 2x2x5

ⓐ 80q55q ⓑ −625353 ⓒ 80m745m4

ⓐ 50m72m ⓑ 125023 ⓒ 486y92y34

ⓐ 5|m3| ⓑ 553
ⓒ 3|y|3y24

ⓐ 72n112n ⓑ 16263 ⓒ 160r105r34

Writing Exercises

Explain why x4 = x2. Then explain why x16 = x8.

Answers will vary.

Explain why 7 + 9 is not equal to 7 + 9.

Explain how you know that x105 = x2.

Answers will vary.

Explain why −644 is not a real number but −643 is.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Simplify Rational Exponents

By the end of this section, you will be able to:

- Simplify expressions with a1n
- Simplify expressions with amn
- Use the properties of exponents to simplify expressions with rational exponents

Before you get started, take this readiness quiz.

Add: 715 + 512.
If you missed this problem, review [link].

5360

Simplify: (4x2y5)3.
If you missed this problem, review [link].

64x6y15

Simplify: $5-3$.
If you missed this problem, review [link].

1125

Simplify Expressions with a1n

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(am)n=am\cdot n$ when *m* and *n* are whole numbers. Let’s assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8p)3=8$. We will use the Power Property of Exponents to find the value of *p*.

$(8p)3=8$Multiply the exponents on the left.$83p=8$Write the exponent 1 on the right.$83p=81$Since the bases are the same, the

exponents must be equal.3p=1Solve forp.p=13

So (813)3=8. But we know also (83)3=8. Then it must be that 813=83.

This same logic can be used for any positive integer exponent *n* to show that a1n=an.

Rational Exponent a1n
If an is a real number and n≥2, then
a1n=an

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Write as a radical expression: ⓐ x12 ⓑ y13 ⓒ z14.

We want to write each expression in the form an.

ⓐ

	x12
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	x

ⓑ

	y13
The denominator of the exponent is 3, so the index is 3.	y3

ⓒ

	z14
The denominator of the exponent is 4, so the index is 4.	z4

Write as a radical expression: ⓐ t12 ⓑ m13 ⓒ r14.

ⓐ t ⓑ m3 ⓒ r4

Write as a radial expression: ⓐ b16 ⓑ z15 ⓒ p14.

ⓐ b6 ⓑ z5 ⓒ p4

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Write with a rational exponent: ⓐ 5y ⓑ 4x3 ⓒ 35z4.
We want to write each radical in the form a1n. ⓐ

	5y
No index is shown, so it is 2. The denominator of the exponent will be 2.	(5y)12

Put parentheses around the entire expression 5y.

ⓑ

4x3

The index is 3, so the denominator of the exponent is 3. Include parentheses (4x).

(4x)13

ⓒ

35z4

The index is 4, so the denominator of the exponent is 4. Put parentheses only around the 5z since 3 is not

3(5z)14

under the radical sign.

Write with a rational exponent: ⓐ 10m ⓑ 3n5 ⓒ 36y4.

ⓐ (10m)12 ⓑ (3n)15
ⓒ 3(6y)14

Write with a rational exponent: ⓐ 3k7 ⓑ 5j4 ⓒ 82a3.

ⓐ (3k)17 ⓑ (5j)14
ⓒ 8(2a)13

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	25614	
Rewrite as a fourth root.	2564	
Recognize 256 is a perfect fourth power.	444	
Simplify.	4	

Simplify: ⓐ 3612 ⓑ 813 ⓒ 1614.

ⓐ 6 ⓑ 2 ⓒ 2

Simplify: ⓐ 10012 ⓑ 2713 ⓒ 8114.

ⓐ 10 ⓑ 3 ⓒ 3

Be careful of the placement of the negative signs in the next example. We will need to use the property a−n=1an in one case.

Simplify: ⓐ $(-16)14$ ⓑ -1614 ⓒ $(16)-14$.

ⓐ

	$(-16)14$
Rewrite as a fourth root.	-164
	$(-2)44$
Simplify.	No real solution.

ⓑ

	-1614
The exponent only applies to the 16. Rewrite as a fouth root.	-164
Rewrite 16 as 24.	-244
Simplify.	-2

ⓒ

	$(16)-14$
Rewrite using the property $a-n=1an$.	$1(16)14$
Rewrite as a fourth root.	1164
Rewrite 16 as 24.	1244
Simplify.	12

Simplify: ⓐ $(-64)-12$ ⓑ -6412 ⓒ $(64)-12$.

ⓐ No real solution ⓑ -8
ⓒ 18

Simplify: ⓐ $(-256)14$ ⓑ -25614 ⓒ $(256)-14$.

ⓐ No real solution ⓑ -4
ⓒ 14

Simplify Expressions with amn

We can look at amn in two ways. Remember the Power Property tells us to multiply the exponents and so (a1n)m and (am)1n both equal amn. If we write these expressions in radical form, we get
$amn=(a1n)m=(an)mandamn=(am)1n=amn$

This leads us to the following definition.

Rational Exponent amn
For any positive integers *m* and *n*,
$amn=(an)mandamn=amn$

Which form do we use to simplify an expression?

We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Write with a rational exponent: ⓐ y3 ⓑ (2x3)4 ⓒ (3a4b)3.

We want to use amn = amn to write each radical in the form amn.

ⓐ

[redacted]

ⓑ

$\left(\sqrt[3]{2x}\right)^4$

The numerator of the exponent is the exponent, 4.

The denominator of the exponent is the index of the radical, .

$(2x)$

ⓒ

$\sqrt{\left(\frac{3a}{4b}\right)^3}$

The numerator of the exponent is the exponent, 3.

The denominator of the exponent is the index of the radical, .

$\left(\frac{3a}{4b}\right)$

Write with a rational exponent: ⓐ x5 ⓑ (3y4)3 ⓒ (2m3n)5.

ⓐ x52 ⓑ (3y)34 ⓒ (2m3n)52

Write with a rational exponent: ⓐ a25 ⓑ (5ab3)5 ⓒ (7xyz)3.

ⓐ a25 ⓑ (5ab)53
ⓒ (7xyz)32

Remember that a − n = 1an. The negative sign in the exponent does not change the sign of the expression.

Simplify: ⓐ 12523 ⓑ 16 − 32 ⓒ 32 − 25.

We will rewrite the expression as a radical first using the defintion, amn = (an)m. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	12523
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	(1253)2
Simplify.	(5)2
	25

ⓑ We will rewrite each expression first using a −n=1an and then change to radical form.

	16−32
Rewrite using a −n=1an	11632
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	1(16)3
Simplify.	143

	164

ⓒ

	32 − 25
Rewrite using $a^{-n} = \frac{1}{a^n}$.	13225
Change to radical form.	1(325)2
Rewrite the radicand as a power.	1(255)2
Simplify.	122
	14

Simplify: ⓐ 2723 ⓑ 81 − 32 ⓒ 16 − 34.

ⓐ 9 ⓑ 1729 ⓒ 18

Simplify: ⓐ 432 ⓑ 27 − 23 ⓒ 625 − 34.

ⓐ 8 ⓑ 19 ⓒ 1125

Simplify: ⓐ −2532 ⓑ −25 − 32 ⓒ (−25)32.

ⓐ

	−2532
Rewrite in radical form.	−(25)3
Simplify the radical.	−(5)3
Simplify.	−125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{(\sqrt{25})^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$(\sqrt{-25})^3$
There is no real number whose square root is -25.	Not a real number.

Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

ⓐ -64 ⓑ -164 ⓒ not a real number

Simplify: ⓐ -8132 ⓑ $-81-32$ ⓒ $(-81)-32$.

ⓐ -729 ⓑ -1729 ⓒ not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Propertyam·an = am + nPower Property(am)n = am·nProduct to a Power(ab)m = ambmQuotient Propertyaman = am − n,a ≠ 0Zero Exponent Definitiona0 = 1,a ≠ 0Quotient to a Power Property(ab)m = ambm,b ≠ 0Negative Exponent Propertya − n = 1an,a ≠ 0

We will apply these properties in the next example.

Simplify: ⓐ x12·x56 ⓑ (z9)23 ⓒ x13x53.

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

	x12·x56
The bases are the same, so we add the exponents.	x12 + 56
Add the fractions.	x86

Simplify the exponent.	$x^{\frac{4}{3}}$

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

	$(z^9)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9\cdot\frac{2}{3}}$
Simplify.	z^6

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

	$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
	$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$

Simplify.	1x43	

Simplify: ⓐ x16·x43 ⓑ (x6)43 ⓒ x23x53.

ⓐ x32 ⓑ x8 ⓒ 1x

Simplify: ⓐ y34·y58 ⓑ (m9)29 ⓒ d15d65.

ⓐ y118 ⓑ m2 ⓒ 1d

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

Simplify: ⓐ (27u12)23 ⓑ (m23n12)32.

ⓐ

	(27u12)23
First we use the Product to a Power Property.	(27)23(u12)23
Rewrite 27 as a power of 3.	(33)23(u12)23
To raise a power to a power, we multiply the exponents.	(32)(u13)
Simplify.	9u13

ⓑ

	(m23n12)32
First we use the Product to a Power Property.	(m23)32(n12)32

To raise a power to a power, we multiply the exponents.	mn34

Simplify: ⓐ (32x13)35 ⓑ (x34y12)23.

ⓐ 8x15 ⓑ x12y13

Simplify: ⓐ (81n25)32 ⓑ (a32b12)43.

ⓐ 729n35 ⓑ a2b23

We will use both the Product Property and the Quotient Property in the next example.

Simplify: ⓐ $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

ⓐ

	$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$
Use the Product Property in the numerator, add the exponents.	$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$
Use the Quotient Property, subtract the exponents.	$x^{\frac{8}{4}}$
Simplify.	x^2

ⓑ Follow the order of operations to simplify inside the parenthese first.

	$\left(\frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$
Use the Quotient	$\left(16x^{\frac{6}{3}}y^{\frac{6}{6}}\right)^{\frac{1}{2}}$

Property, subtract the exponents.	
Simplify.	(16x2y)12
Use the Product to a Power Property, multiply the exponents.	4xy12

Simplify: ⓐ m23·m−13m−53 ⓑ (25m16n116m23n−16)12.

ⓐ m2 ⓑ 5nm14

Simplify: ⓐ u45·u−25u−135 ⓑ (27x45y16x15y−56)13.

ⓐ u3 ⓑ 3x15y13

Access these online resources for additional

instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

Key Concepts

- **Rational Exponent a1n**
    - If an is a real number and n ≥ 2, then a1n = an.
- **Rational Exponent amn**
    - For any positive integers *m* and *n*, amn = (an)m and amn = amn
- **Properties of Exponents**
    - If *a, b* are real numbers and *m, n* are rational numbers, then
        - **Product Property** am·an = am + n
        - **Power Property** (am)n = am·n
        - **Product to a Power** (ab)m = ambm

- **Quotient Property** aman=am−n,a≠0
- **Zero Exponent Definition** a0=1, a≠0
- **Quotient to a Power Property** (ab)m=ambm,b≠0
- **Negative Exponent Property** a−n=1an,a≠0

Practice Makes Perfect

Simplify expressions with a1n

In the following exercises, write as a radical expression.

ⓐ x12 ⓑ y13 ⓒ z14

ⓐ x ⓑ y3 ⓒ z4

ⓐ r12 ⓑ s13 ⓒ t14

ⓐ u15 ⓑ v19 ⓒ w120

ⓐ u5 ⓑ v9 ⓒ w20

ⓐ g17 ⓑ h15 ⓒ j125

In the following exercises, write with a rational exponent.

ⓐ x7 ⓑ y9 ⓒ f5

ⓐ x17 ⓑ y19 ⓒ f15

ⓐ r8 ⓑ s10 ⓒ t4

ⓐ 7c3 ⓑ 12d7 ⓒ 26b4

ⓐ (7c)13 ⓑ (12d)17
ⓒ 2(6b)14

ⓐ 5x4 ⓑ 9y8 ⓒ 73z5

ⓐ 21p ⓑ 8q4 ⓒ 436r6

ⓐ (21p)12 ⓑ (8q)14
ⓒ 4(36r)16

ⓐ 25a3 ⓑ 3b ⓒ 40c8

In the following exercises, simplify.

ⓐ 8112 ⓑ 12513 ⓒ 6412

ⓐ 9 ⓑ 5 ⓒ 8

ⓐ 62514 ⓑ 24315 ⓒ 3215

ⓐ 1614 ⓑ 1612 ⓒ 62514

ⓐ 2 ⓑ 4 ⓒ 5

ⓐ 6413 ⓑ 3215 ⓒ 8114

ⓐ $(-216)13$ ⓑ -21613 ⓒ $(216)-13$

ⓐ -6 ⓑ -6 ⓒ 16

ⓐ $(-1000)13$ ⓑ -100013 ⓒ $(1000)-13$

ⓐ $(-81)14$ ⓑ -8114 ⓒ $(81)-14$

ⓐ not real ⓑ -3 ⓒ 13

ⓐ (−49)12 ⓑ −4912 ⓒ (49)−12

ⓐ (−36)12 ⓑ −3612 ⓒ (36)−12

ⓐ not real ⓑ −6 ⓒ 16

ⓐ (−16)14 ⓑ −1614 ⓒ 16−14

ⓐ (−100)12 ⓑ −10012 ⓒ (100)−12

ⓐ not real ⓑ −10 ⓒ 110

ⓐ (−32)15 ⓑ (243)−15 ⓒ −12513

Simplify Expressions with amn

In the following exercises, write with a rational exponent.

ⓐ m5 ⓑ (3y3)7 ⓒ (4x5y)35

ⓐ m52 ⓑ (3y)73 ⓒ (4x5y)35

ⓐ r74 ⓑ (2pq5)3 ⓒ (12m7n)34

ⓐ u25 ⓑ (6x3)5 ⓒ (18a5b)74

ⓐ u25 ⓑ (6x)53 ⓒ (18a5b)74

ⓐ a3 ⓑ (21v4)3 ⓒ (2xy5z)24

In the following exercises, simplify.

ⓐ 6452 ⓑ 81−32 ⓒ (−27)23

ⓐ 32,768 ⓑ 1729 ⓒ 9

ⓐ 2532 ⓑ 9−32 ⓒ (−64)23

ⓐ 3225 ⓑ 27−23 ⓒ (−25)12

ⓐ 4 ⓑ 19 ⓒ not real

ⓐ 10032 ⓑ 49−52 ⓒ (−100)32

ⓐ −932 ⓑ −9−32 ⓒ (−9)32

ⓐ −27 ⓑ −127 ⓒ not real

ⓐ −6432 ⓑ −64−32 ⓒ (−64)32

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify. Assume all variables are positive.

ⓐ c14·c58 ⓑ (p12)34 ⓒ r45r95

ⓐ c78 ⓑ p9 ⓒ 1r

ⓐ 652·612 ⓑ (b15)35 ⓒ w27w97

ⓐ y12·y34 ⓑ (x12)23 ⓒ m58m138

ⓐ y54 ⓑ x8 ⓒ 1m

ⓐ q23·q56 ⓑ (h6)43 ⓒ n35n85

ⓐ (27q32)43 ⓑ (a13b23)32

ⓐ 81q2 ⓑ a12b

ⓐ (64s37)16 ⓑ (m43n12)34

ⓐ (16u13)34 ⓑ (4p13q12)32

ⓐ 8u14 ⓑ 8p12q34

ⓐ (625n83)34 ⓑ (9x25y35)52

ⓐ r52·r − 12r − 32 ⓑ (36s15t − 32s − 95t12)12

ⓐ r72 ⓑ 6st

ⓐ a34·a − 14a − 104 ⓑ (27b23c − 52b − 73c12)13

ⓐ c53·c − 13c − 23 ⓑ (8x53y − 1227x − 43y52)13

ⓐ c2 ⓑ 2x3y

ⓐ m74·m − 54m − 24 ⓑ (16m15n3281m95n

−12)14

Writing Exercises

Show two different algebraic methods to simplify 432. Explain all your steps.

Answers will vary.

Explain why the expression (−16)32 cannot be evaluated.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to

improve?

Add, Subtract, and Multiply Radical Expressions
By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Before you get started, take this readiness quiz.

1. Add: $3x2+9x-5-(x2-2x+3)$.
   If you missed this problem, review [link].
2. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review [link].
3. Simplify: $(9-5y)2$.
   If you missed this problem, review [link].

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that 3x + 8x is 11x. Similarly we add 3x + 8x and the result is 11x.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

Simplify: ⓐ 22 − 72 ⓑ 5y3 + 4y3 ⓒ 7x4 − 2y4.

ⓐ

22 − 72Since the radicals are like, we subtract thecoefficients. − 52

ⓑ

5y3 + 4y3Since the radicals are like, we add thecoefficients.9y3

ⓒ

7x4 − 2y4

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

Simplify: ⓐ 82 − 92 ⓑ 4x3 + 7x3 ⓒ 3x4 − 5y4.

ⓐ −2 ⓑ 11x3
ⓒ 3x4 − 5y4

Simplify: ⓐ 53 − 93 ⓑ 5y3 + 3y3 ⓒ 5m4 − 2m3.

ⓐ −43 ⓑ 8y3
ⓒ 5m4 − 2m3

For radicals to be like, they must have the same index and radicand. When the radicands contain

more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Simplify: ⓐ 25n − 65n + 45n ⓑ 3xy4 + 53xy4 − 43xy4.

ⓐ

25n − 65n + 45nSince the radicals are like, we combine them.05nSimplify.0

ⓑ

3xy4 + 53xy4 − 43xy4Since the radicals are like, we combine them.23xy4

Simplify: ⓐ 7x − 77x + 47x ⓑ 45xy4 + 25xy4 − 75xy4.

ⓐ −27x ⓑ −5xy4

Simplify: ⓐ 43y − 73y + 23y ⓑ 67mn3 + 7mn3 − 47mn3.

ⓐ −3y ⓑ 37mn3

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Simplify: ⓐ 20 + 35 ⓑ 243 − 3753 ⓒ 12484 − 232434.

ⓐ

20 + 35Simplify the radicals, when possible.4·5 + 3525 + 35Combine the like radicals.55

ⓑ

243 − 3753Simplify the radicals.83·33 − 1253·33233 − 533Combine the

like radicals. − 333

ⓒ
12484 − 232434Simplify the radicals.12164·34 − 23814·3412·2·34 − 23·3·3434 − 234C the like radicals. − 34

Simplify: ⓐ 18 + 62 ⓑ 6163 − 22503 ⓒ 23813 − 12243.

ⓐ 92 ⓑ 223 ⓒ 33

Simplify: ⓐ 27 + 43 ⓑ 453 − 7403 ⓒ 121283 − 53543.

ⓐ 73 ⓑ − 1053 ⓒ − 323

In the next example, we will remove both constant and variable factors from the radicals. Now that we

have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption throughout the rest of this chapter.

Simplify: ⓐ 950m2 − 648m2 ⓑ 54n53 − 16n53.

ⓐ

950m2 − 648m2Simplify the radicals.925m2·2 − 616m2·39·5m·2 − 6·4m·345m2 − 24m radicals are not like and so cannot becombined.

ⓑ

54n53 − 16n53Simplify the radicals.27n33·2n23 − 8n33·2n233n2n23 − 2n2n23Comb the like radicals.n2n23

Simplify: ⓐ 32m7 − 50m7 ⓑ 135x73 − 40x73.

ⓐ $-m32m$ ⓑ $x25x3$

Simplify: ⓐ $27p3-48p3$ ⓑ $256y53-32n53$.

ⓐ $-p3p$
ⓑ $4y4y23-2n4n23$

Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots
For any real numbers, an and bn, and for any integer $n \geq 2$
abn = an·bnandan·bn = abn

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply 4x·3y we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

Simplify: ⓐ (62)(310) ⓑ (−543)(−463).

ⓐ
(62)(310)Multiply using the Product Property.1820Simplify the radical.184·5Simplify.18·2·5365

ⓑ
(−543)(−463)Multiply using the Product

Property.20243Simplify the radical.2083·33Simplify.20·2·334033

Simplify: ⓐ (32)(230) ⓑ (2183)(− 363).

ⓐ 1215 ⓑ − 1823

Simplify: ⓐ (33)(36) ⓑ (− 493)(363).

ⓐ 272 ⓑ − 3623

We follow the same procedures when there are variables in the radicands.

Simplify: ⓐ (106p3)(43p) ⓑ (220y24)(328y34).

ⓐ
(106p3)(43p)Multiply.4018p4Simplify the radical.409p4·2Simplify.40·3p2·3120p23

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

(220y24)(328y34)Multiply.64·5·4·7y54Simplify the radical.616y44·35y4Simplify.6·2y35y4Multiply.12y35y4

Simplify: ⓐ (66x2)(830x4) ⓑ (−412y34)(−8y34).

ⓐ 36x35 ⓑ 8y3y24

Simplify: ⓐ (26y4)(1230y) ⓑ (−49a34)(327a24).

ⓐ 144y25y ⓑ −363a4

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Simplify: ⓐ 6(2+18) ⓑ 93(5−183).

ⓐ
6(2+18)Multiply.12+108Simplify.4·3+36·3Simplify.23
like radicals.83

ⓑ
93(5−183)Distribute.593−1623Simplify.593−273·63S

Simplify: ⓐ 6(1+36) ⓑ 43(−2−63).

ⓐ 18+6 ⓑ −243−233

Simplify: ⓐ 8(2−58) ⓑ 33(−93−63).

ⓐ −40+42 ⓑ −3−183

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Simplify: ⓐ (3−27)(4−27) ⓑ (x3−2)(x3+4).

ⓐ
(3−27)
(4−27)Multiply12−67−87+4·7Simplify.12−67−87+
like terms.40−147

ⓑ
(x3 − 2)
(x3 + 4)Multiply.x23 + 4x3 − 2x3 − 8Combine like terms.x23 + 2x3 − 8

Simplify: ⓐ (6 − 37)(3 + 47) ⓑ (x3 − 2)(x3 − 3).

ⓐ −66 + 157
ⓑ x23 − 5x3 + 6

Simplify: ⓐ (2 − 311)(4 − 11) ⓑ (x3 + 1)(x3 + 3).

ⓐ 41 − 1411
ⓑ x23 + 4x3 + 3

Simplify: $(32-5)(2+45)$.

$(32-5)$
$(2+45)$Multiply.$3\cdot2+1210-10-4\cdot5$Simplify.$6+1210-$
like terms.$-14+1110$

Simplify: $(53-7)(3+27)$

$1+921$

Simplify: $(6-38)(26+8)$

$-12-203$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products
Binomial SquaresProduct of Conjugates(a
$+ b)2 = a2 + 2ab + b2(a + b)(a - b) = a2 - b2(a$
$- b)2 = a2 - 2ab + b2$

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

Simplify: ⓐ $(2+3)2$ ⓑ $(4-25)2$.

Be sure to include the 2ab term when squaring a binomial.

ⓐ

[redacted]

Multiply, using the Product of Binomial Sq[redacted]

Simplify.

[redacted]

Combine like terms.

[redacted]

ⓑ

[redacted]

Multiply, using the Product of Binomial Sq[redacted]

Simplify.

[redacted]

[redacted]

Combine like terms.

[redacted]

Simplify: ⓐ $(10+2)2$ ⓑ $(1+36)2$.

ⓐ $102+202$ ⓑ $55+66$

Simplify: ⓐ $(6-5)2$ ⓑ $(9-210)2$.

ⓐ $41-125$
ⓑ $121-3610$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Simplify: $(5-23)(5+23)$.

[redacted]

Multiply, using the Product of Conjugates Pa [redacted]

Simplify.

[redacted]

[redacted]

Simplify: $(3-25)(3+25)$

-11

Simplify: $(4+57)(4-57)$.

-159

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

Key Concepts

- **Product Property of Roots**

- For any real numbers, an and bn, and for any integer n ≥ 2
  abn = an·bn and an·bn = abn

- **Special Products**
  Binomial SquaresProduct of Conjugates(a + b)2 = a2 + 2ab + b2(a + b)(a − b) = a2 − b2(a − b)2 = a2 − 2ab + b2

Practice Makes Perfect

Add and Subtract Radical Expressions

In the following exercises, simplify.

ⓐ 82 − 52 ⓑ 5m3 + 2m3 ⓒ 8m4 − 2m4

ⓐ 32 ⓑ 7m3 ⓒ 6m4

ⓐ 72 − 32 ⓑ 7p3 + 2p3 ⓒ 5x3 − 3x3

ⓐ 35 + 65 ⓑ 9a3 + 3a3 ⓒ 52z4 + 2z4

ⓐ 95 ⓑ 12a3 ⓒ 62z4

ⓐ $45+85$ ⓑ $m3-4m3$ ⓒ $n+3n$

ⓐ $32a-42a+52a$ ⓑ $53ab4-33ab4-23ab4$

ⓐ $42a$ ⓑ 0

ⓐ $11b-511b+311b$ ⓑ
$811cd4+511cd4-911cd4$

ⓐ $83c+23c-93c$ ⓑ $24pq3-54pq3+44pq3$

ⓐ -23 ⓑ $4pq3$

ⓐ $35d+85d-115d$ ⓑ $112rs3-92rs3+32rs3$

ⓐ $27-75$ ⓑ $403-3203$ ⓒ $12324+231624$

ⓐ -23 ⓑ -253 ⓒ 324

ⓐ $72-98$ ⓑ $243+813$ ⓒ $12804-234054$

ⓐ $48+27$ ⓑ $543+1283$ ⓒ $654-32804$

ⓐ 73 ⓑ 723 ⓒ 354

ⓐ $45+80$ ⓑ $813-1923$ ⓒ $52804+734054$

ⓐ $72a5-50a5$ ⓑ $980p44-6405p44$

ⓐ a22a ⓑ 0

ⓐ $48b5-75b5$ ⓑ $864q63-3125q63$

ⓐ $80c7-20c7$ ⓑ $2162r104+432r104$

ⓐ 2c35c ⓑ 14r22r24

ⓐ $96d9-24d9$ ⓑ $5243s64+23s64$

$3128y2+4y162-898y2$

4y2

$375y2+8y48-300y2$

Multiply Radical Expressions

In the following exercises, simplify.

ⓐ $(-23)(318)$ ⓑ $(843)(-4183)$

ⓐ -186 ⓑ -6493

ⓐ $(-45)(510)$ ⓑ $(-293)(793)$

ⓐ $(56)(-12)$ ⓑ $(-2184)(-94)$

ⓐ $-302)$ ⓑ 624

ⓐ $(-27)(-214)$ ⓑ $(-384)(-564)$

ⓐ $(412z3)(39z)$ ⓑ $(53x33)(318x33)$

ⓐ $72z23$ ⓑ $45x223$

ⓐ $(32x3)(718x2)$ ⓑ $(-620a23)(-216a33)$

ⓐ $(-27z3)(314z8)$ ⓑ $(28y24)(-212y34)$

ⓐ $-42z52z$ ⓑ $-8y6y4$

ⓐ $(42k5)(-332k6)$ ⓑ $(-6b34)(38b34)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

ⓐ $7(5+27)$ ⓑ $63(4+183)$

ⓐ $14+57$ ⓑ $463+343$

ⓐ $11(8+411)$ ⓑ $33(93+183)$

ⓐ $11(-3+411)$ ⓑ $34(544+184)$

ⓐ $44-311$ ⓑ $324+544$

ⓐ $2(-5+92)$ ⓑ $24(124+244)$

$(7+3)(9-3)$

$60+23$

$(8-2)(3+2)$

ⓐ $(9-32)(6+42)$ ⓑ $(x3-3)(x3+1)$

ⓐ $30+182$ ⓑ $x23-2x3-3$

ⓐ $(3-27)(5-47)$ ⓑ $(x3-5)(x3-3)$

ⓐ $(1+310)(5-210)$ ⓑ $(2x3+6)(x3+1)$

ⓐ $-55+1310$
ⓑ $2x23+8x3+6$

ⓐ $(7-25)(4+95)$ ⓑ $(3x3+2)(x3-2)$

$(3+10)(3+210)$

$23+330$

$(11+5)(11+65)$

$(27-511)(47+911)$

$-439-277$

$(46+713)(86-313)$

ⓐ $(3+5)2$ ⓑ $(2-53)2$

ⓐ $14+65$ ⓑ $79-203$

ⓐ $(4+11)2$ ⓑ $(3-25)2$

ⓐ $(9-6)2$ ⓑ $(10+37)2$

ⓐ $87-186$
ⓑ $163+607$

ⓐ $(5-10)2$ ⓑ $(8+32)2$

$(4+2)(4-2)$

14

$(7+10)(7-10)$

$(4+93)(4-93)$

-227

$(1+82)(1-82)$

$(12-55)(12+55)$

19

$(9-43)(9+43)$

$(3x3+2)(3x3-2)$

$9x23-4$

$(4x3+3)(4x3-3)$

Mixed Practice

$2327+3448$

53

$175k4 - 63k4$

$56162 + 316128$

92

$243 + /813$

$12804 - 234054$

-54

$8134 - 4134 - 3134$

$512c4 - 327c6$

$10c23 - 9c33$

$80a5 - 45a5$

$3575 - 1448$

23

$2193-293$

$864q63-3125q63$

$17q2$

$1111-1011$

$3\cdot 21$

37

$(46)(-18)$

$(743)(-3183)$

-4293

$(412x5)(26x3)$

$(29)2$

29

$(-417)(-317)$

$(-4+17)(-3+17)$

$29-717$

$(38a24)(12a34)$

$(6-32)2$

$54-362$

$3(4-33)$

$33(293+183)$

$6+323$

$(6+3)(6+63)$

Writing Exercises

Explain the when a radical expression is in simplest form.

Answers will vary.

Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

ⓐ Explain why $(-n)2$ is always non-negative, for $n \geq 0$.
ⓑ Explain why $-(n)2$ is always non-positive, for $n \geq 0$.

Answers will vary.

Use the binomial square pattern to simplify $(3+2)2$. Explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

like radicals
: Like radicals are radical expressions with the same index and the same radicand.

Divide Radical Expressions
By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Before you get started, take this readiness quiz.

1. Simplify: 3048.
   If you missed this problem, review [link].
2. Simplify: x2·x4.
   If you missed this problem, review [link].
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review [link].

Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property ‘in reverse’ to simplify a fraction with radicals.

We give the Quotient Property of Radical

Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions
If an and bn are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,
abn = anbnandanbn = abn

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Simplify: ⓐ 72x3162x ⓑ 32x234x53.

ⓐ
72x3162xRewrite using the quotient property,anbn = abn.72x3162xRemove common factors.18·4·x2·x18·9·xSimplify.4x29Simplify

the radical.2x3

ⓑ
32x234x53Rewrite using the quotient property,anbn=abn.32x24x53Simplify the fraction under the radical.8x33Simplify the radical.2x

Simplify: ⓐ 50s3128s ⓑ 56a37a43.
ⓐ 5s8 ⓑ 2a

Simplify: ⓐ 75q5108q ⓑ 72b239b53.
ⓐ 5q26 ⓑ 2b

Simplify: ⓐ 147ab83a3b4 ⓑ −250mn−232m

$-2n43.$

ⓐ
147ab83a3b4Rewrite using the quotient property.147ab83a3b4Remove common factors in the fraction.49b4a2Simplify the radical.7b2a

ⓑ
$-250mn-232m-2n43$Rewrite using the quotient property.$-250mn-22m-2n43$Simplify the fraction under the radical.$-125m3n63$Simplify the radical.$-5mn2$

Simplify: ⓐ 162x10y22x6y6 ⓑ $-128x2y-132x-1y23.$

ⓐ 9x2y2 ⓑ $-4xy$

Simplify: ⓐ 300m3n73m5n ⓑ $-81pq-133p-2q53.$

ⓐ 10n3m ⓑ −3pq2

Simplify: 54x5y33x2y.

54x5y33x2y Rewrite using the quotient property.54x5y33x2y Remove common factors in the fraction.18x3y2 Rewrite the radicand as a product using the largest perfect square factor.9x2y2·2x Rewrite the radical as the product of two radicals.9x2y2·2x Simplify.3xy2x

Simplify: 64x4y52xy3.

4xy2x

Simplify: 96a5b42a3b.

4ab3b

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose

denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(a)2=a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the

denominator in the next example.

Simplify: ⓐ 43 ⓑ 320 ⓒ 36x.

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

$\frac{4}{\sqrt{3}}$	
Multiply both the numerator and denominator by 3.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	

$$\frac{4\sqrt{3}}{3}$$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

$$\sqrt{\frac{3}{20}}$$

The fraction is not a perfect square, so rewrite using the Quotient Property.

$$\frac{\sqrt{3}}{\sqrt{20}}$$

Simplify the denominator.

$$\frac{\sqrt{3}}{2\sqrt{5}}$$

Multiply the numerator and denominator by $\sqrt{5}$.

$$\frac{\sqrt{3}\cdot\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}}$$

Simplify.

$$\frac{\sqrt{15}}{2\cdot 5}$$

Simplify.

$$\frac{\sqrt{15}}{10}$$

©

$$\frac{3}{\sqrt{6x}}$$

Multiply the numerator and denominator by 6x

$$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$$

Simplify.

$$\frac{3\sqrt{6x}}{6x}$$

Simplify.

$$\frac{\sqrt{6x}}{2x}$$

Simplify: ⓐ 53 ⓑ 332 ⓒ 22x.

ⓐ 533 ⓑ 68 ⓒ 2xx

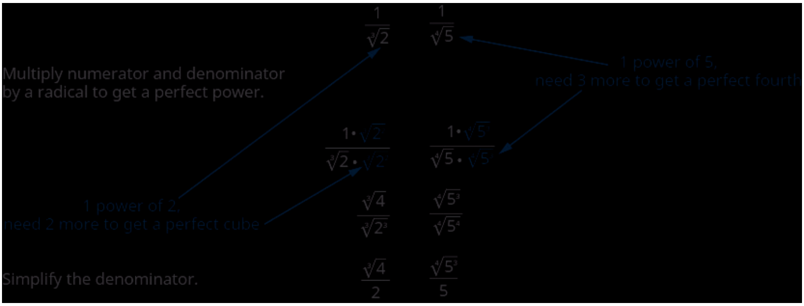
Simplify: ⓐ 65 ⓑ 718 ⓒ 55x.

ⓐ 655 ⓑ 146 ⓒ 5xx

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,



We will use this technique in the next examples.

Simplify ⓐ 163 ⓑ 7243 ⓒ 34x3.

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

$$\frac{1}{\sqrt[3]{6}}$$

The radical in the denominator has one factor of 6.

$$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$$

Multiply both the numerator and denominator by 623, which gives us 2 more factors of 6.

$$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$$

Multiply. Notice the radicand in the denominator has 3 factors of 6.

$$\frac{\sqrt[3]{36}}{6}$$

Simplify the cube root in the denominator.

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

$$\sqrt[3]{\frac{7}{24}}$$

The fraction is not a perfect cube, so rewrite using the Quotient Property.

$$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$$

Simplify the denominator.

$$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$$

Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.

$$\frac{\sqrt[3]{7}\cdot\sqrt[3]{3^2}}{2\sqrt[3]{3}\cdot\sqrt[3]{3^2}}$$

Simplify.

$$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$$

Remember, 333=3.

$$\frac{\sqrt[3]{63}}{2\cdot 3}$$

Simplify.

$$\frac{\sqrt[3]{63}}{6}$$

©

$$\frac{3}{\sqrt[3]{4x}}$$

Rewrite the radicand to show the factors.

$$\frac{3}{\sqrt[3]{2^2 \cdot x}}$$

Multiply the numerator and denominator by 2·x

$$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$$

Th … 3 factors of 2 and 3 factors of x.

Simplify.

$$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$$

Simplify the radical in the denominator.

$$\frac{3\sqrt[3]{2x^2}}{2x}$$

Simplify: ⓐ 173 ⓑ 5123 ⓒ 59y3.

ⓐ 4937 ⓑ 9036 ⓒ 53y233y

Simplify: ⓐ 123 ⓑ 3203 ⓒ 225n3.
ⓐ 432 ⓑ 150310 ⓒ 25n235n

Simplify: ⓐ 124 ⓑ 5644 ⓒ 28x4.
To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor. ⓐ

$$\frac{1}{\sqrt[4]{2}}$$

The radical in the

de $\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$ or has one fac

Multiply both the numerator and denominator by 234, which gives us 3 more factors of 2.

Multiply. Notice the radicand in the de $\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$ or ha ers of 2.

Simplify the fourth root in the de $\frac{\sqrt[4]{8}}{2}$ or.

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

$\sqrt[4]{\frac{5}{64}}$

The fraction is not a

pe $\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$ h power,
so
using the Quotient Property.

Rewrite the radicand in the denominator to sh $\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$ tors.

Simplify the denominator.

$$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$$

Multiply the numerator and denominator by
22 $\frac{\sqrt[4]{5}\cdot\sqrt[4]{2^2}}{2\sqrt[4]{2^2}\cdot\sqrt[4]{2^2}}$
Th e us 4 factors of 2.

Simplify.

$$\frac{\sqrt[4]{5}\cdot\sqrt[4]{4}}{2\sqrt[4]{2^4}}$$

Remember, 244 = 2.

$$\frac{\sqrt[4]{20}}{2\cdot 2}$$

Simplify.

$$\frac{\sqrt[4]{20}}{4}$$

©

$$\frac{2}{\sqrt[4]{8x}}$$

Rewrite the radicand to show the factors.

$$\frac{2}{\sqrt[4]{2^3 \cdot x}}$$

Multiply the numerator and denominator by 2·x

$$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$$

Th 4 factors of 2 and 4 factors of x.

Simplify.

$$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$$

Simplify the radical in the denominator.

$$\frac{2\sqrt[4]{2x^3}}{2x}$$

Simplify the fraction.

$$\frac{\sqrt[4]{2x^3}}{x}$$

Simplify: ⓐ 134 ⓑ 3644 ⓒ 3125x4.

ⓐ 2743 ⓑ 1244 ⓒ 35x345x

Simplify: ⓐ 154 ⓑ 71284 ⓒ 44x4

ⓐ 12545 ⓑ 22448
ⓒ 64x34x

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.
$(a-b)(a+b)(2-5)(2+5)a2-b222-(5)24-5-1$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

Simplify: 52−3.

$$\frac{5}{2-\sqrt{3}}$$

Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$

Multiply the conjugates in the denominator.

$$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$$

Simplify the denominator.

$$\frac{5(2+\sqrt{3})}{4-3}$$

Simplify the denominator.

$$\frac{5(2+\sqrt{3})}{1}$$

Simplify.

$$5(2+\sqrt{3})$$

Simplify: 31 − 5.

−3(1 + 5)4

Simplify: 24 − 6.

4 + 65

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

Simplify: 3u − 6.

$$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$$

Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$$

Multiply the conjugates in the denominator.

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$$

Simplify the denominator.

$$\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6}$$

Simplify: $5x+2$.

$5(x-2)x-2$

Simplify: 10y − 3.
10(y + 3)y − 3

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Simplify: x + 7x − 7.

[redacted]

Multiply the numerator and denominator by the[redacted]

co[redacted]

denominator.

Multiply the conjugates in the denominator.	[redacted]
Simplify the denominator.	[redacted]

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Simplify: $p+2p-2$.

$(p+2)p-22$

Simplify: $q-10q+10$

$(q-10)q-102$

Access these online resources for additional instruction and practice with dividing radical expressions.

- Rationalize the Denominator
- Dividing Radical Expressions and Rationalizing the Denominator
- Simplifying a Radical Expression with a Conjugate
- Rationalize the Denominator of a Radical Expression

Key Concepts

- **Quotient Property of Radical Expressions**
  - If an and bn are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $abn = anbn$ and $anbn = abn$
- **Simplified Radical Expressions**
  - A radical expression is considered simplified if there are:
    - no factors in the radicand that have perfect powers of the index

- no fractions in the radicand
- no radicals in the denominator of a fraction

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

ⓐ 12872 ⓑ 1283543

ⓐ 43 ⓑ 43

ⓐ 4875 ⓑ 813243

ⓐ 200m598m ⓑ 54y232y53

ⓐ 10m27 ⓑ 3y

ⓐ 108n7243n3 ⓑ 54y316y43

ⓐ 75r3108r7 ⓑ 24x7381x43

ⓐ 56r2 ⓑ 2x3

ⓐ 196q484q5 ⓑ 16m4354m3

ⓐ 108p5q23p3q6 ⓑ −16a4b−232a−2b3

ⓐ 6pq2 ⓑ −2a2b

ⓐ 98rs102r3s4 ⓑ −375y4z−233y−2z43

ⓐ 320mn−545m−7n3 ⓑ 16x4y−23−54x −2y43

ⓐ 8m43n4 ⓑ −2x23y2

ⓐ 810c−3d71000cd−1 ⓑ 24a7b−13−81a −2b23

56x5y42xy3

---

x228y

72a3b63ab3

48a3b633a – 1b33

2ab2a3

162x – 3y632x3y – 23

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

ⓐ 106 ⓑ 427 ⓒ 105x

ⓐ 563 ⓑ 239 ⓒ 25xx

ⓐ 83 ⓑ 740 ⓒ 82y

ⓐ 67 ⓑ 845 ⓒ 123p

ⓐ 677 ⓑ 21015 ⓒ 43pp

ⓐ 45 ⓑ 2780 ⓒ 186q

ⓐ 153 ⓑ 5243 ⓒ 436a3

ⓐ 2535 ⓑ 4536 ⓒ 26a233a

ⓐ 133 ⓑ 5323 ⓒ 749b3

ⓐ 1113 ⓑ 7543 ⓒ 33x23

ⓐ 121311 ⓑ 2836 ⓒ 9x3x

ⓐ 1133 ⓑ 31283 ⓒ 36y23

ⓐ 174 ⓑ 5324 ⓒ 44x24

ⓐ 34347 ⓑ 4044 ⓒ 24x24x

ⓐ 144 ⓑ 9324 ⓒ 69x34

ⓐ 194 ⓑ 251284 ⓒ 627a4

ⓐ 943 ⓑ 5044 ⓒ 23a24a

ⓐ 184 ⓑ 271284 ⓒ 1664b24

Rationalize a Two Term Denominator

In the following exercises, simplify.

$81-5$

$-2(1+5)$

$72-6$

$63-7$

$3(3+7)$

$54-11$

$3m-5$

$3(m+5)m-5$

$5n-7$

$2x-6$

$2(x+6)x-6$

$7y+3$

$r+5r-5$

$(r+5)r-52$

$s-6s+6$

$x+8x-8$

$(x+22)x-82$

$m-3m+3$

Writing Exercises

ⓐ Simplify 273 and explain all your steps.
ⓑ Simplify 275 and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

Answers will vary.

Explain what is meant by the word rationalize in the phrase, “rationalize a denominator.”

Explain why multiplying $2x-3$ by its conjugate results in an expression with no radicals.

Answers will vary.

Explain why multiplying 7x3 by x3x3 does not rationalize the denominator.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

rationalizing the denominator
: Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Solve Radical Equations

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Before you get started, take this readiness quiz.

Simplify: $(y-3)2$.
If you missed this problem, review [link].

$y2-6y+9$

Solve: $2x-5=0$.
If you missed this problem, review [link].

$x=52$

Solve $n2-6n+8=0$.
If you missed this problem, review [link].

$n=2 or n=4$

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Radical Equation
An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the

radical.

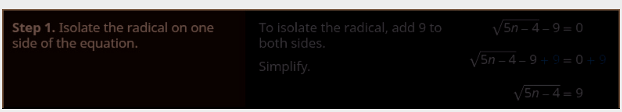
Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

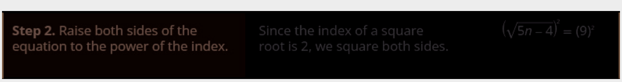
In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index *n* to the *n*th power. This will eliminate the radical.

For $a \geq 0, (an)n = a$.

How to Solve a Radical Equation

Solve: $5n - 4 - 9 = 0$.





Step 3. Solve the new equation.	Remember, $\left(\sqrt{a}\right)^2 = a$.	$5n - 4 = 81$ $5n = 85$ $n = 17$
Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4} - 9 = 0$ $\sqrt{5(17)-4} - 9 \stackrel{?}{=} 0$ $\sqrt{85-4} - 9 \stackrel{?}{=} 0$ $\sqrt{81} - 9 \stackrel{?}{=} 0$ $9 - 9 \stackrel{?}{=} 0$ $0 = 0$ ✓ The solution is $n = 17$.

Solve: $3m + 2 - 5 = 0$.

$m = 233$

Solve: $10z + 1 - 2 = 0$.

$z = 310$

Solve a radical equation with one radical.

Isolate the radical on one side of the equation. Raise both sides of the equation to the power of the index. Solve the new equation. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

Solve: $9k - 2 + 1 = 0$.

[redacted]

To isolate the radical, subtract 1 to both sid[redacted]

Simplify.

[redacted]

Because the square root is equal to a negative number, the equation has no solution.

Solve: $2r-3+5=0$.

no solution

Solve: $7s-3+2=0$.

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares
$(a+b)2=a2+2ab+b2(a-b)2=a2-2ab+b2$

Don't forget the middle term!

Solve: $p-1+1=p$.

[redacted]

To isolate the radical, subtract 1 from both sid[redacted]

Simplify.

[redacted]

Square both sides of the equation.

[redacted]

Simplify, using the

Pr[redacted]
Squares Pattern on the right. Then solve the new equation.

It is a quadratic equation, so get zero on [redacted]

Factor the right side.

[redacted]

Use the Zero Product Property.

[redacted]

Solve each equation.

[redacted]

Check the answers.

[redacted]

The solutions are $p=1, p=2$.

Solve: $x-2+2=x$.

$x = 2, x = 3$

Solve: $y - 5 + 5 = y$.

$y = 5, y = 6$

When the index of the radical is 3, we cube both sides to remove the radical.

$(a3)3 = a$

Solve: $5x + 13 + 8 = 4$.

	$5x + 13 + 8 = 4$
To isolate the radical,	$5x + 13 = -4$

subtract 8 from both sides.	
Cube both sides of the equation.	$(5x+13)3=(-4)3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	

$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$

$\sqrt[3]{5(-13)+1}+8\overset{?}{=}4$

$\sqrt[3]{-64}+8\overset{?}{=}4$

$-4+8\overset{?}{=}4$

$4=4$ ✓

The solution is $x=-13$.

Solve: $4x-33+8=5$
$x=-6$

Solve: $6x-103+1=-3$
$x=-9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(am)n=am\cdot n$, we have for example,

$(x12)2=x,(x13)3=x$

Remember, $x12=x$ and $x13=x3$.

Solve: $(3x-2)14+3=5$.

		$(3x-2)14+3=5$

To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}}=2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4=(2)^4$
Simplify.	$3x-2=16$
Solve the equation.	$3x=18$
	$x=6$
Check the answer.	

$$x=6 \qquad (3x-2)^{\frac{1}{4}}+3=5$$
$$(3\cdot 6-2)^{\frac{1}{4}}+3\stackrel{?}{=}5$$
$$(16)^{\frac{1}{4}}+3\stackrel{?}{=}5$$
$$2+3\stackrel{?}{=}5$$
$$5=5\ \checkmark$$

The solution is $x=6$.

Solve: $(9x+9)^{\frac{1}{4}}-2=1$.

$x=8$

Solve: $(4x-8)14+5=7$.
$x=6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

Solve: $r+4-r+2=0$.

	$r+4-r+2=0$
Isolate the radical.	$r+4=r-2$
Square both sides of the equation.	$(r+4)2=(r-2)2$
Simplify and then solve the equation	$r+4=r2-4r+4$
It is a quadratic	$0=r2-5r$

equation, so get zero on one side.	
Factor the right side.	$0 = r(r-5)$
Use the Zero Product Property.	$0 = r \quad 0 = r - 5$
Solve the equation.	$r = 0 \quad r = 5$
Check your answer.	
	The solution is $r = 5$.

$r = 0, \quad \sqrt{r+4} - r + 2 = 0 \qquad r = 5, \quad \sqrt{r+4} - r + 2 = 0$

$\sqrt{0+4} - 0 + 2 \stackrel{?}{=} 0 \qquad \sqrt{5+4} - 5 + 2 \stackrel{?}{=} 0$

$\sqrt{4} + 2 \stackrel{?}{=} 0 \qquad \sqrt{9} - 3 \stackrel{?}{=} 0$

$4 \neq 0 \qquad 0 = 0 \checkmark$

$r = 0$ is an extraneous solution.

Solve: $m + 9 - m + 3 = 0$.

$m = 7$

Solve: $n + 1 - n + 1 = 0$.

$n = 3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Solve: $3\sqrt{3x-5}-8=4$.

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$
Check the answer.	

$$
\begin{aligned}
x = 7 \qquad 3\sqrt{3x-5}-8 &= 4\\
3\sqrt{3(7)-5}-8 &\stackrel{?}{=} 4\\
3\sqrt{21-5}-8 &\stackrel{?}{=} 4\\
3\sqrt{16}-8 &\stackrel{?}{=} 4\\
3(4)-8 &\stackrel{?}{=} 4\\
4 &= 4\ \checkmark
\end{aligned}
$$

The solution is $x=7$.

Solve: $24a+4-16=16$.
$a=63$

Solve: $32b+3-25=50$.
$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Solve: $4x-33=3x+23$.

The radical terms are isolated.	$4x-33=3x+23$
Since the index is 3, cube both sides of the equation.	$(4x-33)3=(3x+23)3$
Simplify, then solve the new equation.	$4x-3=3x+2$
	$x-3=2$

	$x=5$ The solution is $x=5$.
Check the answer. We leave it to you to show that 5 checks!	

Solve: $5x-43=2x+53$.

$x=3$

Solve: $7x+13=2x-53$.

$x=-65$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both

sides of the equation to the power of the index again.

How to Solve a Radical Equation

Solve: $m+1=m+9$.

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides. Simplify—be very careful as you multiply!	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Step 3. Are there any more radicals? If yes, repeat Step 1 and Step 2 again.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term.	$m+2\sqrt{m}+1=m+9$
	Here, we can easily isolate the radical by dividing both sides by 2.	$2\sqrt{m}=8$
If no, solve the new equation.	Square both sides.	$\sqrt{m}=4$
		$\left(\sqrt{m}\right)^2=(4)^2$
		$m=16$
Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$
		$\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$
		$4+1\stackrel{?}{=}5$
		$5=5\checkmark$
		The solution is $m=16$.

Solve: $3-x=x-3$.
$x=4$

Solve: $x+2=x+16$.
$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a radical equation.

Isolate one of the radical terms on one side of the equation. Raise both sides of the equation to the power of the index. Are there any more radicals? If yes, repeat Step 1 and Step 2 again.

If no, solve the new equation. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Solve: $q-2+3=4q+1$.

$\sqrt{q-2}+3=\sqrt{4q+1}$

The radical on the right is isolated. Square both sides.

$(\sqrt{q-2}+3)^2=(\sqrt{4q+1})^2$

Simplify.

$q-2+6\sqrt{q-2}+9=4q+1$

There is still a radical in the equation so we must repeat the

$6\sqrt{q-2}=3q-6$

previous steps. Isolate the radical.

Square both sides. It would not help to div

Re

squ

the q = 2.

$$(6\sqrt{q-2})^2 = (\overset{a\ -\ b}{3q - 6})^2$$

$$6^2(\sqrt{q-2})^2 = \overset{a^2}{(3q)^2} - \overset{2ab}{2 \cdot 3q \cdot 6} + \overset{b^2}{6^2}$$

Simplify, then solve the new equation.

$$36(q-2) = 9q^2 - 36q + 36$$

Distribute.

$$36q - 72 = 9q^2 - 36q + 36$$

It is a quadratic equation, so get zero on one side.

$$0 = 9q^2 - 72q + 108$$

Factor the right side.

$$0 = 9(q^2 - 8q + 12)$$

$$0 = 9(q-6)(q-2)$$

Use the Zero Product Property.

$$q - 6 = 0 \qquad q - 2 = 0$$

$$q = 6 \qquad q = 2$$

The checks are left to you.

The solutions are q = 6 and q = 2.

Solve: $x-1+2=2x+6$
$x=5$

Solve: $x+2=3x+4$
$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a problem solving strategy for applications

with formulas.

Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information. **Identify** what we are looking for. **Name** what we are looking for by choosing a variable to represent it. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve the equation** using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects
On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula
t=h4.

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting h=64 into the formula.

$$t=\frac{\sqrt{h}}{4}$$

$$t=\frac{\sqrt{64}}{4}$$

Take the square root of 64.

$$t=\frac{8}{4}$$

Simplify the fraction.

$$t=2$$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula t=h4

to find how many seconds it took for the sunglasses to reach the river.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let t = time.
Step 4. Translate into an equation by writing the ap… Su… n inf… $t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$	
Step 5. Solve the equation. $t=\frac{20}{4}$ $t=5$	
Step 6. Check the answer in the problem and make sure it makes sense.	

$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5 \checkmark$	
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula t=h4 to find how many seconds it took for the package to reach the ground.

9 seconds

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula t=h4 to find how many seconds it took for the squeegee to reach the sidewalk.

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

s=24d

After a car accident, the skid marks for one car measured 190 feet. Use the formula s=24d to find the speed of the car before the brakes

were applied. Round your answer to the nearest tenth.

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the [redacted] for the given information.	
Step 5. Solve the equation. [redacted]	
Round to 1 decimal place. [redacted]	

[redacted]

The speed of the car before the brakes were applied
was 67.5 miles per hour.

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = 24d$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

42.7 feet

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula

$s = 24d$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

54.1 feet

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

Key Concepts

- **Binomial Squares**
  $(a+b)2 = a2 + 2ab + b2(a-b)2 = a2 - 2ab + b2$
- **Solve a Radical Equation**

Isolate one of the radical terms on one side of the equation. Raise both sides of the equation to the power of the index. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation. Check the answer in the original equation.

- **Problem Solving Strategy for Applications with Formulas**

  Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information. Identify what we are looking for. Name what we are looking for by choosing a variable to represent it. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. Solve the equation using good algebra techniques. Check the answer in the problem and make sure it makes sense. Answer the question with a complete sentence.

- **Falling Objects**
  - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula t=h4.
- **Skid Marks and Speed of a Car**

- If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s=24d$.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

$5x-6=8$

$x=14$

$4x-3=7$

$5x+1=-3$

---

no solution

$3y-4=-2$

$2x3=-2$

$x = -4$

$4x - 13 = 3$

$2m - 3 - 5 = 0$

$m = 14$

$2n - 1 - 3 = 0$

$6v - 2 - 10 = 0$

$v = 17$

$12u + 1 - 11 = 0$

$4m + 2 + 2 = 6$

$m = 72$

$6n + 1 + 4 = 8$

$2u - 3 + 2 = 0$

---

no solution

$5v - 2 + 5 = 0$

$u - 3 + 3 = u$

$u = 3, u = 4$

$v - 10 + 10 = v$

$r - 1 = r - 1$

$r = 1, r = 2$

$s - 8 = s - 8$

$6x + 43 = 4$

$x = 10$

$11x+43=5$

$4x+53-2=-5$

$x=-8$

$9x-13-1=-5$

$(6x+1)12-3=4$

$x=8$

$(3x-2)12+1=6$

$(8x+5)13+2=-1$

$x=-4$

$(12x-5)13+8=3$

$(12x-3)14-5=-2$

$x = 7$

$(5x - 4)14 + 7 = 9$

$x + 1 - x + 1 = 0$

$x = 3$

$y + 4 - y + 2 = 0$

$z + 100 - z = -10$

$z = 21$

$w + 25 - w = -5$

$32x - 3 - 20 = 7$

$x = 42$

$25x + 1 - 8 = 0$

$28r+1-8=2$

$r=3$

$37y+1-10=8$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

$3u+7=5u+1$

$u=3$

$4v+1=3v+3$

$8+2r=3r+10$

$r=-2$

$10+2c=4c+16$

$5x-13=x+33$

$x = 1$

$8x - 53 = 3x + 53$

$2x2 + 9x - 183 = x2 + 3x - 23$

$x = -8, x = 2$

$x2 - x + 183 = 2x2 - 3x - 63$

$a + 2 = a + 4$

$a = 0$

$r + 6 = r + 8$

$u + 1 = u + 4$

$u = 94$

$x + 1 = x + 2$

$a + 5 - a = 1$

$a = 4$

$-2 = d - 20 - d$

$2x + 1 = 1 + x$

$x = 0x = 4$

$3x + 1 = 1 + 2x - 1$

$2x - 1 - x - 1 = 1$

$x = 1x = 5$

$x + 1 - x - 2 = 1$

$x + 7 - x - 5 = 2$

$x = 9$

$x+5-x-3=2$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=A$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

8.7 feet

Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=A$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=h4$ to find how many seconds it took for the cell phone to reach the ground.

4.7 seconds

Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula t = h4 to find how many seconds it took for the hammer to reach the river.

Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula s = 24d to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

72 feet

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula s = 24d to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

Explain why an equation of the form $x+1=0$ has no solution.

Answers will vary.

ⓐ Solve the equation $r+4-r+2=0$. ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

Use the Complex Number System
By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -7, 0.5–, 73, 3, 81$, list the ⓐ rational numbers, ⓑ irrational numbers, ⓒ real numbers.
   If you missed this problem, review [link].
2. Multiply: $(x-3)(2x+5)$.
   If you missed this problem, review [link].
3. Rationalize the denominator:$55-3$.
   If you missed this problem, review [link].

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a

square root of a negative number we say there is no real number that equals that square root. For example, to simplify -1, we are looking for a real number x so that $x2$ = –1. Since all real numbers squared are positive numbers, there is no real number that equals –1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like 5. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is –1.

Imaginary Unit

The **imaginary unit** *i* is the number whose square is –1.

$i2 = -1 or i = -1$

We will use the imaginary unit to simplify the square roots of negative numbers.

Square Root of a Negative Number

If *b* is a positive real number, then

$-b = bi$

We will use this definition in the next example. Be careful that it is clear that the *i* is not under the radical. Sometimes you will see this written as $-b = ib$ to emphasize the *i* is not under the radical. But the $-b = bi$ is considered standard form.

Write each expression in terms of *i* and simplify if possible:

ⓐ -25 ⓑ -7 ⓒ -12.

ⓐ

-25 Use the definition of the square root ofnegative numbers.$25i$ Simplify.$5i$

ⓑ

-7 Use the definition of the square root ofnegative numbers.7i Simplify.Be careful that it is clear thatiis not under theradical sign.

ⓒ
-12 Use the definition of the square root ofnegative numbers.12i Simplify12.23i

Write each expression in terms of *i* and simplify if possible:

ⓐ -81 ⓑ -5 ⓒ -18.

ⓐ 9i ⓑ 5i ⓒ 32i

Write each expression in terms of *i* and simplify if possible:

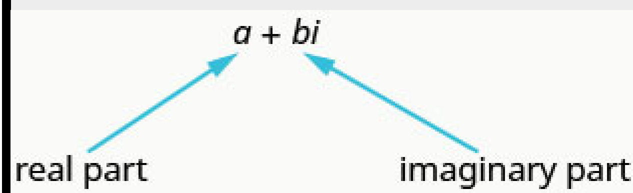
ⓐ -36 ⓑ -3 ⓒ -27.

ⓐ 6i ⓑ 3i ⓒ 33i

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a + bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Complex Number

A **complex number** is of the form $a + bi$, where a and b are real numbers.



A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called

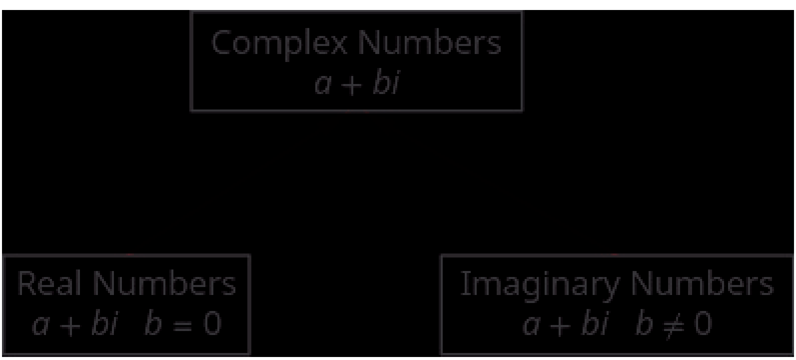
a pure imaginary number.

We summarize this here.

	$a+bi$	
$b=0$	$a+0\cdot ia$	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bibi$	Pure imaginary number

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $-b=bi$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.



Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Add: $-12+-27$.

$-12+-27$ Use the definition of the square root ofnegative numbers.$12i+27i$ Simplify the square roots.$23i+33i$ Add.$53i$

Add: $-8+-32$.

$62i$

Add: $-27+-48$.

$73i$

Remember to add both the real parts and the imaginary parts in this next example.

Simplify: ⓐ $(4-3i)+(5+6i)$ ⓑ $(2-5i)-(5-2i)$.

ⓐ

$(4-3i)+(5+6i)$ Use the Associative Property to put the realparts and the imaginary parts together.$(4+5)+(-3i+6i)$ Simplify.$9+3i$

ⓑ

$(2-5i)-(5-2i)$ Distribute.$2-5i-5+2i$ Use

the Associative Property to put the realparts and the imaginary parts together. $2-5-5i+2i$
Simplify. $-3-3i$

Simplify: ⓐ $(2+7i)+(4-2i)$ ⓑ $(8-4i)-(2-i)$.

ⓐ $6+5i$ ⓑ $6-3i$

Simplify: ⓐ $(3-2i)+(-5-4i)$ ⓑ $(4+3i)-(2-6i)$.

ⓐ $-2-6i$ ⓑ $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Multiply: $2i(7-5i)$.
$2i(7-5i)$ Distribute. $14i-10i2$ Simplify $i2$. $14i-10(-1)$ Multiply. $14i+10$ Write in standard form. $10+14i$

Multiply: $4i(5-3i)$.
$12+20i$

Multiply: $-3i(2+4i)$.

$12+6i$

In the next example, we multiply the binomials using the Distributive Property or FOIL.

Multiply: $(3+2i)(4-3i)$.

$(3+2i)(4-3i)$ Use FOIL. $12-9i+8i-6i^2$ Simplify i^2 and combine like terms. $12-i-6(-1)$ Multiply. $12-i+6$ Combine the real parts. $18-i$

Multiply: $(5-3i)(-1-2i)$.

$-11-7i$

Multiply: $(-4-3i)(2+i)$.

$-5-10i$

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

Multiply: $(3+2i)2$

$$\begin{array}{c} (a+b)^2 \\ (3+2i)^2 \end{array}$$

Use the Product of Binomial Squares Pa... $+b)2=a2+2ab+b2$.

$$\begin{array}{c} a^2+2\ a\ b+b^2 \\ 3^2+2\cdot 3\cdot 2i+(2i)^2 \end{array}$$

Simplify.

[redacted]

Simplify i^2.

[redacted]

Simplify.

[redacted]

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

$-21-20i$

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $-b=bi$. This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Multiply: $-36\cdot-4$.
To multiply square roots of negative numbers, we first write them as complex numbers. $-36\cdot-4$ Write as complex numbers using $-b=bi$.$36i\cdot4i$ Simplify.$6i\cdot2i$ Multiply.$12i2$ Simplify$i2$and multiply.-12

Multiply: $-49\cdot-4$.
-14

Multiply: $-36\cdot-81$.
-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Multiply: $(3--12)(5+-27)$.
To multiply square roots of negative numbers, we first write them as complex numbers. $(3--12)(5+-27)$ Write as complex numbers using $-b=bi.(3-23i)(5+33i)$ Use FOIL.$15+93i-103i-6\cdot3i2$ Combine like terms and simplifyi2.$15-3i-6\cdot(-3)$ Multiply and combine like terms.$33-3i$

Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

$-12-22\sqrt{3}i$

Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Complex Conjugate Pair

A **complex conjugate pair** is of the form $a+bi, a-bi$.

We will multiply a complex conjugate pair in the next example.

Multiply: $(3-2i)(3+2i)$.

$(3-2i)(3+2i)$ Use FOIL. $9+6i-6i-4i^2$
Combine like terms and simplify i^2. $9-4(-1)$
Multiply and combine like terms. 13

Multiply: $(4-3i)\cdot(4+3i)$.

25

Multiply: $(-2+5i)\cdot(-2-5i)$.
29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$. The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$\left(\overset{a}{3}\overset{-}{-}\overset{b}{2i}\right)\left(\overset{a}{3}\overset{+}{+}\overset{b}{2i}\right)$$

$$\overset{a^2}{(3)^2}\overset{-}{-}\overset{b^2}{(2i)^2}$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Notice this is the same result we found in [link].

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a2+b2$

Product of Complex Conjugates
If *a* and *b* are real numbers, then
$(a-bi)(a+bi)=a2+b2$

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

[redacted]

Use the Product of Complex Conjugates Pa[redacted]
$(a$ [redacted]
$+bi)=a2+b2$.

Simplify the squares.

64 + 4

Add.

68

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

109

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

How to Divide Complex Numbers

Divide: $4+3i3-4i$.

Step 1. Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$
Step 3. Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator.	$\frac{12+16i+9i+12i^2}{9+16}$
	Combine like terms.	$\frac{12+25i-12}{25}$
	Simplify.	$\frac{25i}{25}$
	Write the result in standard form.	i

Divide: $2+5i5-2i$.

i

Divide: $1+6i6-i$.

i

We summarize the steps here.

How to divide complex numbers.

Write both the numerator and denominator in standard form. Multiply the numerator and denominator by the complex conjugate of the denominator. Simplify and write the result in standard form.

Divide, writing the answer in standard form: $-35+2i$.

$-35+2i$ Multiply the numerator and denominator by thecomplex conjugate of the denominator. $-3(5-2i)(5+2i)(5-2i)$ Multiply in the numerator and use the Product ofComplex Conjugates Pattern in the denominator. $-15+6i52+22$ Simplify. $-15+6i29$ Write in standard form. $-1529+629i$

Divide, writing the answer in standard form: $41-4i$.

$417+1617i$

Divide, writing the answer in standard form: $-2-1+2i$.

$25+45i$

Be careful as you find the conjugate of the denominator.

Divide: $5+3i4i$.

$5+3i4i$ Write the denominator in standard form.$5+3i0+4i$ Multiply the numerator and denominator bythe complex conjugate of the denominator.$(5+3i)(0-4i)(0+4i)(0-4i)$ Simplify.$(5+3i)(-4i)(4i)(-4i)$ Multiply.$-20i-12i2-16i2$ Simplify thei2.$-20i+1216$ Rewrite in standard form.$1216-2016i$ Simplify the fractions.$34-54i$

Divide: $3+3i2i$.

$32-32i$

Divide: $2+4i5i$.
$45-25i$

Simplify Powers of *i*

The powers of i make an interesting pattern that will help us simplify higher powers of *i*. Let’s evaluate the powers of i to see the pattern.
$i1i2i3i4i-1i2·ii2·i2-1·i(-1)(-1)\ -i1\ i5i6i7i8$
$i4·ii4·i2i4·i3i4·i4\ 1·i1·i21·i31·1\ ii2i31\ \ -1-i$

We summarize this now.
$i1=ii5=i\ i2=-1i6=-1\ i3=-ii7=-i\ i4=1i8=1$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of *i*. Since $i4 = 1$, we rewrite each power, in, as a product using $i4$ to a power and another power of *i*.

We rewrite it in the form $in=(i4)q·ir$, where the exponent, *q*, is the quotient of *n* divided by 4 and the exponent, *r*, is the remainder from this division.

For example, to simplify *i*57, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, 57 = 4·14 + 1. So we write i57 = (14)14·i1 and then simplify from there.

$$\begin{array}{r} 14 \\ 4\overline{)57} \\ 4 \\ \hline 17 \\ 16 \\ \hline 1 \end{array} \qquad \begin{array}{c} i^{57} \\ \\ (1^4)^{14} \cdot i^1 \\ \\ 1 \cdot i \\ \\ i \end{array}$$

Simplify: i86.

i86 Divide 86 by 4 and rewritei86in thein = (i4)q·irform.(14)21·i2

$$\begin{array}{r} 21 \\ 4\overline{)86} \\ 8 \\ \hline 6 \\ 4 \\ \hline 2 \end{array}$$

Simplify.(1)21·(− 1) Simplify.–1

Simplify: i75.

$-i$

Simplify: i92.

1

Access these online resources for additional instruction and practice with the complex number system.

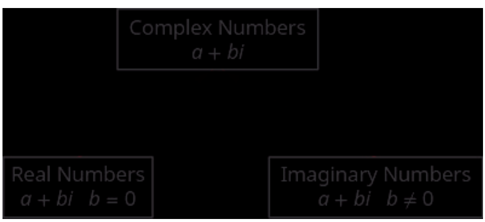
- Expressing Square Roots of Negative Numbers with i
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of i

Key Concepts

- **Square Root of a Negative Number**
  - If *b* is a positive real number, then $-b = bi$

	$a + bi$	
$b = 0$	$a + 0 \cdot ia$	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bibi$	Pure imaginary number

  - A complex number is in **standard form** when written as *a* + *bi*, where *a*, *b* are real numbers.



- **Product of Complex Conjugates**
  - If *a*, *b* are real numbers, then $(a - bi)(a + bi) = a2 + b2$
- **How to Divide Complex Numbers**

Write both the numerator and denominator in

standard form. Multiply the numerator and denominator by the complex conjugate of the denominator. Simplify and write the result in standard form.

Section Exercises

Practice Makes Perfect

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of *i* and simplify if possible.

ⓐ -16 ⓑ -11
ⓒ -8

ⓐ 4i ⓑ 11i ⓒ 22i

ⓐ -121 ⓑ -1 ⓒ -20

ⓐ -100 ⓑ -13 ⓒ -45

ⓐ 10i ⓑ 13i ⓒ 35i

ⓐ -49 ⓑ -15 ⓒ -75

Add or Subtract Complex Numbers In the following exercises, add or subtract.

$-75+-48$

$93i$

$-12+-75$

$-50+-18$

$82i$

$-72+-8$

$(1+3i)+(7+4i)$

$8+7i$

$(6+2i)+(3-4i)$

$(8-i)+(6+3i)$

$14+2i$

$(7-4i)+(-2-6i)$

$(1-4i)-(3-6i)$

$-2+2i$

$(8-4i)-(3+7i)$

$(6+i)-(-2-4i)$

$8+5i$

$(-2+5i)-(-5+6i)$

$(5--36)+(2--49)$

$7-13i$

$(-3+-64)+(5--16)$

$(-7--50)-(-32--18)$

$25-22i$

$(-5+-27)-(-4--48)$

Multiply Complex Numbers

In the following exercises, multiply.

$4i(5-3i)$

$12+20i$

$2i(-3+4i)$

$-6i(-3-2i)$

$-12+18i$

$-i(6+5i)$

$(4+3i)(-5+6i)$

$-38++9i$

$(-2-5i)(-4+3i)$

$(-3+3i)(-2-7i)$

$27+15i$

$(-6-2i)(-3-5i)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

$(3+4i)2$

$-7+24i$

$(-1+5i)2$

$(-2-3i)2$

$-5-12i$

$(-6-5i)2$

In the following exercises, multiply.

$-25\cdot-36$

$30i$

$-4\cdot-16$

$-9\cdot-100$

-30

$-64\cdot-9$

$(-2--27)(4--48)$

$-44+43i$

$(5--12)(-3+-75)$

$(2+-8)(-4+-18)$

$-20-22i$

$(5+-18)(-2--50)$

$(2-i)(2+i)$

5

$(4-5i)(4+5i)$

$(7-2i)(7+2i)$

53

$(-3-8i)(-3+8i)$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

$(7-i)(7+i)$

50

$(6-5i)(6+5i)$

$(9-2i)(9+2i)$

85

$(-3-4i)(-3+4i)$

Divide Complex Numbers

In the following exercises, divide.

$3+4i4-3i$

---

i

$5-2i2+5i$

$2+i3-4i$

$225+1125i$

$3-2i6+i$

$32-3i$

$613+913i$

$24-5i$

$-43-2i$

$-1213-813i$

$-13+2i$

$1+4i3i$

$43-13i$

$4+3i7i$

$-2-3i4i$

$-34+12i$

$-3-5i2i$

Simplify Powers of *i*

In the following exercises, simplify.

$i41$

---

i

$i39$

$i66$

-1

$i48$

$i128$

1

i162

i137

---

i

i255

Writing Exercises

Explain the relationship between real numbers and complex numbers.

Answers will vary.

Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$-7\cdot-7497$

Why is $-64=8i$ but $-643=-4$.

Answers will vary.

Explain how dividing complex numbers is similar to rationalizing a denominator.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Chapter Review Exercises

Simplify Expressions with Roots

Simplify Expressions with Roots

In the following exercises, simplify.

ⓐ 225 ⓑ -16

ⓐ 15 ⓑ -4

ⓐ -169 ⓑ -8

ⓐ 83 ⓑ 814 ⓒ 2435

ⓐ 2 ⓑ 3 ⓒ 3

ⓐ -5123 ⓑ -814 ⓒ -15

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

ⓐ 68 ⓑ 843

ⓐ $8<68<9$
ⓑ $4<843<5$

In the following exercises, approximate each root

and round to two decimal places.

ⓐ 37 ⓑ 843 ⓒ 1254

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

ⓐ a33
ⓑ b77

ⓐ *a* ⓑ |b|

ⓐ a14
ⓑ w24

ⓐ m84
ⓑ n205

ⓐ m2 ⓑ n4

ⓐ 121m20
ⓑ −64a2

ⓐ 216a63
ⓑ 32b205

ⓐ 6a2 ⓑ 2b4

ⓐ 144x2y2
ⓑ 169w8y10
ⓒ 8a51b63

Simplify Radical Expressions

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

125

55

675

ⓐ 6253 ⓑ 1286

ⓐ 553 ⓑ 226

In the following exercises, simplify using absolute value signs as needed.

ⓐ a23
ⓑ b83
ⓒ c138

ⓐ 80s15
ⓑ 96a75
ⓒ 128b76

ⓐ 4|s7|5s ⓑ 2a3a25
ⓒ 2|b|2b6

ⓐ 96r3s3
ⓑ 80x7y63
ⓒ 80x8y94

ⓐ -325
ⓑ -18

ⓐ -2 ⓑ not real

ⓐ $8+96$
ⓑ $2+402$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

ⓐ 7298 ⓑ 24813 ⓒ 6964

ⓐ 67 ⓑ 23 ⓒ 12

ⓐ y4y8 ⓑ u21u115 ⓒ v30v126

300m564

10m23m8

ⓐ 28p7q2
ⓑ 81s8t33
ⓒ 64p15q124

ⓐ 27p2q108p4q3
ⓑ 16c5d7250c2d23
ⓒ 2m9n7128m3n6

ⓐ 12|pq| ⓑ 2cd2d255
ⓒ |mn|262

ⓐ 80q55q
ⓑ −625353
ⓒ 80m745m4

Simplify Rational Exponents

Simplify expressions with a1n

In the following exercises, write as a radical expression.

ⓐ r12 ⓑ s13 ⓒ t14

ⓐ r ⓑ s3 ⓒ t4

In the following exercises, write with a rational exponent.

ⓐ 21p ⓑ 8q4 ⓒ 436r6

In the following exercises, simplify.

ⓐ 62514
ⓑ 24315
ⓒ 3215

ⓐ 5 ⓑ 3 ⓒ 2

ⓐ $(-1{,}000)13$
ⓑ $-1{,}00013$
ⓒ $(1{,}000)-13$

ⓐ $(-32)15$
ⓑ $(243)-15$
ⓒ -12513

ⓐ −2 ⓑ 13 ⓒ −5

Simplify Expressions with amn

In the following exercises, write with a rational exponent.

ⓐ r74
ⓑ (2pq5)3
ⓒ (12m7n)34

In the following exercises, simplify.

ⓐ 2532
ⓑ 9−32
ⓒ (−64)23

ⓐ 125 ⓑ 127 ⓒ 16

ⓐ −6432
ⓑ −64−32
ⓒ (−64)32

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

ⓐ 652·612
ⓑ (b15)35
ⓒ w27w97

ⓐ 63 ⓑ b9 ⓒ 1w

ⓐ a34·a − 14a − 104
ⓑ (27b23c − 52b − 73c12)13

Add, Subtract and Multiply Radical Expressions

Add and Subtract Radical Expressions

In the following exercises, simplify.

ⓐ 72 − 32
ⓑ 7p3 + 2p3
ⓒ 5x3 − 3x3

ⓐ 42 ⓑ 9p3 ⓒ 2x3

ⓐ 11b − 511b + 311b
ⓑ 811cd4 + 511cd4 − 911cd4

ⓐ 48 + 27
ⓑ 543 + 1283
ⓒ 654 − 323204

ⓐ 73 ⓑ 723 ⓒ 354

ⓐ 80c7 − 20c7
ⓑ 2162r104 + 432r104

375y2 + 8y48 − 300y2

37y3

Multiply Radical Expressions

In the following exercises, simplify.

ⓐ (56)(− 12)
ⓑ (− 2184)(− 94)

ⓐ (32x3)(718x2)
ⓑ (−620a23)(−216a33)

ⓐ 126x22 ⓑ 48aa23

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

ⓐ 11(8+411)
ⓑ 33(93+183)

ⓐ (3−27)(5−47)
ⓑ (x3−5)(x3−3)

ⓐ 71−227
ⓑ x23−8x3+15

(27−511)(47+911)

ⓐ (4+11)2

ⓑ $(3-25)2$

ⓐ $27+811$ ⓑ $29-125$

$(7+10)(7-10)$

$(3x3+2)(3x3-2)$

$9x23-4$

Divide Radical Expressions

Divide Square Roots

In the following exercises, simplify.

ⓐ 4875
ⓑ 813243

ⓐ $320mn-545m-7n3$
ⓑ $16x4y-23-54x-2y43$

ⓐ $8m43n4$ ⓑ $-x22y2$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

ⓐ 83 ⓑ 740 ⓒ 82y

ⓐ 1113 ⓑ 7543 ⓒ 33x23

ⓐ 121311 ⓑ 2836 ⓒ 9x3x

ⓐ 144 ⓑ 9324 ⓒ 69x34

Rationalize a Two Term Denominator

In the following exercises, simplify.

$72-6$

$-7(2+6)2$

$5n-7$

$x+8x-8$

$(x+22)x-82$

Solve Radical Equations

Solve Radical Equations

In the following exercises, solve.

$4x-3=7$

$5x+1=-3$

no solution

$4x-13=3$

$u-3+3=u$

$u=3, u=4$

$4x+53-2=-5$

$(8x+5)13+2=-1$

$x = -4$

$y + 4 - y + 2 = 0$

$28r + 1 - 8 = 2$

$r = 3$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

$10 + 2c = 4c + 16$

$2x2 + 9x - 183 = x2 + 3x - 23$

$x = -8, x = 2$

$r + 6 = r + 8$

$x + 1 - x - 2 = 1$

$x = 3$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = A$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = 24d$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

64.8 feet

Use Radicals in Functions

Evaluate a Radical Function

In the following exercises, evaluate each function.

$g(x)=6x+1$, find
ⓐ $g(4)$
ⓑ $g(8)$

$G(x)=5x-1$, find
ⓐ $G(5)$
ⓑ $G(2)$

ⓐ $G(5)=26$ ⓑ $G(2)=3$

$h(x)=x2-43$, find
ⓐ $h(-2)$
ⓑ $h(6)$

For the function
$g(x)=4-4x4$, find
ⓐ $g(1)$
ⓑ $g(-3)$

ⓐ $g(1)=0$ ⓑ $g(-3)=2$

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

$g(x) = 2 - 3x$

$F(x) = x + 3x - 2$

$(2, \infty)$

$f(x) = 4x2 - 163$

$F(x) = 10 - 7x4$

$[710, \infty)$

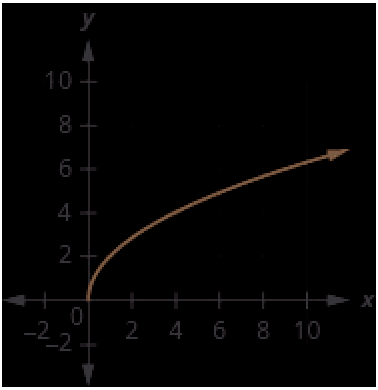
Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

$g(x) = x + 4$

$g(x) = 2x$

ⓐ domain: $[0, \infty)$
ⓑ

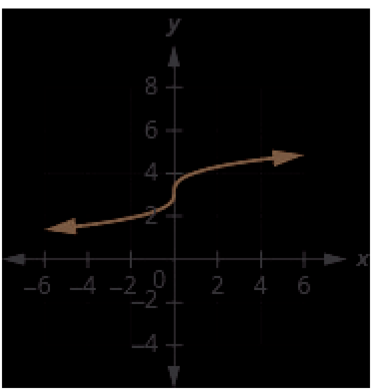


ⓒ range: $[0, \infty)$

$f(x) = x - 13$

$f(x) = x3 + 3$

ⓐ domain: $(-\infty, \infty)$
ⓑ



ⓒ range: $(-\infty, \infty)$

Use the Complex Number System

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of *i* and simplify if possible.

ⓐ -100
ⓑ -13
ⓒ -45

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

$-50 + -18$

$82i$

$(8 - i) + (6 + 3i)$

$(6 + i) - (-2 - 4i)$

$8+5i$

$(-7-\sqrt{-50})-(-32-\sqrt{-18})$

Multiply Complex Numbers

In the following exercises, multiply.

$(-2-5i)(-4+3i)$

$23+14i$

$-6i(-3-2i)$

$\sqrt{-4}\cdot\sqrt{-16}$

-6

$(5-\sqrt{-12})(-3+\sqrt{-75})$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

$(-2-3i)^2$

$-5-12i$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

$(9-2i)(9+2i)$

Divide Complex Numbers

In the following exercises, divide.

$2+i3-4i$

$225+1125i$

$-43-2i$

Simplify Powers of *i*

In the following exercises, simplify.

$i48$

1

$i255$

Practice Test

In the following exercises, simplify using absolute values as necessary.

125x93

5x3

169x8y6

72x8y43

2x2y9x2y3

45x3y4180x5y2

In the following exercises, simplify. Assume all variables are positive.

ⓐ 216 − 14 ⓑ − 4932

ⓐ 14 ⓑ − 343

-45 $x-14 \cdot x54x-34$
$x74$ $(8x23y-52x-73y12)13$ $48x5-75x5$
$-x23x$ $27x2-4x12+108x2$ $212x5 \cdot 36x3$
$36x42$ $43(163-63)$ $(4-33)(5+23)$

$2-73$

1283543

$245xy-445x-4y3$

$7x53y7$

153

$32+3$

$3(2-3)$

$-4\cdot-9$

$-4i(-2-3i)$

$-12+8i$

$4+i3-2i$

$i172$

$-i$

In the following exercises, solve.

$2x+5+8=6$

$x+5+1=x$

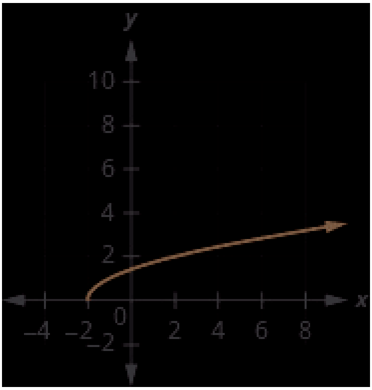
$x=4$

$2x2-6x-233=x2-3x+53$

In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

$g(x)=x+2$

ⓐ domain: $[-2,\infty)$
ⓑ



ⓒ range: $[0, \infty)$

Glossary

complex conjugate pair
: A complex conjugate pair is of the form $a + bi$, $a - bi$.

complex number
: A complex number is of the form $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system
: The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit
: The imaginary unit i is the number whose

square is –1. $i^2 = -1$ or $i = -1$.

standard form

A complex number is in standard form when written as $a + bi$, where *a, b* are real numbers.

Solve Quadratic Equations Using the Square Root Property

By the end of this section, you will be able to:

- Solve quadratic equations of the form ax2 = k using the Square Root Property
- Solve quadratic equations of the form a(x–h)2 = k using the Square Root Property

Before you get started, take this readiness quiz. Simplify: 128. If you missed this problem, review [link].
82

Simplify: 325. If you missed this problem, review [link].
4105

Factor: $9x^2-12x+4$.
If you missed this problem, review [link].

$(3x-2)^2$

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2=k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

		$x^2=9$
Put the equation in standard form.		$x^2-9=0$
Factor the difference of squares.		$(x-3)(x+3)=0$
Use the Zero Product Property.		$x-3=0 \quad x-3=0$
Solve each equation.		$x=3 \quad x=-3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x=4, x=-4$ and $x=5, x=-5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2=m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root**

Property.

> Square Root Property
> If $x^2 = k$, then
> $x = k \text{or} x = -k \text{or} x = \pm k.$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm k$. We read this as *x* equals positive or negative the square root of *k*.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm 9$
	$x = \pm 3$
	So $x = 3 \text{or} x = -3.$

What happens when the constant is not a perfect

square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = 7, x = -7$

We cannot simplify 7, so we leave the answer as a radical.

How to solve a Quadratic Equation of the form $ax^2 = k$ Using the Square Root Property

Solve: $x^2 - 50 = 0$.

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$

Step 2. Use Square Root Property.	Remember to write the $\pm$ symbol.	$x = \pm\sqrt{50}$

Step 3. Simplify the radical.		$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$
	Rewrite to show two solutions.	$x = 5\sqrt{2},\ x = -5\sqrt{2}$

Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $\left(5\sqrt{2}\right)^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^2 - 50 = 0$ $\left(-5\sqrt{2}\right)^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Solve: $x2 - 48 = 0.$

$x = 43, x = -43$

Solve: $y2 - 27 = 0.$

$y = 33, y = -33$

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

Solve a quadratic equation using the square root property.

Isolate the quadratic term and make its coefficient one. Use Square Root Property. Simplify the radical. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Solve: $3z2=108$.

	$3z2=108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$3z23=1083$
Simplify.	$z2=36$
Use the Square Root Property.	$z=\pm 36$
Simplify the radical.	$z=\pm 6$
Rewrite to show two solutions.	$z=6, z=-6$
Check the solutions:	

$$\begin{aligned} 3z^2 &= 108 \\ 3(6)^2 &\stackrel{?}{=} 108 \\ 3(36) &\stackrel{?}{=} 108 \\ 108 &= 108\ \checkmark \end{aligned} \qquad \begin{aligned} 3z^2 &= 108 \\ 3(-6)^2 &\stackrel{?}{=} 108 \\ 3(36) &\stackrel{?}{=} 108 \\ 108 &= 108\ \checkmark \end{aligned}$$

Solve: $2x2=98$.

$x=7, x=-7$

Solve: $5m^2=80$.
$m=4, m=-4$

The Square Root Property states 'If $x^2=k$,' What will happen if $k<0$? This will be the case in the next example.

Solve: $x^2+72=0$.

	$x^2+72=0$
Isolate the quadratic term.	$x^2=-72$
Use the Square Root Property.	$x=\pm\sqrt{-72}$
Simplify using complex numbers.	$x=\pm\sqrt{72}i$

Simplify the radical.	x = ±62i
Rewrite to show two solutions.	x = 62i, x = −62i
Check the solutions:	

$$\begin{array}{rr} x^2+72=0 & x^2+72=0 \\ \left(6\sqrt{2}\,i\right)^2+72\overset{?}{=}0 & \left(6\sqrt{2}\,i\right)^2+72\overset{?}{=}0 \\ 6^2\left(\sqrt{2}\right)^2 i^2+72\overset{?}{=}0 & (-6)^2\left(\sqrt{2}\right)^2 i^2+72\overset{?}{=}0 \\ 36\cdot 2\cdot(-1)+72\overset{?}{=}0 & 36\cdot 2\cdot(-1)+72\overset{?}{=}0 \\ 0=0\,\checkmark & 0=0\,\checkmark \end{array}$$

Solve: c2 + 12 = 0.
c = 23i, c = −23i

Solve: q2 + 24 = 0.
c = 26i, c = −26i

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Solve: 23u2 + 5 = 17.

23u2 + 5 = 17

Isolate the quadratic term.

[redacted]

Multiply by 32 to make the coefficient 1.

[redacted]

Simplify.

[redacted]

Use the Square Root Property.

[redacted]

Simplify the radical.

$$u = \pm\sqrt{9 \cdot 2}$$

Simplify.

$$u = \pm 3\sqrt{2}$$

Rewrite to show two solutions.

$$u = 3\sqrt{2}, \quad u = -3\sqrt{2}$$

Check:

$$\frac{2}{3}u^2 + 5 = 17 \qquad \frac{2}{3}u^2 + 5 = 17$$

$$\frac{2}{3}\left(3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17 \qquad \frac{2}{3}\left(-3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17$$

$$\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17 \qquad \frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$$

$$12 + 5 \stackrel{?}{=} 17 \qquad 12 + 5 \stackrel{?}{=} 17$$

$$17 = 17 \checkmark \qquad 17 = 17 \checkmark$$

Solve: 12x2 + 4 = 24.

x = 210, x = − 210

Solve: $34y2-3=18$.

$y=27, y=-27$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

Solve: $2x2-8=41$.

[redacted]

Isolate the quadratic term.

[redacted]

Divide by 2 to make the coefficient 1.

[redacted]

Simplify.

[redacted]

Use the Square Root Property.

[redacted]

Rewrite the radical as a fraction of square ro[redacted]

Rationalize the denominator.

[redacted]

Simplify.

[redacted]

Rewrite to show two solutions.

[redacted]

Check:
We leave the check for you.

Solve: $5r2 - 2 = 34$.

$r=655,r=-655$

Solve: $3t2+6=70$.
$t=833,t=-833$

Solve Quadratic Equations of the Form $a(x - h)2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)2 = k$ as well. Notice that the quadratic term, x, in the original form $ax2 = k$ is replaced with $(x - h)$.

$ax^2 = k \qquad a(x-h)^2 = k$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the

Square Root Property can be used on $(x - h)2$.

Solve: $4(y-7)2=48$.

	$4(y-7)2=48$
Divide both sides by the coefficient 4.	$(y-7)2=12$
Use the Square Root Property on the binomial	$y-7=\pm 12$
Simplify the radical.	$y-7=\pm 23$
Solve for y.	$y=7\pm 23$
Rewrite to show two solutions.	$y=7+23,\ y=7-23$
Check:	

$$4(y-7)^2=48 \qquad 4(y-7)^2=48$$
$$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48 \qquad 4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$$
$$4(2\sqrt{3})^2 \stackrel{?}{=} 48 \qquad 4(-2\sqrt{3})^2 \stackrel{?}{=} 48$$
$$4(12) \stackrel{?}{=} 48 \qquad 4(12) \stackrel{?}{=} 48$$
$$48=48\ \checkmark \qquad 48=48\ \checkmark$$

Solve: $3(a-3)2=54$.

$a=3+32, a=3-32$

Solve: $2(b+2)2=80$.

$b=-2+210, b=-2-210$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

Solve: $(x-13)2=59$.

	$(x-13)2=59$
Use the Square Root Property.	$x-13=\pm 59$
Rewrite the radical as a fraction of square roots.	$x-13=\pm 59$
Simplify the radical.	$x-13=\pm 53$
Solve for x.	$x=13\pm 53$
Rewrite to show two solutions.	$x=13+53, x=13-53$
Check:	
We leave the check for you.	

Solve: $(x-12)2=54$.

$x=12+52, x=12-52$

Solve: $(y+34)2=716$.

$y = -34+74, y = -34-74$

We will start the solution to the next example by isolating the binomial term.

Solve: $2(x-2)2+3=57$.

	$2(x-2)2+3=57$
Subtract 3 from both sides to isolate the binomial term.	$2(x-2)2=54$
Divide both sides by 2.	$(x-2)2=27$
Use the Square Root Property.	$x-2=\pm 27$
Simplify the radical.	$x-2=\pm 33$
Solve for x.	$x=2\pm 33$
Rewrite to show two solutions.	$x=2+33, x=2-33$
Check:	
We leave the check for	

you.

Solve: $5(a-5)2+4=104$.

$a=5+25, a=5-25$

Solve: $3(b+3)2-8=88$.

$b=-3+42, b=-3-42$

Sometimes the solutions are complex numbers.

Solve: $(2x-3)2=-12$.

	$(2x-3)2=-12$
Use the Square Root Property.	$2x-3=\pm -12$
Simplify the radical.	$2x-3=\pm 23i$
Add 3 to both sides.	$2x=3\pm 23i$
Divide both sides by 2.	$x=3\pm 23i2$
Rewrite in standard form.	$x=32\pm 23i2$
Simplify.	$x=32\pm 3i$
Rewrite to show two solutions.	$x=32+3i, x=32-3i$
Check: We leave the check for you.	

Solve: $(3r+4)2=-8$.

$r=-43+22i3, r=-43-22i3$

Solve: $(2t-8)2=-10$.

$t=4+10i2, t=4-10i2$

The left sides of the equations in the next two examples do not seem to be of the form $a(x - h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Solve: $4n^2+4n+1=16$.
We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm16$
Simplify the radical.	$2n+1=\pm4$
Solve for n.	$2n=-1\pm4$

Divide each side by 2.	2n2 = −1 ± 42 n = −1 ± 42
Rewrite to show two solutions.	n = −1 + 42, n = −1 − 42
Simplify each equation.	n = 32, n = −52
Check:	

$$4n^2+4n+1=16 \qquad\qquad 4n^2+4n+1=16$$

$$4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\overset{?}{=}16 \qquad 4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\overset{?}{=}16$$

$$4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\overset{?}{=}16 \qquad 4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\overset{?}{=}16$$

$$9+6+1\overset{?}{=}16 \qquad 25-10+1\overset{?}{=}16$$

$$16=16\checkmark \qquad 16=16\checkmark$$

Solve: 9m2 − 12m + 4 = 25.
m = 73, m = −1

Solve: 16n2 + 40n + 25 = 4.

$n = -34, n = -74$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property
- Using the Square Root Property to Solve Quadratic Equations

Key Concepts

- Square Root Property
  - If $x2 = k$, then $x = k or x = -k$ or $x = \pm k$

  How to solve a quadratic equation using the square root property.

  Isolate the quadratic term and make its coefficient one. Use Square Root Property. Simplify the radical. Check the solutions.

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

$a2 = 49$

$a = \pm 7$

$b2 = 144$

$r2 - 24 = 0$

$r = \pm 26$

$t2 - 75 = 0$

$u2 - 300 = 0$

$u = \pm 103$

$v2 - 80 = 0$

$4m2 = 36$

$m = \pm 3$

$3n2 = 48$

$43x2 = 48$

$x = \pm 6$

$53y2 = 60$

$x2 + 25 = 0$

$x = \pm 5i$

$y2 + 64 = 0$

$x2 + 63 = 0$

$x = \pm 37i$

$y2 + 45 = 0$

$43x2 + 2 = 110$

$x = \pm 9$

$23y2 - 8 = -2$

$25a2 + 3 = 11$

$a = \pm 25$

$32b2 - 7 = 41$

$7p2 + 10 = 26$

$p = \pm 477$

$2q2 + 5 = 30$

$5y2-7=25$

$y=\pm 4105$

$3x2-8=46$

Solve Quadratic Equations of the Form $a(x-h)2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

$(u-6)2=64$

$u=14, u=-2$

$(v+10)2=121$

$(m-6)2=20$

$m=6\pm 25$

$(n+5)2=32$

$(r-12)2=34$

$r = 12 \pm 32$
$(x + 15)2 = 725$
$(y + 23)2 = 881$
$y = -23 \pm 229$
$(t - 56)2 = 1125$
$(a - 7)2 + 5 = 55$
$a = 7 \pm 52$
$(b - 1)2 - 9 = 39$
$4(x + 3)2 - 5 = 27$
$x = -3 \pm 22$
$5(x + 3)2 - 7 = 68$

$(5c+1)2=-27$

$c=-15\pm 335i$

$(8d-6)2=-24$

$(4x-3)2+11=-17$

$x=34\pm 72i$

$(2y+1)2-5=-23$

$m2-4m+4=8$

$m=2\pm 22$

$n2+8n+16=27$

$x2-6x+9=12$

$x=3\pm 23$

$y2+12y+36=32$

$25x2-30x+9=36$

$x=-35,x=95$

$9y2+12y+4=9$

$36x2-24x+4=81$

$x=-76,x=116$

$64x2+144x+81=25$

Mixed Practice

In the following exercises, solve using the Square Root Property.

$2r2=32$

$r=\pm 4$

$4t2=16$

$(a-4)2=28$

$a=4\pm 27$

$(b+7)2=8$

$9w2-24w+16=1$

$w=1, w=53$

$4z2+4z+1=49$

$a2-18=0$

$a=\pm 32$

$b2-108=0$

$(p-13)2=79$

$p=13\pm 73$

$(q-35)2=34$

$m2+12=0$

$m=\pm 23i$

$n2+48=0.$

$u2-14u+49=72$

$u=7\pm 62$

$v2+18v+81=50$

$(m-4)2+3=15$

$m=4\pm 23$

$(n-7)2-8=64$

$(x+5)2=4$

$x=-3, x=-7$

$(y-4)2=64$

$6c2+4=29$

$c=\pm 566$

$2d2-4=77$

$(x-6)2+7=3$

$x=6\pm 2i$

$(y-4)2+10=9$

Writing Exercises

In your own words, explain the Square Root Property.

Answers will vary.

In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)2=16$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x - h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes

in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Whom can you ask for help?Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Solve Quadratic Equations by Completing the Square

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax2+bx+c=0$ by completing the square

Before you get started, take this readiness quiz.

Expand: $(x+9)2$.
If you missed this problem, review [link].

$x2+18x+81$

Factor $y2-14y+49$.
If you missed this problem, review [link].

$y-72$

Factor $5n2+40n+80$.
If you missed this problem, review [link].

$5n+42$

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.
$(y-7)2=12y-7=\pm 12y-7=\pm 23y=7\pm 23$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)2$ in order to use the Square Root Property.
$x2-10x+25=18(x-5)2=18$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$(x+9)^2 \quad (y-7)^2$$
$$(x+9)(x+9) \quad (y-7)(y-7)$$
$$x^2+9x+9x+81 \quad y^2-7y-7y+49$$
$$x^2+18x+81 \quad y^2-14y+49$$

We restate the patterns here for reference.

Binomial Squares Pattern

If *a* and *b* are real numbers,

$$(a+b)^2 = a^2+2ab+b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2-2ab+b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to "make" a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + \_\_ \end{array}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of $x^2 + 6x$ is a square, x^2. This tells us that $a = x$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 2 \cdot x \cdot b + b^2 \end{array}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 2 \cdot 3 \cdot x + \_\_ \end{array}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2 = 9$.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + 9 \end{array}$$

We can now factor.

$$\begin{array}{c} (a + b)^2 \\ (x + 3)^2 \end{array}$$

So we found that adding 9 to $x^2 + 6x$ ‘completes the square’, and we write it as $(x + 3)^2$.

Complete a square of $x2 + bx$.

Identify *b*, the coefficient of *x*. Find (12b)2, the number to complete the square. Add the (12b)2 to *x*2 + *bx*. Factor the perfect square trinomial, writing it as a binomial squared.

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x2 - 26x$ ⓑ $y2 - 9y$ ⓒ $n2 + 12n$

ⓐ

[redacted]

The coefficient of x is -26.

Find(12b)2.

$(12\cdot(-26))2(13)2169$

Add 169 to the binomial to complete the[redacted]

Factor the perfect square trinomial, wr[redacted]
a binomial squared.

ⓑ

[redacted]

The coefficient of y is -9.

Find(12b)2.

(12·(−9))2(−92)2814

Add 814 to the binomial to complete the[redacted]

Factor the perfect square trinomial, wr[redacted]
a binomial squared.

ⓒ

$x^2 + bx$

$n^2 + \frac{1}{2}n$

The coefficient of n is 12.

Find(12b)2.

(12·12)2(14)2116

Add 116 to the binomial to complete the

$n^2 + \frac{1}{2}n + \frac{1}{16}$

Rewrite as a binomial square.

$\left(n + \frac{1}{4}\right)^2$

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ a2−20a ⓑ m2−5m ⓒ p2+14p

ⓐ (a−10)2 ⓑ (b−52)2

ⓒ (p+18)2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $b^2 - 4b$ ⓑ $n^2 + 13n$ ⓒ $q^2 - 23q$

ⓐ $(b - 2)^2$ ⓑ $(n + 132)^2$
ⓒ $(q - 13)^2$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

$x^2 + 6x = 40$

$x^2 + 6x + \_\_ = 40 + \_\_$

$x^2 + 6x + 9 = 40 + 9$

Add 9 to both sides to complete the square.

$(x + 3)^2 = 49$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

How to Solve a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square

Solve by completing the square: $x^2 + 8x = 48$.

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2 + 8x = 48$

Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2} \cdot 8\right)^2}{\underline{\quad}} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$. Substitute $x = -12$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48\ ✓$ $x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48\ ✓$

Solve by completing the square: x2 + 4x = 5.

$x = -5, x = -1$

Solve by completing the square: $y2 - 10y = -9$.

$y = 1, y = 9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a quadratic equation of the form $x2 + bx + c = 0$ by completing the square.

Isolate the variable terms on one side and the constant terms on the other. Find $(12 \cdot b)2$, the number needed to complete the square. Add it to both sides of the equation. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right Use the Square Root Property. Simplify the radical and then solve the two resulting equations. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Solve by completing the square: $x2+4x=-21$.

[redacted]

The variable terms are on the [redacted] and square it. $(12(4))2=4$

Add 4 to both sides.

[redacted]

Factor the perfect square tri[redacted] writing it as a binomial

squared.	
Use the Square Root Property.	[redacted]
Simplify using complex numbers.	[redacted]
Subtract 2 from each side.	[redacted]
Rewrite to show two sol	[redacted]
We leave the check to you.	

Solve by completing the square: y2 − 10y = −35.

y = 5 ± 10i

Solve by completing the square: z2 + 8z = −19.

$z=-4+3i, z=-4-3i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Solve by completing the square: $y^2-18y=-6$.

[redacted]

The variable terms are on the left side.

Take half of -18 and square it.

$\left(\frac{1}{2}(-18)\right)^2=81$

[redacted]

Add 81 to both sides.

[redacted]

Factor the perfect square trinomial, wr[redacted] squared.

Use the Square Root Property.

[redacted]

Simplify the radical.

[redacted]

Solve for y.

[redacted]

Check.

[redacted]

Another way to check this would be to use a calculator. Evaluate y2 − 18yfor both of the solutions. The answer should be −6.

Solve by completing the square: $x2 - 16x = -16$.

$x = 8 + 43, x = 8 - 43$

Solve by completing the square: $y2 + 8y = 11$.

$y = -4 + 33, y = -4 - 33$

We will start the next example by isolating the variable terms on the left side of the equation.

Solve by completing the square: $x2 + 10x + 4 = 15$.

[redacted]

Isolate the variable terms on the left side. Su[redacted] constant terms on the right side.

Take half of 10 and square it.

$(12(10))2=25$

[redacted]

Add 25 to both sides.

[redacted]

Factor the perfect square trinomial, wr[redacted] a binomial squared.

Use the Square Root Property.

[redacted]

Simplify the radical.

[redacted]

Solve for x.

[redacted]

Rewrite to show two solutions.

[redacted]

Solve the equations.

[redacted]

Check:

[redacted]

Solve by completing the square: $a^2+4a+9=30$.

$a=-7, a=3$

Solve by completing the square: $b^2+8b-4=16$.

$b=-10, b=2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Solve by completing the square: n2 = 3n + 11.

[redacted]

Subtract 3n to get the variable terms on the left side.

[redacted]

Take half of −3 and square it.

(12(−3))2 = 94

[redacted]

Add 94 to both sides.

[redacted]

Factor the perfect square tri[redacted] writing it as a binomial squared.

Add the fractions on the rig[redacted]

Use the Square Root Property.

[redacted]

Simplify the radical.

[redacted]

Solve for *n*.

[redacted]

Rewrite to show two sol[redacted]

Check:
We leave the check for you!

Solve by completing the square: $p2=5p+9$.
$p=52+612, p=52-612$

Solve by completing the square: $q2=7q-3$.
$q=72+372, q=72-372$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says "If $a \cdot b=0$, then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Solve by completing the square: $(x-3)(x+5)=9$.

[redacted]

We multiply the binomials on the left.

[redacted]

Add 15 to isolate the constant terms on the right.

[redacted]

Take half of 2 and square it. $(12\cdot(2))2 = 1$

[redacted]

Add 1 to both sides.

[redacted]

Factor the perfect square trinomial, writing it as a binomial squared.

[redacted]

Use the Square Ro[redacted] Property. Solve for x.

[redacted]

Rewrite to show two so[redacted] Simplify.

[redacted]

Check: We leave the check for you!

Solve by completing the square: $(c-2)(c+8)=11$.

$c=-9, c=3$

Solve by completing the square: $(d-7)(d+3)=56$.

$d=11, d=-7$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Solve by completing the square: $3x^2-12x-15=0$.

To complete the square, we need the coefficient of x2 to be one. If we factor out the coefficient of x2 as a common factor, we can continue with solving the equation by completing the square.

[redacted]

Factor out the greatest common factor.

[redacted]

Divide both sides by 3 to isolate the trinomial wi[redacted]

Simplify.

[redacted]

Add 5 to get the constant terms on the [redacted]

Take half of 4 and square it.

$(12(-4))2=4$

[redacted]

Add 4 to both sides.

[redacted]

Factor the perfect square trinomial, wr[redacted] as a binomial squared.

Use the Square Root Property.

[redacted]

Solve for x.

[redacted]

Rewrite to show two solutions.

[redacted]

Simplify.

[redacted]

Check:

[redacted]

Solve by completing the square: $2m^2+16m+14=0$.
$m=-7, m=-1$

Solve by completing the square: $4n^2-24n-56=8$.
$n=-2, n=8$

To complete the square, the coefficient of the x^2 must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Solve by completing the square: $2x^2-3x=20$.

To complete the square we need the coefficient of x2 to be one. We will divide both sides of the equation by the coefficient of $x2$. Then we can continue with solving the equation by completing the square.

[redacted]

Divide both sides by 2 to ge[redacted] coefficient of x2 to be 1.

Simplify.

[redacted]

Take half of -32 and square it.

$(12(-32))2 = 916$

[redacted]

Add 916 to both sides.

[redacted]

Factor the perfect square tri[redacted]
writing it as a binomial squared.

Add the fractions on the rig[redacted]

Use the Square Root Property.

[redacted]

Simplify the radical.

[redacted]

Solve for x.

[redacted]

Rewrite to show two sol[redacted]

Simplify.

[redacted]

Check:
We leave the check for you!

Solve by completing the square: $3r2-2r=21$.
$r=-73,r=3$

Solve by completing the square: $4t2+2t=20$.
$t=-52,t=2$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

Solve a quadratic equation of the form $ax2+bx+c=0$ by completing the square.

Divide by a to make the coefficient of x^2 term 1. Isolate the variable terms on one side and the constant terms on the other. Find $(12\cdot b)2$, the

number needed to complete the square. Add it to both sides of the equation. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right Use the Square Root Property. Simplify the radical and then solve the two resulting equations. Check the solutions.

Solve by completing the square: $3x2+2x=4$.

Again, our first step will be to make the coefficient of *x*2 one. By dividing both sides of the equation by the coefficient of *x*2, we can then continue with solving the equation by completing the square.

[redacted]

Divide both sides by 3 to ma[redacted]

coefficient of x2 equal 1. Simplify.

[redacted]

Take half of 23 and square it. (12·23)2 = 19

[redacted]

Add 19 to both sides.

[redacted]

Factor the perfect square tri[redacted] writing it as a binomial squared.

Use the Square Root Property.

[redacted]

Simplify the radical.

[redacted]

Solve for x .

[redacted]

Rewrite to show two

sol $x=-\frac{1}{3}+\frac{\sqrt{13}}{3}, x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$

Check:
We leave the check for you!

Solve by completing the square: $4x2+3x=2$.
$x=-38+418, x=-38-418$

Solve by completing the square: $3y2-10y=-5$.
$y=53+103, y=53-103$

Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

Key Concepts

- Binomial Squares Pattern
  If *a* and *b* are real numbers,

$(a+b)^2 = a^2+2ab+b^2$	$(a+b)^2$	=	a^2	+	$2ab$	+	b^2
	$(\text{binomial})^2$		$(\text{first term})^2$		2 • (product of terms)		$(\text{second term})^2$
$(a-b)^2 = a^2-2ab+b^2$	$(a-b)^2$	=	a^2	−	$2ab$	+	b^2
	$(\text{binomial})^2$		$(\text{first term})^2$		2 • (product of terms)		$(\text{second term})^2$

- How to Complete a Square

  Identify *b*, the coefficient of *x*. Find (12b)2, the number to complete the square. Add the (12b)2 to $x2 + bx$ Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form

$ax^2 + bx + c = 0$ by completing the square.

Divide by a to make the coefficient of x^2 term 1. Isolate the variable terms on one side and the constant terms on the other. Find $(\frac{1}{2}\cdot b)^2$, the number needed to complete the square. Add it to both sides of the equation. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right. Use the Square Root Property. Simplify the radical and then solve the two resulting equations. Check the solutions.

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ m^2-24m ⓑ x^2-11x ⓒ p^2-13p

ⓐ $(m-12)^2$ ⓑ $(x-\frac{11}{2})^2$
ⓒ $(p-\frac{1}{6})^2$

ⓐ $n2-16n$ ⓑ $y2+15y$ ⓒ $q2+34q$

ⓐ $p2-22p$ ⓑ $y2+5y$ ⓒ $m2+25m$

ⓐ $(p-11)2$ ⓑ $(y+52)2$
ⓒ $(m+15)2$

> ⓐ $q2-6q$ ⓑ $x2-7x$ ⓒ $n2-23n$

Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

$u2+2u=3$

$u=-3, u=1$

$z2+12z=-11$

$x2-20x=21$

$x=-1, x=21$

$y2 - 2y = 8$

$m2 + 4m = -44$

$m = -2 \pm 210i$

$n2 - 2n = -3$

$r2 + 6r = -11$

$r = -3 \pm 2i$

$t2 - 14t = -50$

$a2 - 10a = -5$

$a = 5 \pm 25$

$b2 + 6b = 41$

$x2 + 5x = 2$

$x = -52 \pm 332$

$y2 - 3y = 2$

$u2 - 14u + 12 = -1$

$u = 1, u = 13$

$z2 + 2z - 5 = 2$

$r2 - 4r - 3 = 9$

$r = -2, r = 6$

$t2 - 10t - 6 = 5$

$v2 = 9v + 2$

$v = 92 \pm 892$

$w2 = 5w - 1$

$x2-5=10x$

$x=5\pm30$

$y2-14=6y$

$(x+6)(x-2)=9$

$x=-7, x=3$

$(y+9)(y+7)=80$

$(x+2)(x+4)=3$

$x=-5, x=-1$

$(x-2)(x-6)=5$

Solve Quadratic Equations of the form $ax2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

$3m2+30m-27=6$

$m=-11, m=1$

$2x2-14x+12=0$

$2n2+4n=26$

$n=-1\pm14$

$5x2+20x=15$

$2c2+c=6$

$c=-2, c=32$

$3d2-4d=15$

$2x2+7x-15=0$

$x=-5, x=32$

$3x2-14x+8=0$

$2p2+7p=14$

$p=-74\pm1614$

$3q2-5q=9$

$5x2-3x=-10$

$x=310\pm19110i$

$7x2+4x=-3$

Writing Exercises

Solve the equation $x2+10x=-25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

Answers will vary.

Solve the equation $y2+8y=48$ by completing the square and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2+bx+c=0$ by completing the square.			
solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Solve Quadratic Equations Using the Quadratic Formula
By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Before you get started, take this readiness quiz.

1. Evaluate $b2-4ab$ when $a=3$ and $b=-2$.
   If you missed this problem, review [link].
2. Simplify: 108.
   If you missed this problem, review [link].
3. Simplify: 50.
   If you missed this problem, review [link].

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last

section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes’. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’, so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

	[redacted]	
Isolate the variable terms on one side.	[redacted]	

Make the coefficient of x2 equal to 1, by di[redacted]

Simplify.

[redacted]

To complete the square, find (12·ba)2 and add it to both sides of the equation.

(12ba)2 = b24a2

[redacted]

The left side is a perfect square, factor it.

[redacted]

Find the common denominator of the right sid[redacted]

fractions with the common denominator.

Simplify.

[redacted]

Combine to one fraction.

[redacted]

Use the square root

pr

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify the radical.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Add $-\frac{b}{2a}$ to both sides of the equation.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Combine the terms on the right side.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

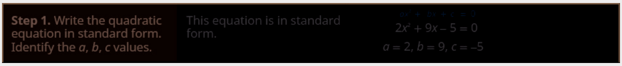
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of *a*, *b*, and *c* from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

How to Solve a Quadratic Equation Using the Quadratic Formula

Solve by using the Quadratic Formula: 2x2 + 9x − 5 = 0.



Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$

Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Step 3. Simplify the fraction, and solve for *x*.	$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$

Step 4. Check the solutions.	Put each answer in the original equation to check.	$2x^2 + 9x - 5 = 0$
	Substitute $x = \frac{1}{2}$.	$2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$
		$\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$
		$\frac{10}{2} - 5 \stackrel{?}{=} 0$
		$5 - 5 \stackrel{?}{=} 0$
		$0 = 0$ ✓
		$2x^2 + 9x - 5 = 0$
	Substitute $x = -5$.	$2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$
		$2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$
		$50 - 45 - 5 \stackrel{?}{=} 0$
		$0 = 0$ ✓

Solve by using the Quadratic Formula: $3y2-5y+2=0$.

$y=1, y=23$

Solve by using the Quadratic Formula: $4z2+2z-6=0$.

$z=1, z=-32$

Solve a quadratic equation using the quadratic formula.

Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, and c. Write the Quadratic Formula. Then substitute in the values of a, b, and c. Simplify. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x =$".

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Write the equation in standard form by ad[redacted]
5 to each side.

This equation is now in standard form.

[redacted]

Identify the values of a, b, c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a, b, c.

[redacted]

Simplify.

[redacted]

[redacted]

[redacted]

Rewrite to show two solutions.

[redacted]

Simplify.

$$x = \frac{10}{2}, \quad x = \frac{2}{2}$$

$$x = 5, \quad x = 1$$

Check:

$x^2 - 6x + 5 = 0$	$x^2 - 6x + 5 = 0$
$5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$	$1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$
$25 - 30 + 5 \stackrel{?}{=} 0$	$1 - 6 + 5 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

Solve by using the Quadratic Formula: $a2 - 2a = 15$.

$a = -3, a = 5$

Solve by using the Quadratic Formula: $b2 + 24 = -10b$.

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

Solve by using the Quadratic Formula: $2x2+10x+11=0$.

[redacted]

This equation is in standard form.

[redacted]

Identify the values of *a*, *b*, and *c*.

[redacted]

Write the Quadratic

Fo [redacted]

Then substitute in the values of *a*, *b*, and *c*.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical.

[redacted]

Factor out the common factor in the nu [redacted]

Remove the common factors.

[redacted]

Rewrite to show two solutions.

[redacted]

Check:
We leave the check for you!

Solve by using the Quadratic Formula: $3m2+12m+7=0$.

$m=-6+153, m=-6-153$

Solve by using the Quadratic Formula: $5n2+4n-4=0$.

$n=-2+265, n=-2-265$

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Solve by using the Quadratic Formula: $3p2+2p+9=0$.

[redacted]

This equation is in standard form

[redacted]

Identify the values of a,b,c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a,b,c.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical using complex nu

[redacted]

Simplify the radical.

[redacted]

Factor the common factor in the nu

[redacted]

Remove the common factors.

$$p=\frac{-1\pm\sqrt{26}\,i}{3}$$

Rewrite in standard a + bi form.

$$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$$

Write as two solutions.

$$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$$

Solve by using the Quadratic Formula: 4a2 − 2a + 8 = 0.

a = 14 + 314i,a = 14 − 314i

Solve by using the Quadratic Formula: 5b2 + 2b + 4 = 0.

b = −15 + 195i,b = −15 − 195i

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Our first step is to get the equation in standard form.

[redacted]

Distribute to get the equation in standard form.

[redacted]

This equation is now in standard form

[redacted]

Identify the values of a,b,c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a,b,c.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical.

[redacted]

Factor the common factor in the nu[redacted]

Remove the common factors.

[redacted]

Write as two solutions.

[redacted]

Check:
We leave the check for you!

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

$x=-1+6, x=-1-6$

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

$y=1+2, y=1-2$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Solve by using the Quadratic Formula:

12u2 + 23u = 13.

Our first step is to clear the fractions.

[redacted]

Multiply both sides by the LCD, 6, to clear the fra[redacted]

Multiply.

[redacted]

Subtract 2 to get the equation in standard for[redacted]

Identify the values of *a*, *b*, and *c*.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of *a*, *b*, and *c*.

[redacted]

Simplify.

[redacted]

[redacted]

Simplify the radical.

[redacted]

Factor the common factor in the [redacted]

Remove the common factors.

[redacted]

Rewrite to show two solutions.

[redacted]

Check:
We leave the check for you!

Solve by using the Quadratic Formula: $14c2-13c=112$.

$c=2+73, c=2-73$

Solve by using the Quadratic Formula: $19d2-12d=-13$.

$d=9+334, d=9-334$

Think about the equation $(x - 3)2 = 0$. We know from the Zero Product Property that this equation has only one solution,
$x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

Solve by using the Quadratic Formula: $4x2-20x=-25$.

[redacted]

Add 25 to get the equation in standard for[redacted]

Identify the values of *a*, *b*, and *c*.

[redacted]

Write the quadratic formula.

[redacted]

Then substitute in the values of *a*, *b*, and *c*.

[redacted]

Simplify.

[redacted]

Simplify the radical.

[redacted]

Simplify the fraction.

[redacted]

Check:
We leave the check for you!

Did you recognize that $4x^2 - 20x + 25$ is a

perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

$r = -5$

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

$t = 45$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

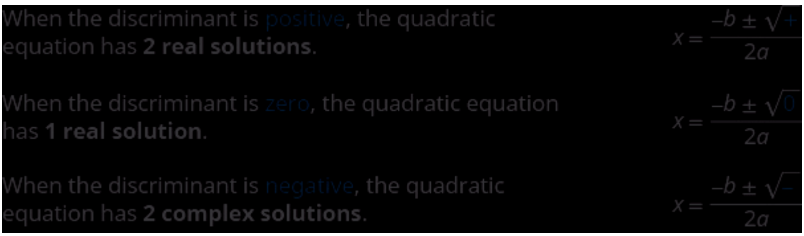
When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

> Discriminant
>
> In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
>
> the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant b^2-4ac	Value of the Discriminant	Number and Type of solutions
$2x^2+9x-5=0$	$9^2-4\cdot2(-5)$	121	2 real
$4x^2-20x+25=0$	$(-20)^2-4\cdot4\cdot25$	0	1 real
$3p^2+2p+9=0$	$2^2-4\cdot3\cdot9$	-104	2 complex


When the discriminant is positive, the quadratic equation has **2 real solutions**. $x=\frac{-b\pm\sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x=\frac{-b\pm\sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x=\frac{-b\pm\sqrt{-}}{2a}$

Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2+bx+c=0$, $a\neq0$,

- If $b^2-4ac>0$, the equation has 2 real solutions.
- if $b^2-4ac=0$, the equation has 1 real solution.
- if $b^2-4ac<0$, the equation has 2 complex solutions.

Determine the number of solutions to each quadratic equation.

ⓐ $3x2+7x-9=0$ ⓑ $5n2+n+4=0$ ⓒ $9y2-6y+1=0$.

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

$3x2+7x-9=0$ The equation is in standard form, identifya,b,andc.$a=3,b=7,c=-9$ Write the discriminant.$b2-4ac$ Substitute in the values ofa,b,andc.$(7)2-4\cdot 3\cdot(-9)$ Simplify.$49+108$ 157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

$5n2+n+4=0$ The equation is in standard form, identifya,b,andc.$a=5,b=1,c=4$ Write the discriminant.$b2-4ac$ Substitute in the values ofa,b,andc.$(1)2-4\cdot 5\cdot 4$ Simplify.$1-80$ -79

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ
$9y2-6y+1=0$ The equation is in standard form, identify$a,b,$and$c.$ $a=9,b=-6,c=1$ Write the discriminant.$b2-4ac$ Substitute in the values of$a,b,$and$c.$$(-6)2-4·9·1$
Simplify.$36-36$ 0

Since the discriminant is 0, there is 1 real solution to the equation.

Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m2-3m+6=0$ ⓑ $5z2+6z-2=0$ ⓒ $9w2+24w+16=0.$

ⓐ 2 complex solutions; ⓑ 2 real solutions; ⓒ 1 real solution

Determine the number and type of solutions to each quadratic equation.

ⓐ $b2+7b-13=0$ ⓑ $5a2-6a+10=0$ ⓒ

$4r2 - 20r + 25 = 0.$

ⓐ 2 real solutions; ⓑ 2 complex solutions; ⓒ 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so

we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the most appropriate method to solve a quadratic equation.

Try **Factoring** first. If the quadratic factors easily, this method is very quick. Try the **Square Root Property** next. If the equation fits the form $ax^2=k$ or $a(x-h)^2=k$, it can easily be solved by using the Square Root Property. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how

to solve each quadratic equation.

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2=17$ ⓑ $4x^2-12x+9=0$ ⓒ $8u^2+6u=11$.

ⓐ

$5z^2=17$

Since the equation is in the $ax^2=k$, the most appropriate method is to use the Square Root Property.

ⓑ

$4x^2-12x+9=0$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

$8u^2+6u=11$ Put the equation in standard form. $8u^2+6u-11=0$

While our first thought may be to try

factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x2+6x+8=0$ ⓑ $(n-3)2=16$ ⓒ $5p2-6p=9$.

ⓐ factoring; ⓑ Square Root Property; ⓒ Quadratic Formula

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a2+3a-9=0$ ⓑ $4b2+4b+1=0$ ⓒ $5c2=125$.

ⓐ Quadratic Forumula;
ⓑ Factoring or Square Root Property ⓒ

Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

Key Concepts

- Quadratic Formula
    - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- How to solve a quadratic equation using the Quadratic Formula.

  Write the quadratic equation in standard form,

$ax^2 + bx + c = 0$. Identify the values of a, b, c. Write the Quadratic Formula. Then substitute in the values of a, b, c. Simplify. Check the solutions.

- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
    - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
        - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
        - if $b^2 - 4ac = 0$, the equation has 1 real solution.
        - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
    - Factoring
    - Square Root Property
    - Completing the Square
    - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.

  Try Factoring first. If the quadratic factors easily, this method is very quick. Try the **Square Root Property** next. If the equation fits

the form $ax2 = k$ or $a(x - h)2 = k$, it can easily be solved by using the Square Root Property. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

$4m2+m-3=0$

$m=-1,m=34$

$4n2-9n+5=0$

$2p2-7p+3=0$

$p=12,p=3$

$3q2+8q-3=0$

$p2+7p+12=0$

$p=-4,p=-3$

$q2+3q-18=0$

$r2-8r=33$

$r=-3,r=11$

$t2+13t=-40$

$3u2+7u-2=0$

$u=-7\pm736$

$2p2+8p+5=0$

$2a2-6a+3=0$

$a=3\pm32$

$5b2+2b-4=0$

$x2+8x-4=0$

$x=-4\pm25$

$y2+4y-4=0$

$3y2+5y-2=0$

$y=-2, y=13$

$6x2+2x-20=0$

$2x2+3x+3=0$

$x=-34\pm154i$

$2x2-x+1=0$

$8x2-6x+2=0$

$x = 38 \pm 78i$

$8x2 - 4x + 1 = 0$

$(v + 1)(v - 5) - 4 = 0$

$v = 2 \pm 213$

$(x + 1)(x - 3) = 2$

$(y + 4)(y - 7) = 18$

$y = -4, y = 7$

$(x + 2)(x + 6) = 21$

$13m2 + 112m = 14$

$m = -1, m = 34$

$13n2 + n = -12$

$34b2+12b=38$

$b=-2\pm116$

$19c2+23c=3$

$16c2+24c+9=0$

$c=-34$

$25d2-60d+36=0$

$25q2+30q+9=0$

$q=-35$

$16y2+8y+1=0$

Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

In the following exercises, determine the number of real solutions for each quadratic equation.

ⓐ $4x2-5x+16=0$ ⓑ $36y2+36y+9=0$ ⓒ $6m2+3m-5=0$

ⓐ no real solutions ⓑ 1
ⓒ 2

ⓐ $9v2-15v+25=0$ ⓑ $100w2+60w+9=0$ ⓒ $5c2+7c-10=0$

ⓐ $r2+12r+36=0$ ⓑ $8t2-11t+5=0$ ⓒ $3v2-5v-1=0$

ⓐ 1 ⓑ no real solutions
ⓒ 2

ⓐ $25p2+10p+1=0$ ⓑ $7q2-3q-6=0$ ⓒ $7y2+2y+8=0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

ⓐ $x2-5x-24=0$ ⓑ $(y+5)2=12$ ⓒ $14m2+3m=11$

ⓐ factor ⓑ square root
ⓒ Quadratic Formula

ⓐ $(8v+3)2=81$ ⓑ $w2-9w-22=0$ ⓒ $4n2-10=6$

ⓐ $6a2+14=20$ ⓑ $(x-14)2=516$ ⓒ $y2-2y=8$

ⓐ Quadratic Formula
ⓑ square root ⓒ factor

ⓐ $8b2+15b=4$ ⓑ $59v2-23v=1$ ⓒ $(w+43)2=29$

Writing Exercises

Solve the equation $x2+10x=120$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

Answers will vary.

Solve the equation $12y2+23y=24$

ⓐ by completing the square

ⓑ using the Quadratic Formula

ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don’t get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

discriminant

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

Solve Quadratic Equations in Quadratic Form

By the end of this section, you will be able to:

- Solve equations in quadratic form

Before you get started, take this readiness quiz.

Factor by substitution: $y4-y2-20$.
If you missed this problem, review [link].

$y2+4y2-5$

Factor by substitution: $(y-4)2+8(y-4)+15$.
If you missed this problem, review [link].

$(y-1)(y+1)$

Simplify: ⓐ $x12 \cdot x14$ ⓑ $(x13)2$ ⓒ $(x-1)2$.
If you missed this problem, review [link].

ⓐ $x34$; ⓑ $x23$; ⓒ $x-2$

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the ax^2+bx+c form. So we factored by substitution allowing us to make it fit the ax^2+bx+c form. We used the standard u for the substitution.

To factor the expression x^4-4x^2-5, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x2)2=x4$.) So we let $u=x^2$ and factored.

[redacted]

[redacted]

Let u = x2 and substitute.	

[redacted]

Factor the trinomial.	

[redacted]

Replace *u* with x2.	

[redacted]

Similarly, sometimes an equation is not in the $ax2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax2 + bx + c = 0$, the middle term has a variable, x, and its square, $x2$, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

How to Solve Equations in Quadratic Form

Solve: $6x4 - 7x2 + 2 = 0$

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2} \quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$

Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x=\pm\sqrt{\frac{1}{2}}$ $\quad x=\pm\sqrt{\frac{2}{3}}$ $x=\pm\frac{\sqrt{2}}{2}$ $\quad x=\pm\frac{\sqrt{6}}{3}$ There are four solutions. $x=\frac{\sqrt{2}}{2}$ $\quad x=\frac{\sqrt{6}}{3}$ $x=-\frac{\sqrt{2}}{2}$ $\quad x=-\frac{\sqrt{6}}{3}$

Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x=\frac{\sqrt{2}}{2}$ $6x^4-7x^2+2=0$ $6\left(\frac{\sqrt{2}}{2}\right)^4-7\left(\frac{\sqrt{2}}{2}\right)^2+2\stackrel{?}{=}0$ $6\left(\frac{4}{16}\right)-7\left(\frac{2}{4}\right)+2\stackrel{?}{=}0$ $\frac{3}{2}-\frac{7}{2}+\frac{4}{2}\stackrel{?}{=}0$ $0=0\,\checkmark$ We leave the other checks to you!

Solve: $x4-6x2+8=0.$

$x=2,x=-2,x=2,x=-2$

Solve: $x4-11x2+28=0.$

$x=7, x=-7, x=2, x=-2$

We summarize the steps to solve an equation in quadratic form.

Solve equations in quadratic form.

Identify a substitution that will put the equation in quadratic form. Rewrite the equation with the substitution to put it in quadratic form. Solve the quadratic equation for *u*. Substitute the original variable back into the results, using the substitution. Solve for the original variable. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

Solve: $(x-2)^2+7(x-2)+12=0$.

[redacted]

Prepare for the substitution.

[redacted]

Let u = x − 2 and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Replace u with x − 2.

[redacted]

Solve for x.

[redacted]

Check:

$x=-1$	$x=-2$
$(x-2)^2+7(x-2)+12=0$	$(x-2)^2+7(x-2)+12=0$
$(-1-2)^2+7(-1-2)+12\stackrel{?}{=}0$	$(-2-2)^2+7(-2-2)+12\stackrel{?}{=}0$
$(-3)^2+7(-3)+12\stackrel{?}{=}0$	$(-4)^2+7(-4)+12\stackrel{?}{=}0$
$9-21+12\stackrel{?}{=}0$	$16-28+12\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

Solve: $(x-5)2+6(x-5)+8=0$.

$x=3, x=1$

Solve: $(y-4)2+8(y-4)+15=0$.

$y=-1, y=1$

In the next example, we notice that $(x)2=x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

Solve: $x - 3x + 2 = 0$.

The x in the middle term, is squared in the first term $(x)2 = x$. If we let $u = x$ and substitute, our trinomial will be in $ax2 + bx + c = 0$ form.

[redacted]

Rewrite the trinomial to prepare for the su [redacted]

Let $u = x$ and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

$u = 2, \qquad u = 1$

Replace u with x.

$\sqrt{x} = 2, \qquad \sqrt{x} = 1$

Solve for x, by squaring both sides.

$x = 4, \qquad x = 1$

Check:

$x = 4$	$x = 1$
$x - 3\sqrt{x} + 2 = 0$	$x - 3\sqrt{x} + 2 = 0$
$4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0$	$1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0$
$4 - 6 + 2 \stackrel{?}{=} 0$	$1 - 3 + 2 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

Solve: $x - 7x + 12 = 0$.

$x = 9, x = 16$

Solve: $x - 6x + 8 = 0$.

$x=4, x=16$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.
The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $(x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

[redacted]

Rewrite the trinomial to prepare for the

[redacted]

substitution.

Let u = x13 and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Replace *u* with x13.

[redacted]

Solve for x by cubing both sides.

[redacted]

[redacted]

Check:

[redacted]

Solve: $x23 - 5x13 - 14 = 0$.

$x = -8, x = 343$

Solve: $x12 - 8x14 + 15 = 0$.

$x = 81, x = 625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

Solve: $3x - 2 - 7x - 1 + 2 = 0$.

The $x-1$ in the middle term is squared in the first term $(x-1)2=x-2$. If we let $u=x-1$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

[redacted]

Rewrite the trinomial to prepare for the substitution.

[redacted]

Let $u=x-1$ and substitute.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Replace u with $x-1$.

[redacted]

Solve for x by taking the rec

$x = 3, \qquad x = \frac{1}{2}$

x − 1 = 1x.

Check:

$x = 3$	$x = \frac{1}{2}$
$3x^{-2} - 7x^{-1} + 2 = 0$	$3x^{-2} - 7x^{-1} + 2 = 0$
$3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0$	$3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0$
$3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0$	$3(4) - 7(2) + 2 \stackrel{?}{=} 0$
$\frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0$	$12 - 14 + 2 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

Solve: 8x − 2 − 10x − 1 + 3 = 0.

x = 43x = 2

Solve: 6x − 2 − 23x − 1 + 20 = 0.

$x = 25, x = 34$

Access this online resource for additional instruction and practice with solving quadratic equations.

- Solving Equations in Quadratic Form

Key Concepts

- How to solve equations in quadratic form.

  Identify a substitution that will put the equation in quadratic form. Rewrite the equation with the substitution to put it in quadratic form. Solve the quadratic equation for u. Substitute the original variable back into the results, using the substitution. Solve for the original variable. Check the solutions.

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

$$x4-7x2+12=0$$

$$x=\pm 3, x=\pm 2$$

$$x4-9x2+18=0$$

$$x4-13x2-30=0$$

$$x=\pm 15, x=\pm 2i$$

$$x4+5x2-36=0$$

$$2x4-5x2+3=0$$

$$x=\pm 1, x=\pm 62$$

$$4x4-5x2+1=0$$

$$2x4-7x2+3=0$$

$x = \pm 3, x = \pm 22$

$3x4 - 14x2 + 8 = 0$

$(x-3)2 - 5(x-3) - 36 = 0$

$x = -1, x = 12$

$(x+2)2 - 3(x+2) - 54 = 0$

$(3y+2)2 + (3y+2) - 6 = 0$

$x = -53, x = 0$

$(5y-1)2 + 3(5y-1) - 28 = 0$

$(x2+1)2 - 5(x2+1) + 4 = 0$

$x = 0, x = \pm 3$

$(x2-4)2 - 4(x2-4) + 3 = 0$

$2(x2-5)2-5(x2-5)+2=0$

$x= \pm 222, x= \pm 7$

$2(x2-5)2-7(x2-5)+6=0$

$x-x-20=0$

$x=25$

$x-8x+15=0$

$x+6x-16=0$

$x=4$

$x+4x-21=0$

$6x+x-2=0$

$x=14$

$6x+x-1=0$

$10x-17x+3=0$

$x=125,\ x=94$

$12x+5x-3=0$

$x23+9x13+8=0$

$x=-1,x=-512$

$x23-3x13=28$

$x23+4x13=12$

$x=8,x=-216$

$x23-11x13+30=0$

$6x23-x13=12$

$$x = 278, x = -6427$$

$$3x23 - 10x13 = 8$$

$$8x23 - 43x13 + 15 = 0$$

$$x = 27512, x = 125$$

$$20x23 - 23x13 + 6 = 0$$

$$x - 8x12 + 7 = 0$$

$$x = 1, x = 49$$

$$2x - 7x12 = 15$$

$$6x - 2 + 13x - 1 + 5 = 0$$

$$x = -2, x = -35$$

$$15x - 2 - 26x - 1 + 8 = 0$$

$8x-2-2x-1-3=0$

$x=-2, x=43$

$15x-2-4x-1-4=0$

Writing Exercises

Explain how to recognize an equation in quadratic form.

Answers will vary.

Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Solve Applications of Quadratic Equations

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review [link].
2. Solve: $2x+1+1x-1=1x2-1$.
   If you missed this problem, review [link].
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review [link].

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we

will take another look at applications.

Let’s first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy.

Read the problem. Make sure all the words and ideas are understood. **Identify** what we are looking for. **Name** what we are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important

information. Then, translate the English sentence into an algebraic equation. **Solve** the equation using algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one *n*, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

Consecutive even integersConsecutive odd integers 64,66,6877,79,81 n1steven integern1stodd integer n + 22ndconsecutive even integern + 22ndconsecutive odd integer n + 43rdconsecutive even integern + 43rdconsecutive odd integer

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

The product of two consecutive odd integers is 195. Find the integers.
Step 1. Readthe problem. Step 2. Identifywhat we are looking for.We are looking for two consecutive odd integers. Step 3. Namewhat we are looking for.Let$n=$the first odd integer. $n+2=$the next odd integer Step 4. Translateinto an equation. State the problem in one sentence.“The product of two consecutive odd integers is 195.” The product of the first odd integer andthe second odd integer is 195. Translate into an equation.Step 5. Solvethe equation. Distribute.Write the equation in standard form.Factor.Use the Zero Product Property.Solve each equation.$n(n+2)=195$ $n2+2n=195$ $n2+2n-195=0$ $(n+15)(n-13)=0$ $n+15=0n-13=0n=-15,n=13$ There are two values ofnthat are solutions. This will give us two pairs of consecutive odd integersfor our solution. First odd integer$n=13$First odd integer$n=-15$ next odd integer$n+2$next odd integer$n+2$ $13+2-15+2$ $15-13$ Step 6. Checkthe answer. Do these pairs work? Are they consecutive odd integers? 13,15yes$-13,-15$yes Is their product 195? $13\cdot15=195$yes $-13(-15)=195$yes Step 7. Answerthe question.Two consecutive odd integers whose product is 195 are 13, 15 and $-13,-15$.

The product of two consecutive odd integers is 99. Find the integers.

The two consecutive odd integers whose product is 99 are 9, 11, and -9, -11

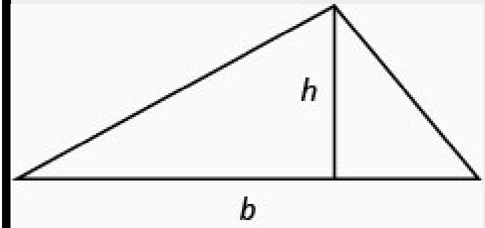
The product of two consecutive even integers is 168. Find the integers.

The two consecutive even integers whose product is 128 are 12, 14 and -12, -14.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

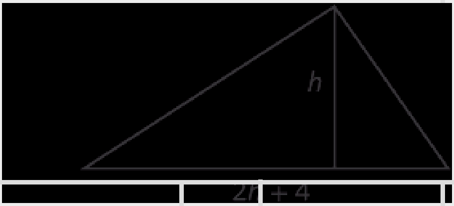
For a triangle with base, b, and height, h, the area, A, is given by the formula $\mathrm{A} = 12\mathrm{bh}$.



Recall that when we solve geometric applications, it is helpful to draw the figure.

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Step 1. Read the problem. Draw a picture.		



Step 2. Identify what we are looking for.	We are looking for the base and height.
Step 3. Name what we are looking for.	Let h = the height of the triangle. $2h + 4$ = the base of the triangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a triangle.	$A=12bh$
Step 5. Solve the equation. Substitute in the values.	$120=12(2h+4)h$
Distribute. This is a quadratic equation, rewrite it in	$120=h2+2h$ $h2+2h-120=0$

standard form.	
Factor.	$(h-10)(h+12)=0$
Use the Zero Product Property.	$h-10=0 \quad h+12=0$
Simplify.	$h=10, h=-12$
Since *h* is the height of a window, a value of $h=-12$ does not make sense.	
The height of the triangle $h=10$.	
The base of the triangle $2h+4$. $2\cdot10+4$ 24	
Step 6. Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the

base is 24 feet.

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

The height of the triangle is 12 inches and the base is 76 inches.

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

The height of the triangle is 11 feet and the base is 20 feet.

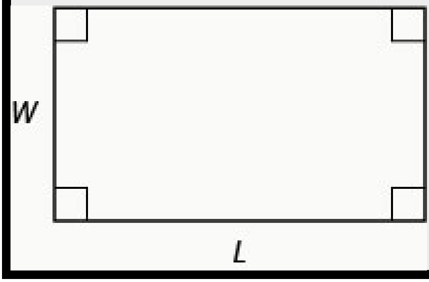
In the two preceding examples, the number in the

radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Area of a Rectangle

For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.



Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot

less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Step 1. **Read** the problem. Dr[redacted]	
Step 2. **Identify** what we are looking for.	We are looking for the length and width.
Step 3. **Name** what we are looking for.	Let w = the width of the rectangle. $3w-1=$ the length of the rectangle
Step 4. **Translate** into an equation. We know the area. Write the formula for the area of a rectangle.	[redacted]
Step 5. **Solve** the equation. Substitute in the values.	[redacted]
Distribute.	[redacted]

This is a quadratic equation; rewrite it in standard form.

[redacted]

Solve the equation using the Quadratic Formula.

Identify the a,b,c values.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a,b,c.

[redacted]

Simplify.

[redacted]

[redacted]

Rewrite to show two solutions.

[redacted]

Approximate the answers using a calculator.

We eliminate the negative solution the width.

[redacted]

Step 6. **Check** the

answer.
Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.

Step 7. **Answer** the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

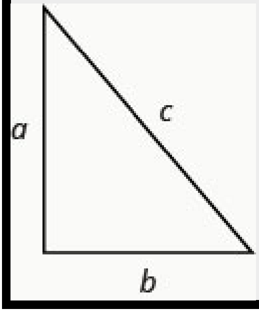
The length of the garden is approximately 18 feet and the width 11 feet.

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot.?
The length of the tablecloth is approximatel 11.8 feet and the width 6.8 feet.

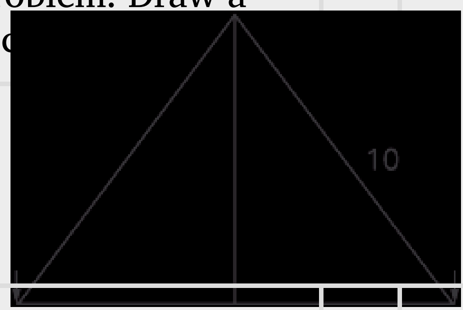
The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

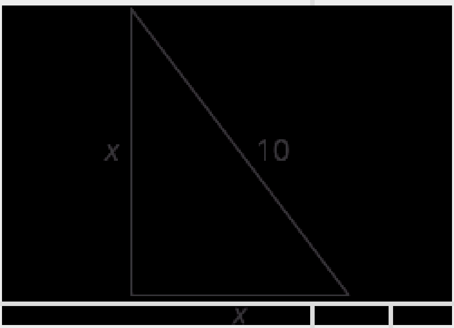
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a2 + b2 = c2$.



Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Step 1. Read the problem. Draw a pic	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the

	pole. Let $x=$ the height of the pole. $x=$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them.



Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*. Write the Pythagorean Theorem.	$a2+b2=c2$
Step 5. Solve the equation. Substitute.	$x2+x2=102$
Simplify.	$2x2=100$
Divide by 2 to isolate the variable.	$2x22=1002$

Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm 50$
Simplify the radical.	$x = \pm 52$
Rewrite to show two solutions.	$x = 52, x = -52$
	If we approximate this number to the nearest tenth with a calculator, we find $x \approx 7.1$.
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

The distance between the opposite corners is approximately 7.2 feet.

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile motion

The height in feet, h , of an object shot upwards into the air with initial velocity, v0, after t seconds

is given by the formula
$h = -16t^2 + v_0 t$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation. Use the	

fo[redacted]

Step 5. Solve the equation.

We [redacted]

v0 is 130 feet per second.

The height is 260 feet.

Substitute the values.

This is a quadratic equation, rewrite it in sta[redacted]

Solve the equation using the Quadratic Formula.

Identify the values of a,b,c.

[redacted]

Write the Quadratic Formula.

[redacted]

Then substitute in the values of a,b,c.

[redacted]

Simplify.

[redacted]

[redacted]

Rewrite to show two solutions.

$t = \frac{130 + \sqrt{260}}{32}$, $t = \frac{130 - \sqrt{260}}{32}$

Approximate the answer with a calculator.

$t \approx 4.6$ seconds, $t \approx 3.6$ seconds

Step 6. Check the answer.

The check is left to you.

Step 7. Answer the question.

The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

The ball will reach 48 feet on its way up after approximately .6 second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

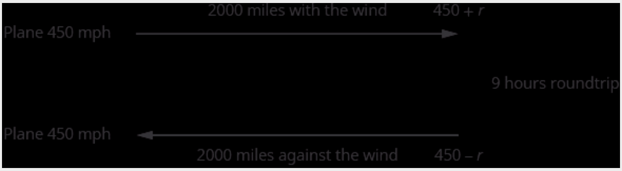
	Rate	• Time	= Distance

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula t=Dr.

Some uniform motion problems are also modeled by quadratic equations.

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Write in the rates.
Write in the distances.

Type	Rate	·	Time	=	Distance
Headwind	$450 - r$		$\frac{2000}{450-r}$		2000
Tailwind	$450 + r$		$\frac{2000}{450+r}$		2000
			9		

Si
fo
t a
We divide the distance by the rate in each row, and place the expression in the time column.

We know the times add to 9 and so we write our equation.

$$\frac{2000}{450-r}+\frac{2000}{450+r}=9$$

We multiply both sides $(450-r)(450+r)$

by the LCD.	$(2000450-r + 2000450+r)= 9(450-r)(450+r)$
Simplify.	$2000(450+r)+2000(450-r)= 9(450-r)(450+r)$
Factor the 2,000.	$2000(450+r + 450-r)= 9(4502-r2)$
Solve.	$2000(900)= 9(4502-r2)$
Divide by 9.	$2000(100)= 4502-r2$
Simplify.	$200000= 202500-r2$ $-2500= -r2$ $50=r$The speed of the jet stream.
Check: Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450+50=500$mph$2000500=4$hours Headwind $450-50=400$ mph$2000400=5$ hours The times add to 9 hours, so it checks.	
	The speed of the jet stream was 50 mph.

MaryAnne just returned from a visit with her grandchildren back east . The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

The speed of the jet stream was 100 mph.

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let x = the number of hours for Press #2

to

En

for

	Number of hours needed to complete the job.	Part of job completed/hour
Press #1	$x + 12$	$\frac{1}{x+12}$
Press #2	x	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

Press #2, and when they work together.

The part completed by Press #1 plus the part

co

eq

Work completed by

Press #1 + Press #2 = Together

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

amount completed together.

Translate to an equation.

Solve.

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

Multiply by the LCD, 8x(x + 12).

$$8x(x+12)\left(\frac{1}{x+12} + \frac{1}{x}\right) = \left(\frac{1}{8}\right)8x(x+12)$$

Simplify.

$$8x + 8(x+12) = x(x+12)$$

$$8x + 8x + 96 = x^2 + 12x$$

$$0 = x^2 - 4x - 96$$

Solve.

[redacted]

[redacted]

[redacted]

Since the idea of negative hours does not [redacted] the value $x = 12$.

[redacted]

Write our sentence answer.

Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as

possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

The red hose take 6 hours and the green hose take 3 hours alone.

Access these online resources for additional instruction and practice with solving applications

modeled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

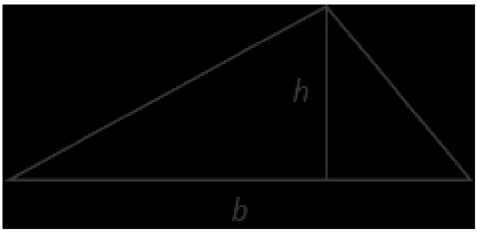
Key Concepts

- Methods to Solve Quadratic Equations
    - Factoring
    - Square Root Property
    - Completing the Square
    - Quadratic Formula
- How to use a Problem-Solving Strategy.

  Read the problem. Make sure all the words and ideas are understood. **Identify** what we are looking for. **Name** what we are looking for. Choose a variable to represent that quantity. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation. **Solve** the equation using good algebra techniques. **Check** the answer in the problem

and make sure it makes sense. **Answer** the question with a complete sentence.

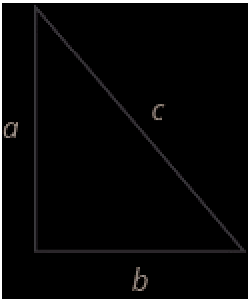
- Area of a Triangle
  - For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A=12bh$.



- Area of a Rectangle
  - For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.

- Pythagorean Theorem

- In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.



- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h = -16t^2 + v_0t$.

Practice Makes Pefect

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

The product of two consecutive odd numbers is 255. Find the numbers.

Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.

The product of two consecutive even numbers is 360. Find the numbers.

The product of two consecutive even numbers is 624. Find the numbers.

The first and second consecutive odd numbers are 24 and 26, and -26 and -24.

The product of two consecutive odd numbers is 1,023. Find the numbers.

The product of two consecutive odd numbers is 483. Find the numbers.

Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

The product of two consecutive even numbers is 528. Find the numbers.

In the following exercises, solve using any method.

Round your answers to the nearest tenth, if needed.

A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.

The width of the triangle is 5 inches and the height is 18 inches.

The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.

The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?

The base is 24 feet and the height of the triangle is 10 feet.

A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?

The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.

The length of the driveway is 15.0 feet and the width is 3.3 feet.

A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.

The length of table is 8 feet and the width is 3 feet.

The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?

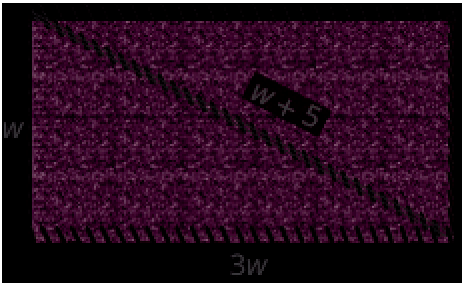
The hypotenuse of a right triangle is twice the

length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

The length of the legs of the right triangle are 3.2 and 9.6 cm.

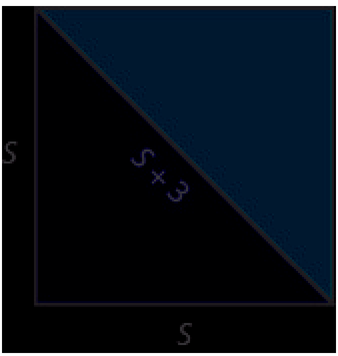
The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.

A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.



The length of the diagonal fencing is 7.3 yards.

Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

The ladder will reach 24.5 feet on the side of

the house.

John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0t$ to determine when the height of the firework rocket will be 1200 feet.

The arrow will reach 400 feet on its way up in 2.8 seconds and on the way down in 11 seconds.

An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0t$, to determine when height of the arrow will be 400 feet.

A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

The bullet will take 70 seconds to hit the ground.

A stone is dropped from a 196-foot platform. Use the formula $h = -16t^2 + v_0t + 196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0 = 0$.)

The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

The speed of the wind was 49 mph.

The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

The speed of the current was 4.3 mph.

Rick paddled up the river, spent the night camping, and and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.

Two gardeners can do the weekly yard maintenance in 8 minutes if they work

together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintainence individually?

It takes two hours for two machines to manufacture 10,000 parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture 10,000 parts alone?

Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the pool?

Writing Exercises

Make up a problem involving the product of two consecutive odd integers.

ⓐ Start by choosing two consecutive odd integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Answers will vary.

Make up a problem involving the product of two consecutive even integers.

ⓐ Start by choosing two consecutive even integers. What are your integers?

ⓑ What is the product of your integers?

ⓒ Solve the equation $n(n + 2) = p$, where p is the product you found in part (b).

ⓓ Did you get the numbers you started with?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this

section.

I can…	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

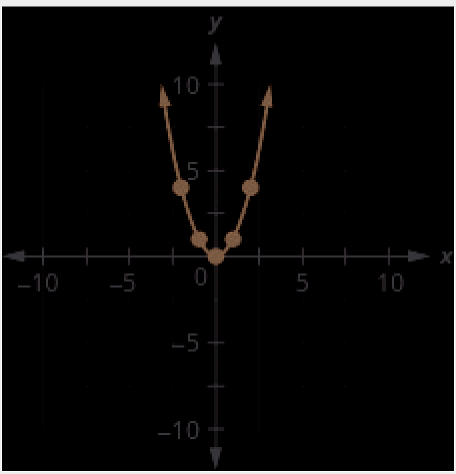
Graph Quadratic Functions Using Properties

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

Before you get started, take this readiness quiz.

Graph the function f(x) = x2 by plotting points.
If you missed this problem, review [link].



Solve: $2x2+3x-2=0$.
If you missed this problem, review [link].

$x=12,x=-2$

Evaluate $-b2a$ when $a = 3$ and $b = -6$.
If you missed this problem, review [link].

1

Recognize the Graph of a Quadratic Function

Previously we very briefly looked at the function $f(x)=x2$, which we called the square function. It was one of the first non-linear functions we looked

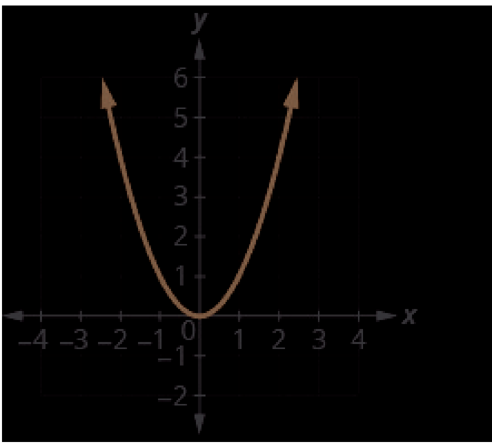
at. Now we will graph functions of the form $f(x)=ax2+bx+c$ if $a\neq0$. We call this kind of function a quadratic function.

Quadratic Function

A **quadratic function**, where *a*, *b*, and *c* are real numbers and $a\neq0$, is a function of the form

$f(x)=ax2+bx+c$

We graphed the quadratic function $f(x)=x2$ by plotting points.



x	$f(x)=x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

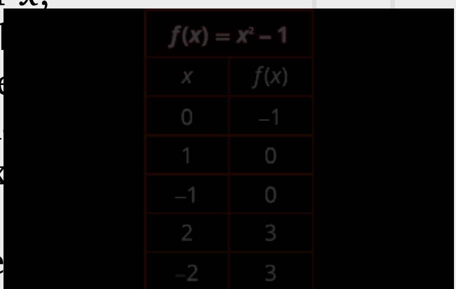
Every quadratic function has a graph that looks like this. We call this figure a **parabola**.

Let’s practice graphing a parabola by plotting a few points.

Graph $f(x)=x^2-1$.

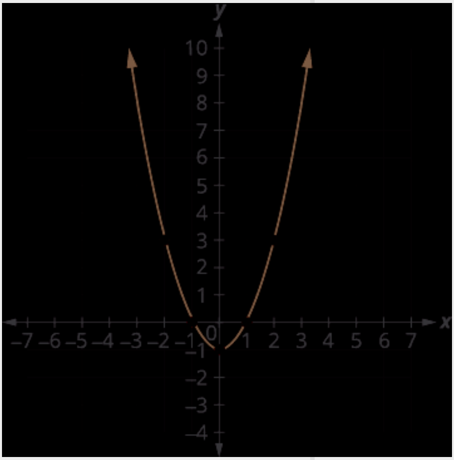
We will graph the function by plotting points.

Choose integer values for x, su[redacted]

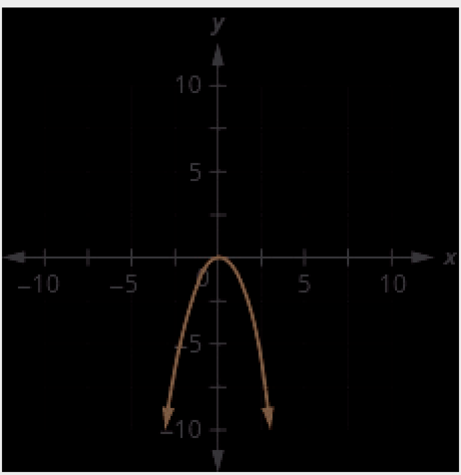


Re[redacted] the ordered pairs in the chart.

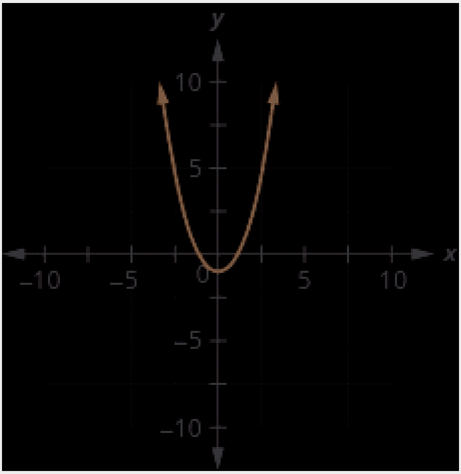
Plot the points, and then connect them with a smooth curve. The result will be the graph of the function $f(x)=x^2-1$.



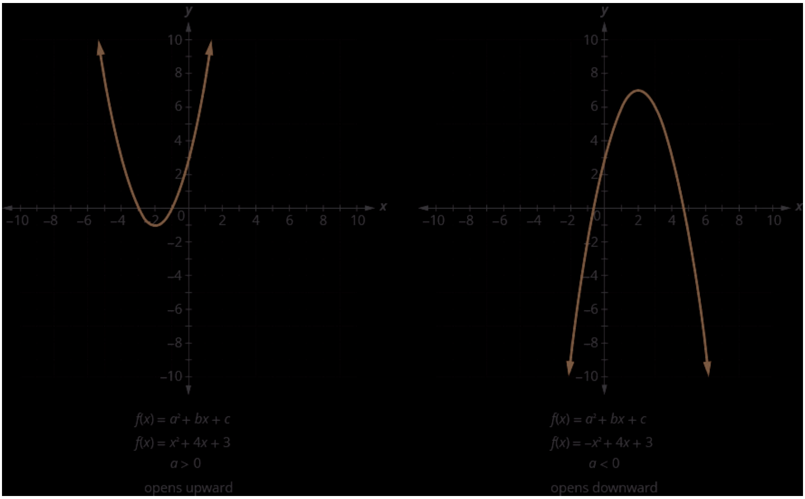
Graph f(x) = − x2..



Graph $f(x) = x^2 + 1$.



All graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$ are parabolas that open upward or downward. See [link].



Notice that the only difference in the two functions is the negative sign before the quadratic term ($x2$ in the equation of the graph in [link]). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Parabola Orientation

For the graph of the quadratic function $f(x) = ax2 + bx + c$, if

- $a > 0$, the parabola opens upward 
- $a < 0$, the parabola opens downward 

Determine whether each parabola opens upward or downward:

ⓐ $f(x) = -3x2 + 2x - 4$ ⓑ $f(x) = 6x2 + 7x - 9$.

ⓐ

Find the value of “*a*”.

[redacted]

Since the “*a*” is negative, the parabola will open downward.

ⓑ

Find the value of “*a*”.

[redacted]

Since the “a” is positive, the parabola will open upward.

Determine whether the graph of each function is a parabola that opens upward or downward:

ⓐ $f(x)=2x2+5x-2$ ⓑ $f(x)=-3x2-4x+7$.

ⓐ up; ⓑ down

Determine whether the graph of each function is a parabola that opens upward or downward:

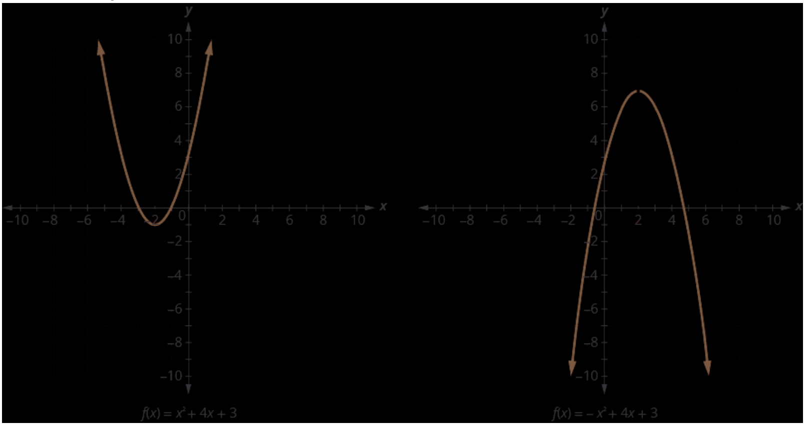
ⓐ $f(x)=-2x2-2x-3$ ⓑ $f(x)=5x2-2x-1$.

ⓐ down; ⓑ up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at [link]. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry. See [link].



The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $f(x) = ax2 + bx + c$ is $x = -b2a$.

So to find the equation of symmetry of each of the

parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$f(x) = ax^2 + bx + c$

$f(x) = x^2 + 4x + 3$

axis of symmetry

$$x = -\frac{b}{2a}$$

$$x = -\frac{4}{2 \cdot 1}$$

$$x = -2$$

$f(x) = ax^2 + bx + c$

$f(x) = -x^2 + 4x + 3$

axis of symmetry

$$x = -\frac{b}{2a}$$

$$x = -\frac{4}{2(-1)}$$

$$x = 2$$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its
x-coordinate is $-\frac{b}{2a}$. To find the *y*-coordinate of the vertex we substitute the value of the *x*-coordinate into the quadratic function.

$f(x) = x^2 + 4x + 3$
axis of symmetry is $x = -2$
vertex is $(-2, \_\_\_)$

$f(x) = x^2 + 4x + 3$
$f(x) = (-2)^2 + 4(-2) + 3$
$f(x) = -1$
vertex is $(-2, -1)$

$f(x) = -x^2 + 4x + 3$
axis of symmetry is $x = 2$
vertex is $(2, \_\_\_)$

$f(x) = -x^2 + 4x + 3$
$f(x) = -(2)^2 + 4(2) + 3$
$f(x) = 7$
vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
- the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

For the graph of $f(x) = 3x^2 - 6x + 2$ find:

ⓐ the axis of symmetry ⓑ the vertex.

ⓐ

[redacted]

The axis of symmetry is the vertical line $x=-b2a$.

Substitute the values of a,b into the eq [redacted]

Simplify.

[redacted]

The axis of symmetry is the line $x=1$.

ⓑ

	$f(x) = 3x^2 - 6x + 2$
The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find $f(1)$.	$f(1) = 3(1)^2 - 6(1) + 2$
Simplify.	$f(1) = 3 \cdot 1 - 6 + 2$
The result is the *y*-coordinate.	$f(1) = -1$
	The vertex is $(1, -1)$.

For the graph of $f(x) = 2x^2 - 8x + 1$ find:

ⓐ the axis of symmetry ⓑ the vertex.

ⓐ $x = 2$; ⓑ $(2, -7)$

For the graph of $f(x) = 2x^2 - 4x - 3$ find:

ⓐ the axis of symmetry ⓑ the vertex.

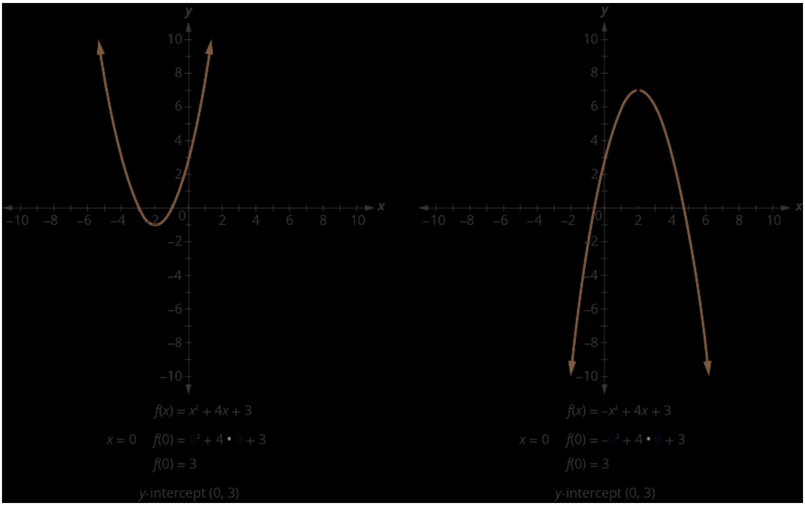
ⓐ $x=1$; ⓑ $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the *y*-intercept the value of *x* is zero. So to find the *y*-intercept, we substitute $x = 0$ into the function.

Let’s find the *y*-intercepts of the two parabolas shown in [link].



An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x) = 0$. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

$$f(x) = ax^2 + bx + c$$
$$0 = ax^2 + bx + c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose function is $f(x) = x^2 + 4x + 3$.

[redacted]

Let f(x) = 0.

[redacted]

Factor.

[redacted]

Use the Zero Product Property.

[redacted]

Solve.

[redacted]

The x-intercepts are $(-1,0)$ and $(-3,0)$.

Now we will find the x-intercepts of the parabola whose function is $f(x) = -x^2 + 4x + 3$.

[redacted]

Let f(x) = 0.

[redacted]

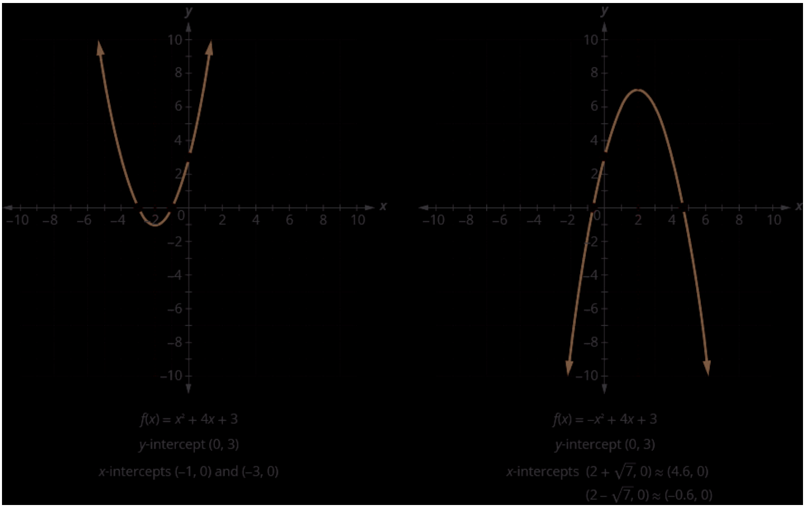
This quadratic does not factor, so

we ... Formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	
$a=-1, b=4, c=3$	
$x=\frac{-4\pm\sqrt{4^2-4(-1)(3)}}{2(-1)}$	
Simplify.	
$x=\frac{-4\pm\sqrt{28}}{-2}$	
$x=\frac{-4\pm 2\sqrt{7}}{-2}$	
$x=\frac{-2(2\pm\sqrt{7})}{-2}$	
$x=2\pm\sqrt{7}$	
	The *x*-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

We will use the decimal approximations of the *x*-intercepts, so that we can locate these points on the graph,

$(2+\sqrt{7},0)\approx(4.6,0)$ $(2-\sqrt{7},0)\approx(-0.6,0)$

Do these results agree with our graphs? See [link].



Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is $f(x)=ax2+bx+c$:

y-interceptx-interceptsLet$x=0$and solve for$f(x)$.Let$f(x)=0$and solve for$x$.

Find the intercepts of the parabola whose function is $f(x)=x2-2x-8$.

To find the ***y***-intercept, let x = [redacted] solve for f(x).

[redacted]

[redacted]

When $x = 0$, then $f(0) = -8$. The *y*-intercept is the point $(0, -8)$.

To find the ***x***-intercept, let f(x[redacted] solve for x.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

When $f(x) = 0$,

then $x=4$ or $x=-2$.
The x-intercepts are the points $(4,0)$ and $(-2,0)$.

Find the intercepts of the parabola whose function is $f(x)=x2+2x-8$.

y-intercept: $(0,\ -8)$ x-intercepts $(-4,0),(2,0)$

Find the intercepts of the parabola whose function is $f(x)=x2-4x-12$.

y-intercept: $(0,\ -12)$ x-intercepts $(-2,0),(6,0)$

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

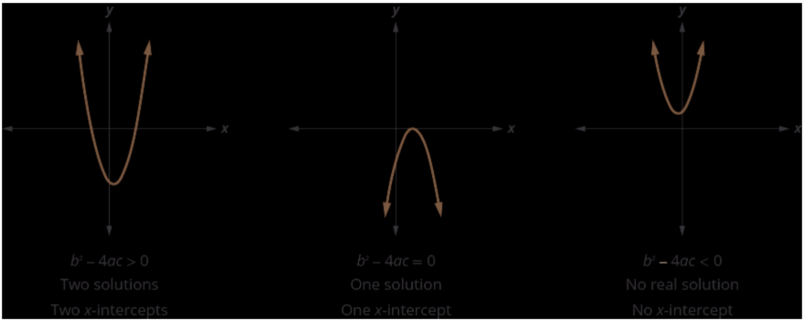
We are now looking at quadratic functions of the form $f(x) = ax^2 + bx + c$. The graphs of these functions are parabolas. The x-intercepts of the parabolas occur where $f(x) = 0$.

For example:
Quadratic equationQuadratic function $x2-2x$
$-15=0(x-5)(x+3)=0x-5=0x+3=0x=5x=$
$-3\text{Let}f(x)=0.f(x)=x2-2x-150=x2-2x$
$-150=(x-5)(x+3)x-5=0x+3=0x=5x=$
$-3(5,0)\text{and}(-3,0)x\text{-intercepts}$

The solutions of the quadratic function are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the form $ax2+bx+c=0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



Before you to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Find the intercepts of the parabola for the function $f(x) = 5x2 + x + 4$.

$f(x) = 5x^2 + x + 4$

To find the *y*-intercept, let $x = 0$ and solve for f(x).

$f(0) = 5 \cdot 0^2 + 0 + 4$

[redacted]

When x = 0, then f(0) = 4.
The *y*-intercept is the point (0,4).

To find the *x*-intercept, let f(x) = 0 and so [redacted]

[redacted]

Find the value of the discriminant to predict the number of solutions which is also the number of *x*-intercepts.
b2 − 4ac12 − 4·5·41 − 80 − 79

Since the value of the discriminant is negative, there is no real solution to the equation.
There are no *x*-intercepts.

Find the intercepts of the parabola whose function is $f(x)=3x2+4x+4$.

y-intercept: (0, 4) no *x*-intercept

Find the intercepts of the parabola whose function is $f(x)=x2-4x-5$.

y-intercept: $(0,\ -5)$ *x*-intercepts $(-1,\ 0),(5,\ 0)$

Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

How to Graph a Quadratic Function Using Properties

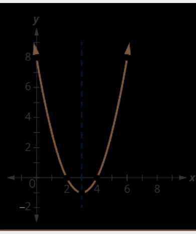
Graph $f(x) = x^2 - 6x + 8$ by using its properties.

Step 1. Determine whether the parabola opens upward or downward.	Look at a in the equation. $f(x) = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	$f(x) = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**

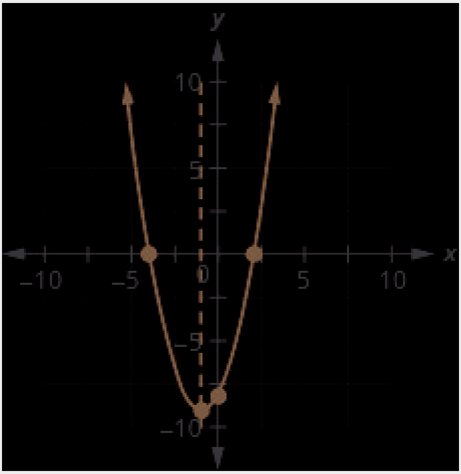
Step 2. Find the axis of symmetry.	$f(x) = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**

Step 3. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $f(x) = x^2 - 6x + 8$ $f(3) = (3)^2 - 6(3) + 8$ $f(3) = -1$ **The vertex is (3, –1).**

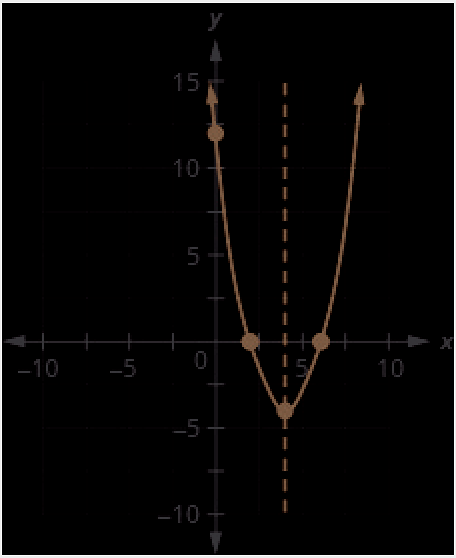
Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find $f(0)$.	y-intercept $f(x) = x^2 - 6x + 8$ $f(0) = (0)^2 - 6(0) + 8$ $f(0) = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept: **The point is (6, 8).**

Step 5. Find the *x*-intercepts. Find additional points if needed.	We solve $f(x) = 0$. We can solve this quadratic equation by factoring.	*x*-intercepts $f(x) = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The *x*-intercepts are (2, 0) and (4, 0).**
Step 6. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

Graph $f(x) = x^2 + 2x - 8$ by using its properties.



Graph $f(x) = x^2 - 8x + 12$ by using its properties.



We list the steps to take in order to graph a quadratic function here.

To graph a quadratic function using properties.

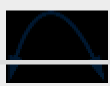
Determine whether the parabola opens upward or downward. Find the equation of the axis of symmetry. Find the vertex. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry. Find the *x*-intercepts. Find additional points if needed. Graph the parabola.

We were able to find the *x*-intercepts in the last example by factoring. We find the *x*-intercepts in the next example by factoring, too.

Graph $f(x) = x2 + 6x - 9$ by using its properties.

[redacted]

Since a is -1, the parabola opens downward.



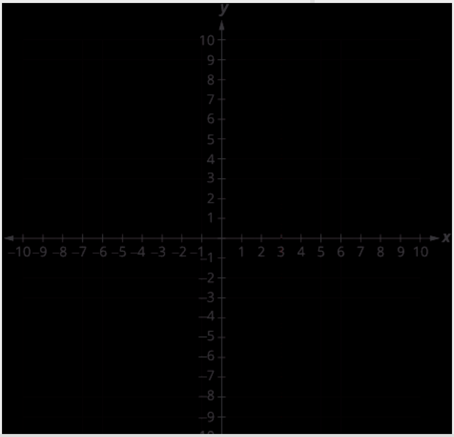
To find the equation of the axis of symmetry, use [redacted]

x= −b2a.

[redacted]

[redacted]

The axis of symmetry is $x = 3$.
The vertex is on the line $x = 3$.



[redacted]

Find f(3).

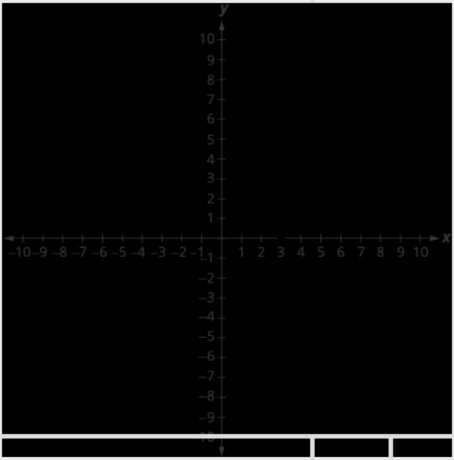
[redacted]

[redacted]

[redacted]

[redacted]

The vertex is (3,0).



The y-intercept occurs when $x = 0$. Find $f(0)$.

[redacted]

Substitute $x = 0$.

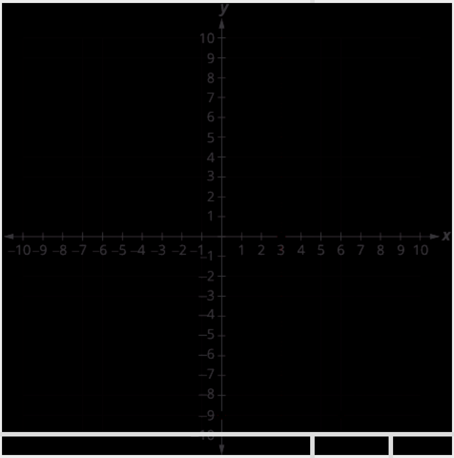
[redacted]

Simplify.

[redacted]

The y-intercept is $(0, -9)$.

The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.



Point symmetric to the y intercept is $(6, -9)$

The x-intercept occurs when $f(x) = 0$.

[redacted]

Find $f(x) = 0$.

[redacted]

Factor the GCF.

[redacted]

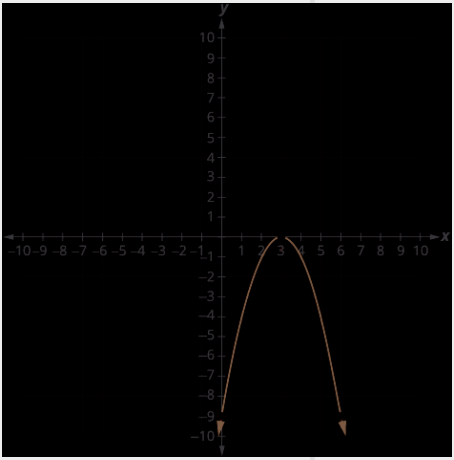
Factor the trinomial.

[redacted]

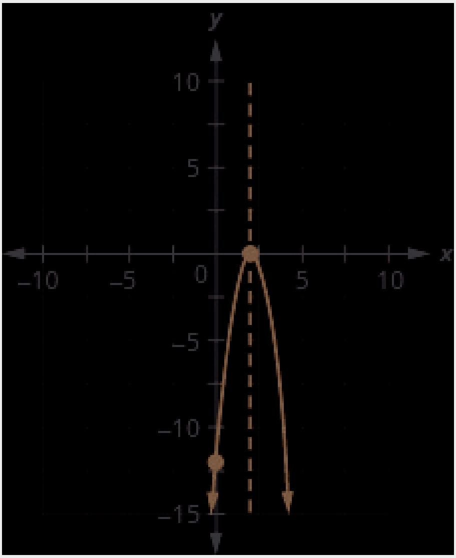
Solve for x.

[redacted]

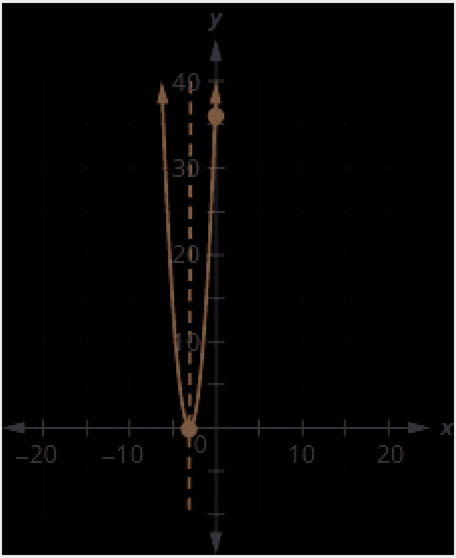
Connect the points to graph the parabola.



Graph $f(x) = -3x^2 + 12x - 12$ by using its properties.



Graph $f(x) = 4x^2 + 24x + 36$ by using its properties.



For the graph of $f(x) = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

How many x-intercepts would you expect to see on the graph of $f(x) = x^2 + 4x + 5$?

Graph $f(x) = x^2 + 4x + 5$ by using its properties.

[redacted]

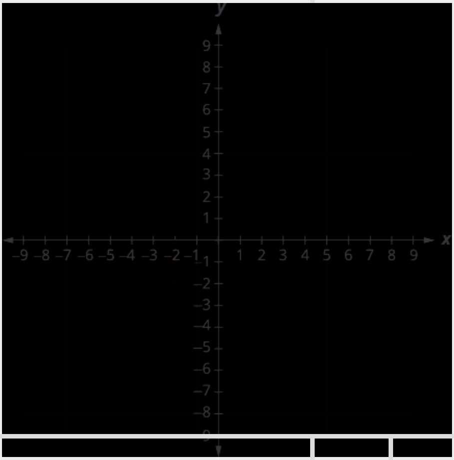
Since a is 1, the parabola opens upward.

To find the axis of symmetry, find $x = -$ [redacted]

[redacted]

[redacted]

The equation of the axis of symmetry is $x = -2$.



The vertex is on the line $x = -2$.
Find $f(x)$ when $x = -2$.

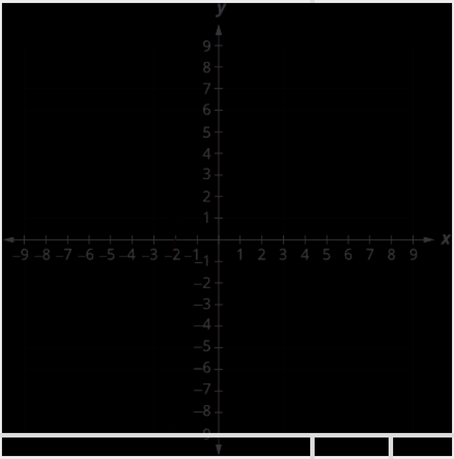
[redacted]

[redacted]

[redacted]

[redacted]

The vertex is $(-2, 1)$.



The *y*-intercept occurs when $x = 0$.

[redacted]

Find f(0).

[redacted]

Simplify.

[redacted]

The *y* intercept is (0,5).

The point $(-4,5)$ is two units to the left of the
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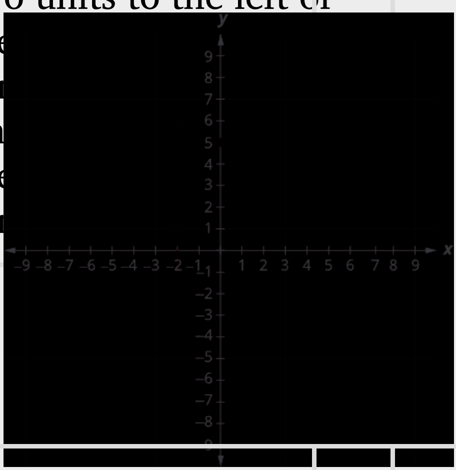
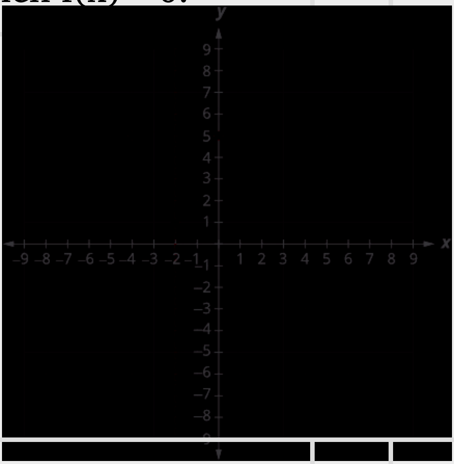


Point symmetric to the y intercept is $(-4,5)$.

The x-intercept occurs when $f(x) = 0$.



Find $f(x) = 0$.

[redacted]

Test the discriminant.

[redacted]

[redacted]

[redacted]

[redacted]

Since the value of the discriminant is

negative, there is
no real solution and so
no x intercept.
Connect the points to
graph the parabola.
Yo
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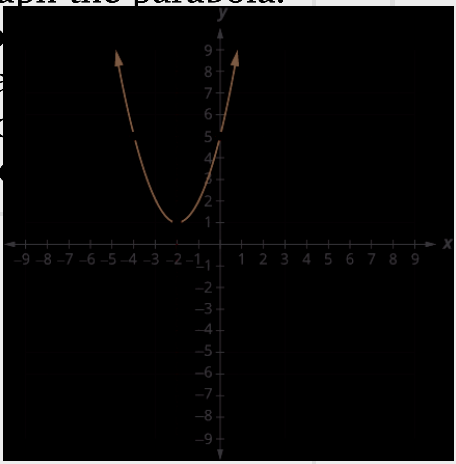
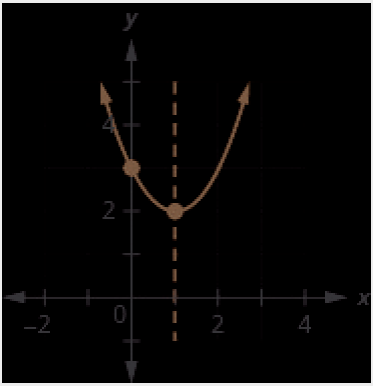
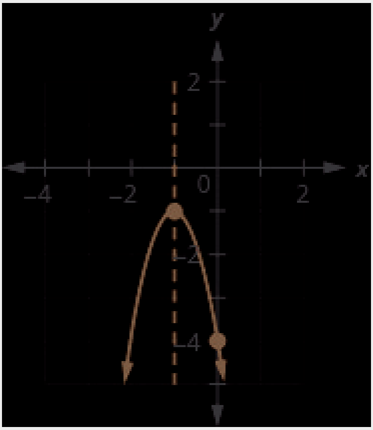


Graph $f(x) = x^2 - 2x + 3$ by using its properties.



Graph $f(x) = -3x^2 - 6x - 4$ by using its properties.



Finding the y-intercept by finding $f(0)$ is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the x-intercepts.

Graph $f(x) = 2x2 - 4x - 3$ by using its properties.

[redacted]

Since a is 2, the parabola opens upward.

[redacted]

To find the equation of the axis of symmetry, use [redacted]

x= −b2a.

[redacted]

[redacted]

The equation of the axis of symmetry is $x = 1$.

The vertex is on the line $x = 1$.

[redacted]

Find $f(1)$.

[redacted]

[redacted]

[redacted]

The vertex is $(1, -5)$.

The *y*-intercept occurs when $x = 0$.

[redacted]

Find $f(0)$.

[redacted]

Simplify.

[redacted]

The *y*-intercept is $(0, -3)$.

The point $(0, -3)$ is one unit to the left of the line of

Point symmetric to the *y*-intercept is $(2, -3)$

symmetry.

The point one unit to the right of the line of symmetry is $(2, -3)$.

The x-intercept occurs when $y = 0$.

[redacted]

Find $f(x) = 0$.

[redacted]

Use the Quadratic Formula.

[redacted]

Substitute in the values of a,b, and c.

[redacted]

Simplify.

[redacted]

Simplify inside the radical.

[redacted]

Simplify the radical.

[redacted]

Factor the GCF.

[redacted]

Remove common factors.

[redacted]

Write as two equations.

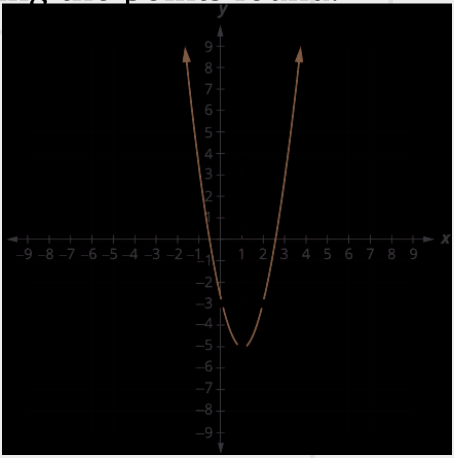
[redacted]

Approximate the values.

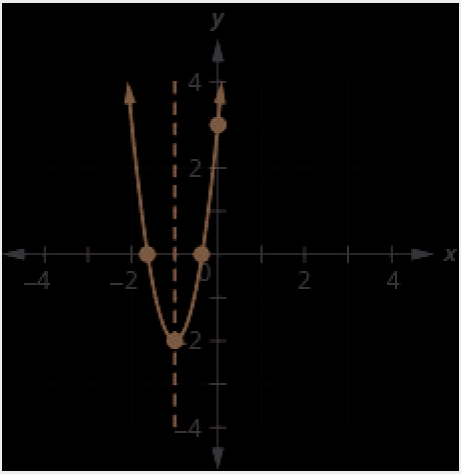
[redacted]

The approximate values of the ***x*****-intercepts are** $(2.5,0)$ **and** $(-0.6,0)$.

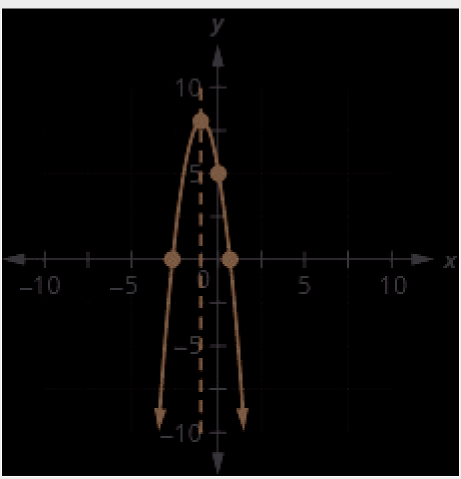
Graph the parabola using the points found.



Graph $f(x) = 5x^2 + 10x + 3$ by using its properties.

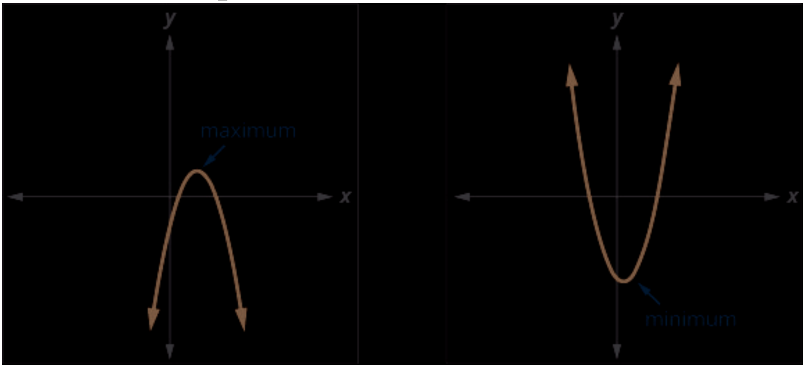


Graph $f(x) = -3x^2 - 6x + 5$ by using its properties.



Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The *y*-coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See [link].



Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the

- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

Find the minimum or maximum value of the quadratic function f(x) = x2 + 2x − 8.

[redacted]

Since a is positive, the parabola opens upward.
The quadratic equation has a minimum.
Find the equation of the axis of symmetry.

[redacted]

[redacted]

[redacted]

The equation of the axis of symmetry is $x = -1$.

The vertex is on the line $x = -1$.

[redacted]

Find $f(-1)$.

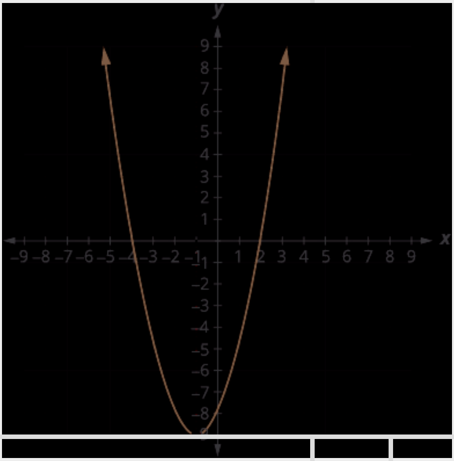
[redacted]

[redacted]

[redacted]

The vertex is $(-1, -9)$.

Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.
The minimum value of the quadratic is -9 and it occurs when $x = -1$.



Show the graph to verify the result.

Find the maximum or minimum value of the quadratic function f(x) = x2 − 8x + 12.

The minimum value of the quadratic function is -4 and it occurs when $x = 4$.

Find the maximum or minimum value of the quadratic function f(x) = −4x2 + 16x − 11.

The maximum value of the quadratic function is 5 and it occurs when $x = 2$.

We have used the formula
$h(t) = -16t^2 + v_0t + h_0$

to calculate the height in feet, h , of an object shot upwards into the air with initial velocity, v_0, after t seconds .

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex (t, h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

The quadratic equation $h(t) = -16t^2 + 176t + 4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height? ⓑ Find the maximum height of the volleyball.

	$h(t) = -16t^2 + 176t + 4$
Since a is negative, the parabola opens downward.	
The quadratic function has a maximum.	

ⓐ

Find the equation of the axis of symmetry.	$t = -\frac{b}{2a}$ $t = -\frac{176}{2(-16)}$ $t = 5.5$
	The equation of the axis of symmetry is $t = 5.5$.
The vertex is on the line $t = 5.5$.	The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find h(5.5).	$h(t) = -16t2 + 176t + 4$ $h(t) =$ $-16(5.5)2 + 176(5.5) + 4$ $h(t) = 488$
Use a calculator to simplify.	The vertex is (5.5,488).

Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

Solve, rounding answers to the nearest tenth.

The quadratic function $h(t) = -16t2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h(t) = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height?

It will 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- Quadratic Functions: Axis of Symmetry and Vertex
- Finding x- and y-intercepts of a Quadratic Function
- Graphing Quadratic Functions
- Solve Maxiumum or Minimum Applications
- Quadratic Applications: Minimum and Maximum

Key Concepts

- Parabola Orientation
    - For the graph of the quadratic function f(x) = ax2 + bx + c, if
        - $a > 0$, the parabola opens upward.
        - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function f(x) = ax2 + bx + c is a parabola where:
    - the axis of symmetry is the vertical line x = − b2a.
    - the vertex is a point on the axis of symmetry, so its *x*-coordinate is − b2a.
    - the *y*-coordinate of the vertex is found by substituting x = − b2a into the quadratic equation.
- Find the Intercepts of a Parabola
    - To find the intercepts of a parabola whose function is f(x) = ax2 + bx + c: y-interceptx-interceptsLetx = 0and solve forf(x).Letf(x) = 0and solve forx.

- How to graph a quadratic function using properties.

  Determine whether the parabola opens upward or downward. Find the equation of the axis of symmetry. Find the vertex. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry. Find the *x*-intercepts. Find additional points if needed. Graph the parabola.

- Minimum or Maximum Values of a Quadratic Equation

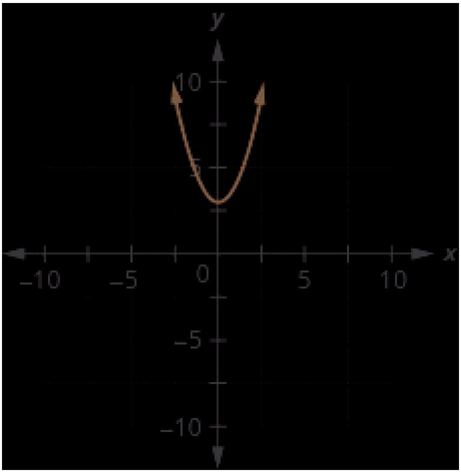
  - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

Practice Makes Perfect

Recognize the Graph of a Quadratic Function

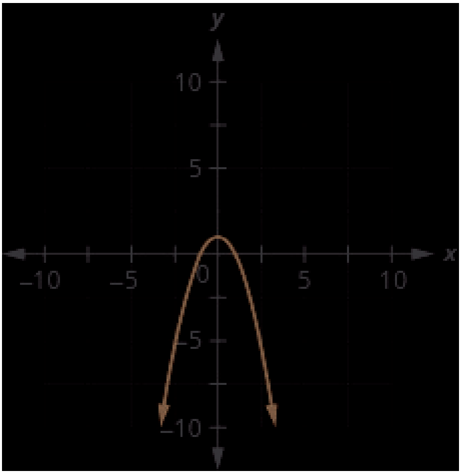
In the following exercises, graph the functions by plotting points.

$f(x)=x2+3$



$f(x) = x2 - 3$

$y = -x2 + 1$



$f(x)=-x2-1$

For each of the following exercises, determine if the parabola opens up or down.

ⓐ $f(x)=-2x2-6x-7$ ⓑ $f(x)=6x2+2x+3$

ⓐ down ⓑ up

ⓐ $f(x)=4x2+x-4$ ⓑ $f(x)=-9x2-24x-16$

ⓐ $f(x)=-3x2+5x-1$ ⓑ $f(x)=2x2-4x+5$

ⓐ down ⓑ up

ⓐ $f(x)=x2+3x-4$ ⓑ $f(x)=-4x2-12x-9$

Find the Axis of Symmetry and Vertex of a Parabola

In the following functions, find ⓐ the equation of the axis of symmetry and ⓑ the vertex of its graph.

$f(x)=x2+8x-1$

ⓐ $x=-4$; ⓑ $(-4,\ -17)$

$f(x)=x2+10x+25$

$f(x)=-x2+2x+5$

ⓐ $x=1$; ⓑ $(1,6)$

$f(x)=-2x2-8x-3$

Find the Intercepts of a Parabola

In the following exercises, find the intercepts of the parabola whose function is given.

$f(x)=x2+7x+6$

---

y-intercept: (0, 6); x-intercept $(-1, 0),(-6, 0)$

$f(x) = x2 + 10x - 11$

$f(x) = x2 + 8x + 12$

---

y-intercept: (0, 12); x-intercept $(-2, 0),(-6, 0)$

$f(x) = x2 + 5x + 6$

$f(x) = -x2 + 8x - 19$

---

y-intercept: $(0, -19)$; x-intercept: none

$f(x) = -3x2 + x - 1$

$f(x) = x2 + 6x + 13$

---

y-intercept: (0, 13); x-intercept: none

$f(x) = x2 + 8x + 12$

$f(x)=4x2-20x+25$

---

y-intercept: (0,25); *x*-intercept (52,0)

$f(x)=-x2-14x-49$

$f(x)=-x2-6x-9$

---

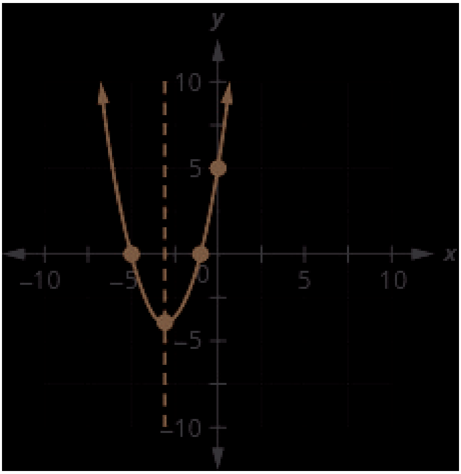
y-intercept: $(0,-9)$; *x*-intercept $(-3,\ 0)$

$f(x)=4x2+4x+1$

Graph Quadratic Functions Using Properties

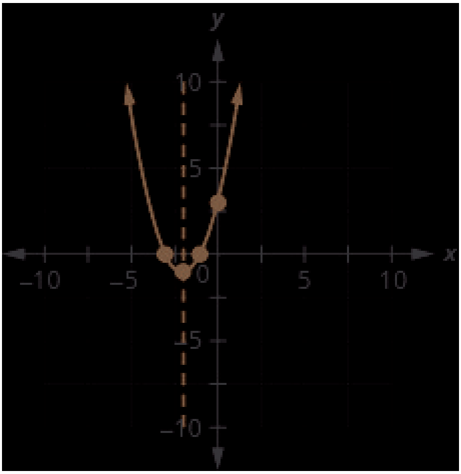
In the following exercises, graph the function by using its properties.

$f(x)=x2+6x+5$



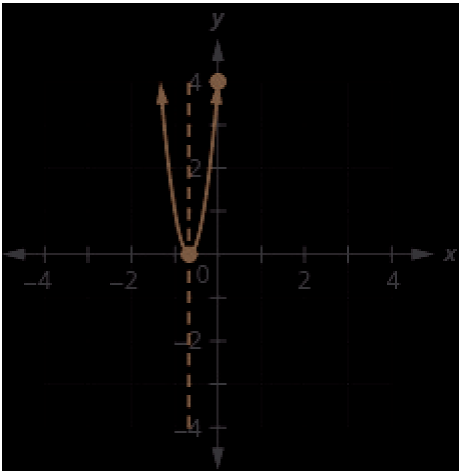
$f(x) = x2 + 4x - 12$

$f(x) = x2 + 4x + 3$



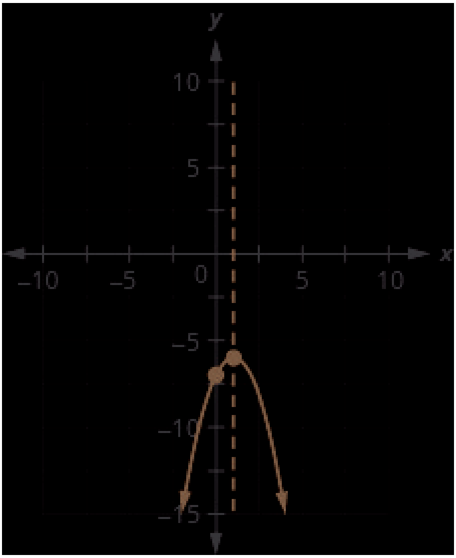
$f(x) = x2 - 6x + 8$

$f(x) = 9x2 + 12x + 4$



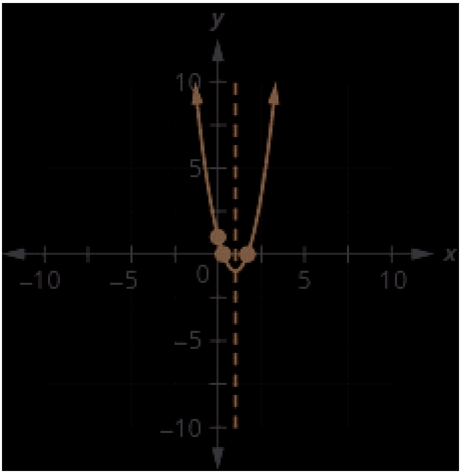
$f(x) = -x2 + 8x - 16$

$f(x) = -x2 + 2x - 7$



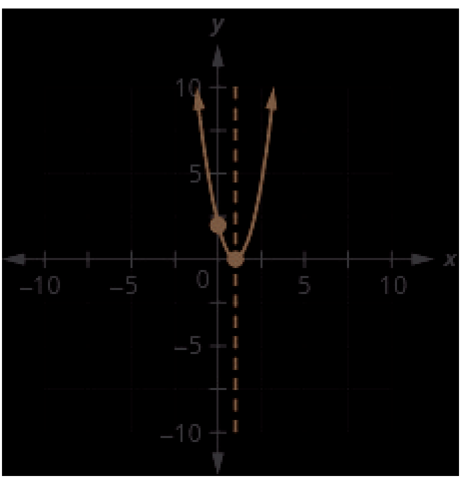
$f(x) = 5x2 + 2$

$f(x) = 2x2 - 4x + 1$



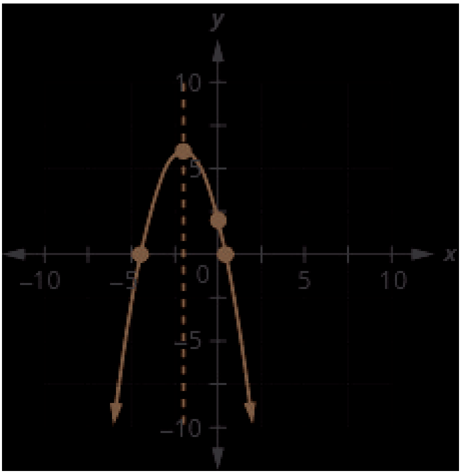
$f(x) = 3x2 - 6x - 1$

$f(x) = 2x2 - 4x + 2$



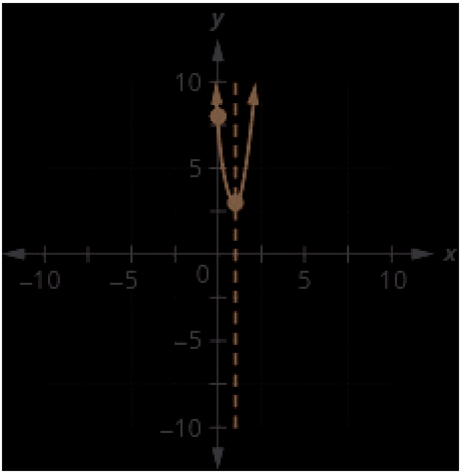
$f(x) = -4x2 - 6x - 2$

$f(x) = -x2 - 4x + 2$



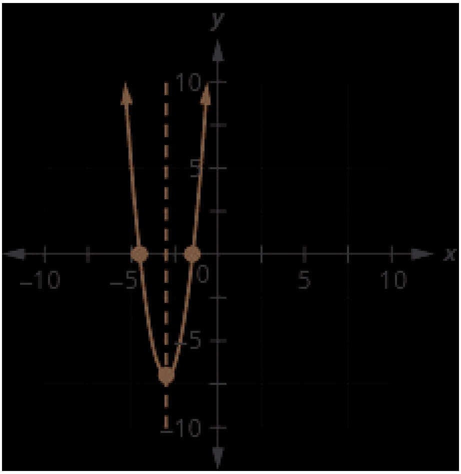
$f(x) = x2 + 6x + 8$

$f(x) = 5x2 - 10x + 8$



$f(x) = -16x2 + 24x - 9$

$f(x) = 3x2 + 18x + 20$



$f(x) = -2x2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value of each function.

$f(x) = 2x2 + x - 1$

The minimum value is -98 when $x = -14$.

$y = -4x2 + 12x - 5$

$y = x2 - 6x + 15$

The minimum value is 6 when $x = 3$.

$y = -x2 + 4x - 5$

$y = -9x2 + 16$

The maximum value is 16 when $x = 0$.

$y = 4x2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic function $h(t) = -16t^2 + 168t + 45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

In 5.3 sec the arrow will reach maximum height of 486 ft.

A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic function $h(t) = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic function $h(t) = -16t^2 + 109t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

In 3.4 seconds the ball will reach its maximum

height of 185.6 feet.

A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic function $h(t) = -16t^2 + 122t + 0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic function $R(x) = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

A selling price of $20 per computer will give the maximum revenue of $400.

A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic function $R(x) = -x^2 + 100x$ is used to find the R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find

the amount of the maximum revenue.

A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70 - x$ boots a week. Use the quadratic function $R(x) = -x2 + 70x$ to find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue per day.

A selling price of $35 per pair of boots will give a maximum revenue of $1,225.

A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8 - x$ cell phones per day. Use the quadratic function $R(x) = -x2 + 8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The

quadratic equation $A(x) = x120 - x2$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

The length of one side along the river is 120 feet and the maximum are is 7,200 square feet.

A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic function $A(x) = x50 - x2$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the "walls." He wants to maximize the area using 80 feet of fencing. The quadratic function $A(x) = x(80 - 2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

The maximum area of the patio is 800 feet.

A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic function $A(x)=x150-x2$ determine the maximum area of the fenced in yard.

Writing Exercise

How do the graphs of the functions $f(x)=x2$ and $f(x)=x2-1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

Answers will vary.

Explain the process of finding the vertex of a parabola.

Explain how to find the intercepts of a parabola.

Answers will vary.

How can you use the discriminant when you are graphing a quadratic function?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

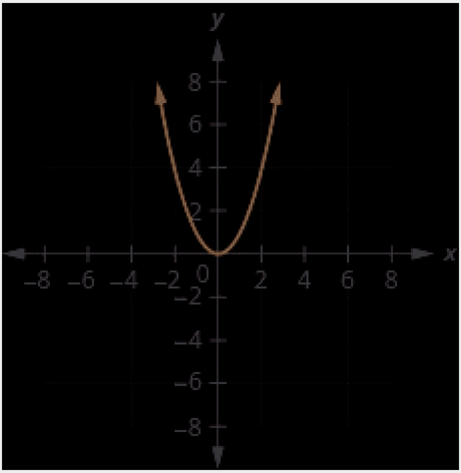
Glossary

quadratic function
: A quadratic function, where *a*, *b*, and *c* are real numbers and $a \neq 0$, is a function of the form $f(x) = ax2 + bx + c$.

Graph Quadratic Functions Using Transformations

Before you get started, take this readiness quiz.

Graph the function f(x) = x2 by plotting points.
If you missed this problem, review [link].



Factor completely: y2 − 14y + 49.
If you missed this problem, review [link].

$y - 72$

Factor completely: $2x^2 - 16x + 32$.
If you missed this problem, review [link].

$2(x - 4)^2$

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of $f(x) = x^2$ and 'moving' it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function $f(x) = x^2$ by plotting points. Then we will see what effect adding a constant, *k*, to the equation will have on the graph of the new function

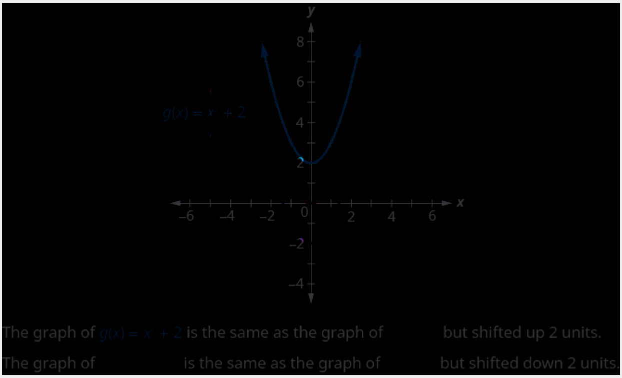
$f(x) = x^2 + k.$

Graph $f(x) = x^2, g(x) = x^2 + 2$, and $h(x) = x^2 - 2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Plotting points will help us see the effect of the constants on the basic $f(x) = x^2$ graph. We fill in the chart for all three functions.

x	$f(x) = x^2$	$(x, f(x))$	$g(x) = x^2 + 2$	$(x, g(x))$		
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

The *g*(*x*) values are two more than the *f*(*x*) values. Also, the *h*(*x*) values are two less than the *f*(*x*) values. Now we will graph all three functions on the same rectangular coordinate system.

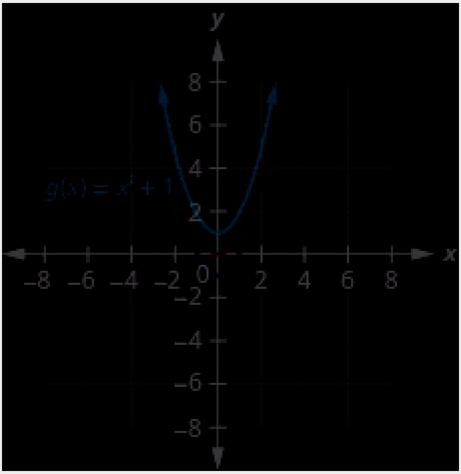


The graph of $g(x)=x2+2$ is the same as the graph of $f(x)=x2$ but shifted up 2 units.

The graph of $h(x)=x2-2$ is the same as the graph of $f(x)=x2$ but shifted down 2 units.

ⓐ Graph $f(x)=x2, g(x)=x2+1$, and $h(x)=x2-1$ on the same rectangular coordinate system.
ⓑ Describe what effect adding a constant to the function has on the basic parabola.

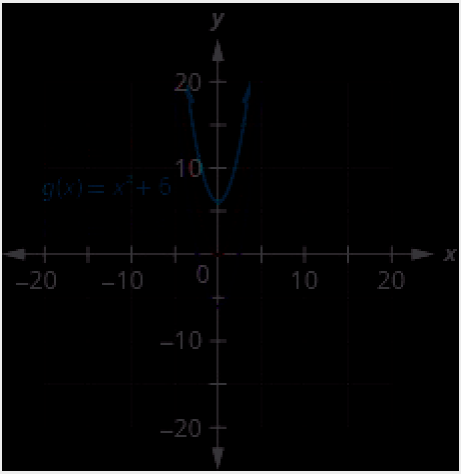
ⓐ



ⓑ The graph of $g(x)=x2+1$ is the same as the graph of $f(x)=x2$ but shifted up 1 unit. The graph of $h(x)=x2-1$ is the same as the graph of $f(x)=x2$ but shifted down 1 unit.

ⓐ Graph $f(x)=x2, g(x)=x2+6$, and $h(x)=x2-6$ on the same rectangular coordinate system.
ⓑ Describe what effect adding a constant to the function has on the basic parabola.

ⓐ



ⓑ The graph of h(x) = x2 + 6 is the same as the graph of f(x) = x2 but shifted up 6 units. The graph of h(x) = x2 − 6 is the same as the graph of f(x) = x2 but shifted down 6 units.

The last example shows us that to graph a quadratic function of the form f(x) = x2 + k, we take the basic parabola graph of f(x) = x2 and vertically shift it up (k > 0) or shift it down (k < 0).

This transformation is called a vertical shift.

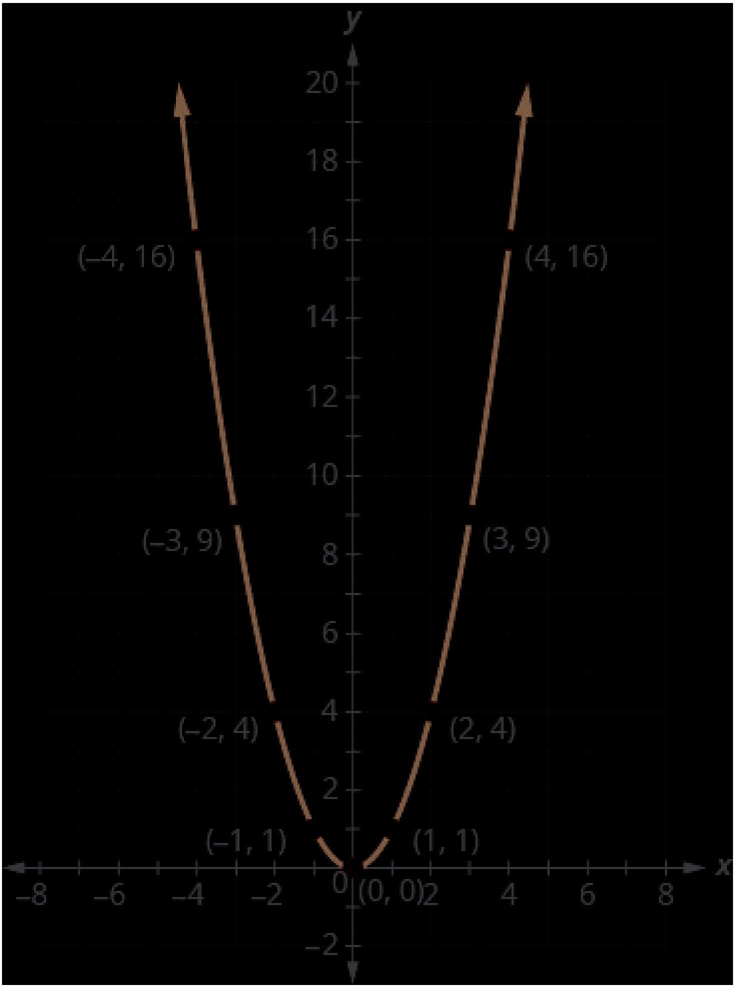
Graph a Quadratic Function of the form f(x) = x2 + k Using a Vertical Shift

The graph of $f(x)=x2+k$ shifts the graph of $f(x)=x2$ vertically k units.

- If $k > 0$, shift the parabola vertically up k units.
- If $k < 0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph functions of the form $f(x)=x2+k$. We just start with the basic parabola of $f(x)=x2$ and then shift it up or down.

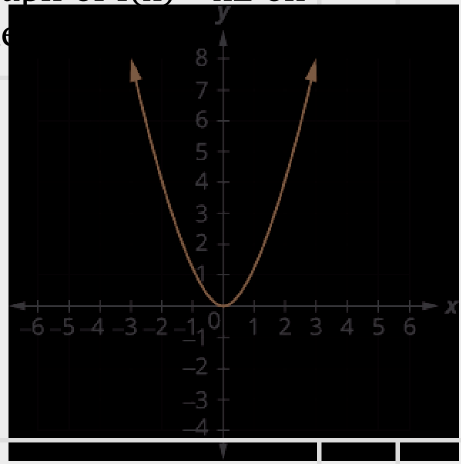
It may be helpful to practice sketching $f(x)=x2$ quickly. We know the values and can sketch the graph from there.



Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

Graph f(x) = x2 − 3 using a vertical shift.

We first draw the graph of f(x) = x2 on the



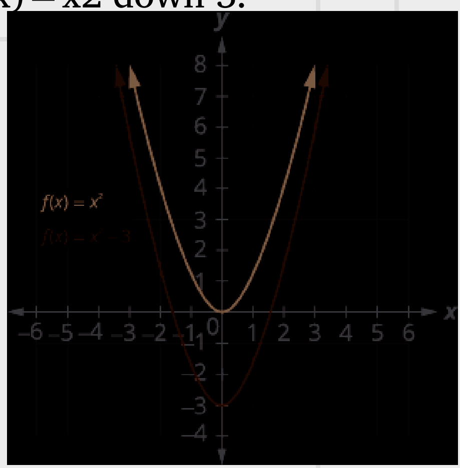
Determine k.

$$f(x) = x^2 + k$$

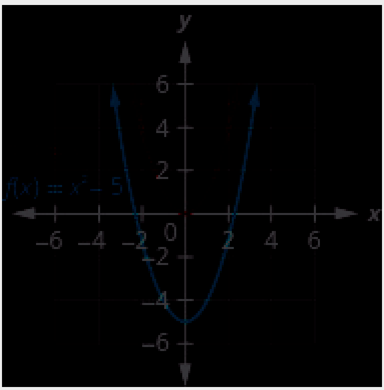
$$f(x) = x^2 - 3$$

$$k = -3$$

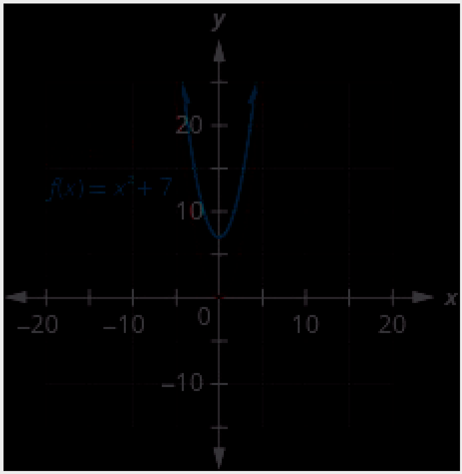
Shift the graph f(x) = x2 down 3.



Graph $f(x) = x2 - 5$ using a vertical shift.



Graph $f(x) = x2 + 7$ using a vertical shift.



Graph Quadratic Functions of the form $f(x)=(x-h)2$

In the first example, we graphed the quadratic function $f(x)=x2$ by plotting points and then saw the effect of adding a constant *k* to the function had on the resulting graph of the new function $f(x)=x2+k$.

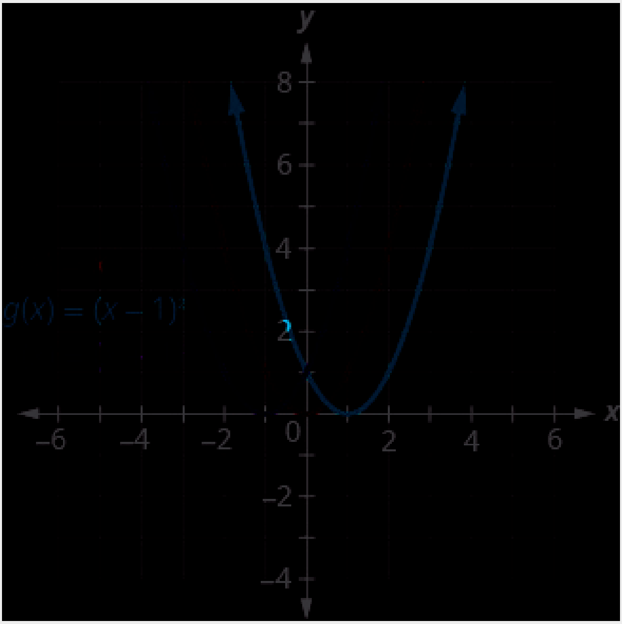
We will now explore the effect of subtracting a constant, *h*, from *x* has on the resulting graph of the new function $f(x)=(x-h)2$.

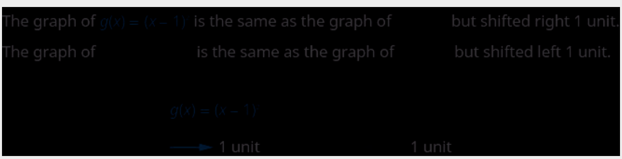
Graph $f(x)=x^2, g(x)=(x-1)^2$, and $h(x)=(x+1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

Plotting points will help us see the effect of the constants on the basic $f(x)=x^2$ graph. We fill in the chart for all three functions.

x			$g(x)=(x-1)^2$	$(x, g(x))$		
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

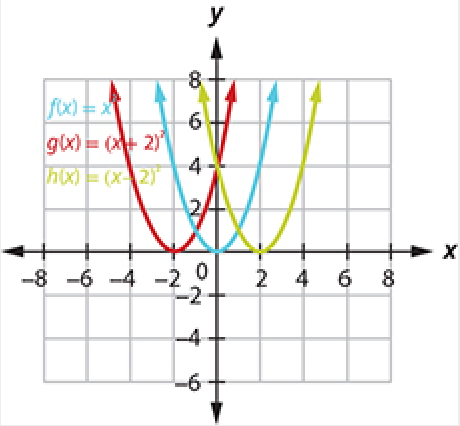
The $g(x)$ values and the $h(x)$ values share the common numbers 0, 1, 4, 9, and 16, but are shifted.





ⓐ Graph $f(x)=x2, g(x)=(x+2)2$, and $h(x)=(x-2)2$ on the same rectangular coordinate system.
ⓑ Describe what effect adding a constant to the function has on the basic parabola.

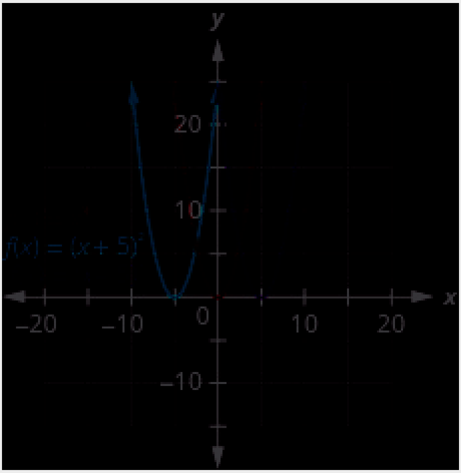
ⓐ



ⓑ The graph of g(x) = (x + 2)2 is the same as the graph of f(x) = x2 but shifted left 2 units. The graph of h(x) = (x − 2)2 is the same as the graph of f(x) = x2 but shift right 2 units.

ⓐ Graph f(x) = x2, g(x) = x2 + 5, and h(x) = x2 − 5 on the same rectangular coordinate system.
ⓑ Describe what effect adding a constant to the function has on the basic parabola.

ⓐ



ⓑ The graph of g(x) = (x + 5)2 is the same as the graph of f(x) = x2 but shifted left 5 units. The graph of h(x) = (x − 5)2 is the same as the graph of f(x) = x2 but shifted right 5 units.

The last example shows us that to graph a quadratic function of the form f(x) = (x − h)2, we take the basic parabola graph of f(x) = x2 and shift it left (*h* > 0) or shift it right (*h* < 0).

This transformation is called a horizontal shift.

Graph a Quadratic Function of the form $f(x)=(x-h)^2$ Using a Horizontal Shift

The graph of $f(x)=(x-h)^2$ shifts the graph of $f(x)=x^2$ horizontally h units.

- If $h>0$, shift the parabola horizontally right h units.
- If $h<0$, shift the parabola horizontally left $|h|$ units.

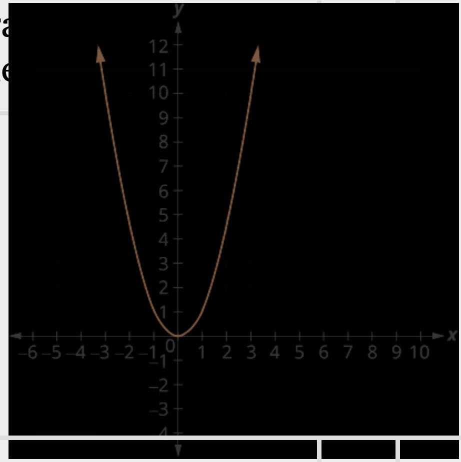
Now that we have seen the effect of the constant, h, it is easy to graph functions of the form $f(x)=(x-h)^2$. We just start with the basic parabola of $f(x)=x^2$ and then shift it left or right.

The next example will require a horizontal shift.

Graph $f(x)=(x-6)^2$ using a horizontal shift.

We first draw the		

gra
the

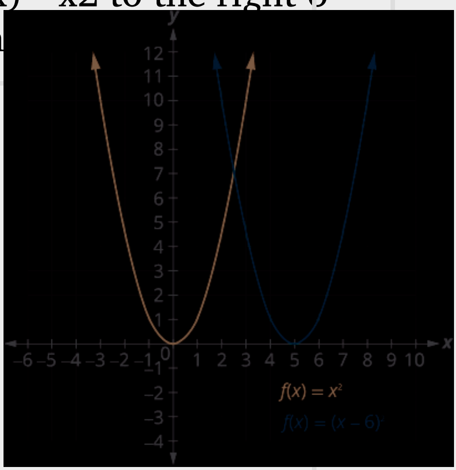


Determine *h*.

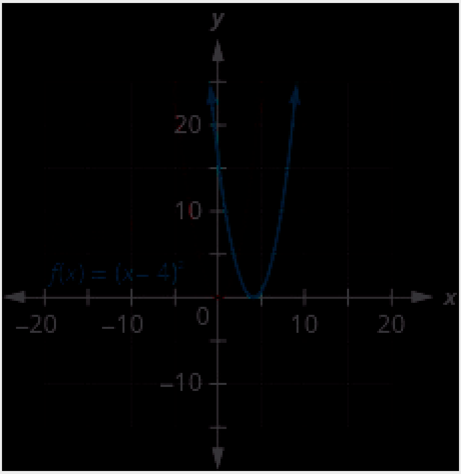
[redacted]

[redacted]

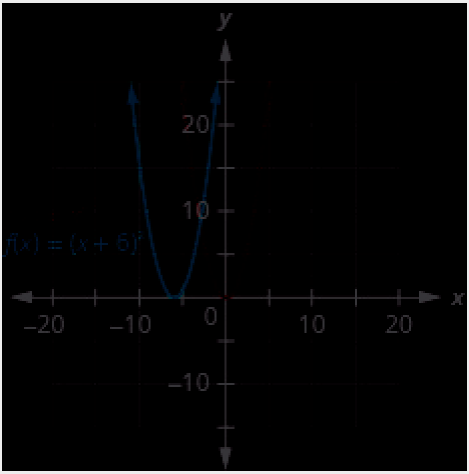
Shift the graph
f(x) = x2 to the right 6
un



Graph $f(x) = (x - 4)2$ using a horizontal shift.



Graph $f(x) = (x + 6)2$ using a horizontal shift.



Now that we know the effect of the constants *h* and *k*, we will graph a quadratic function of the form $f(x)=(x-h)2+k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

Graph $f(x)=(x+1)2-2$ using transformations.

This function will involve two transformations and we need a plan.

Let’s first identify the constants *h*, *k*.

$$f(x) = (x+1)^2 - 2$$

$$f(x) = (x-h)^2 + k$$

$$f(x) = (x-(-1))^2 + (-2)$$

$$h = -1 \quad k = -2$$

The *h* constant gives us a horizontal shift and the *k* gives us a vertical shift.

$f(x) = x^2$ → $f(x) = (x+1)^2$ → $f(x) = (x+1)^2 - 2$

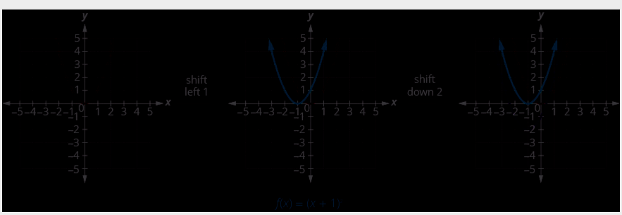
$h = -1$: Shift left 1 unit

$k = -2$: Shift down 2 units

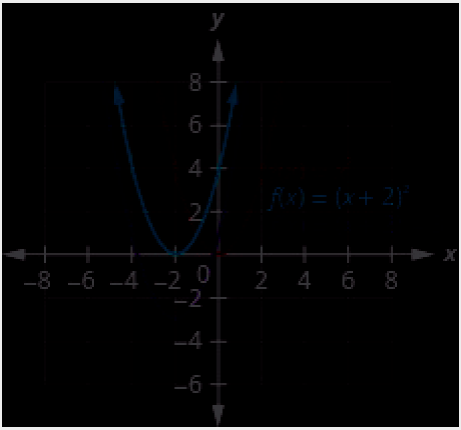
We first draw the graph of f(x) = x2 on the grid.

To graph $f(x) = (x+1)^2$, shift the graph to the left 1 unit.

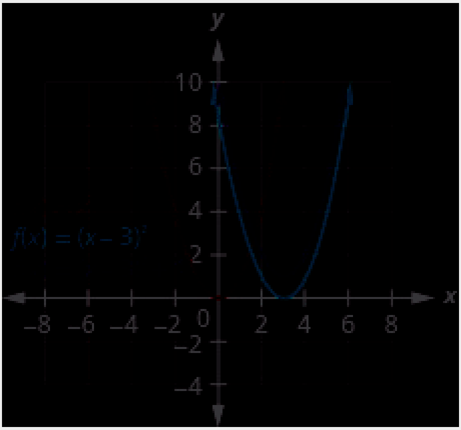
To graph , shift the graph $f(x) = (x+1)^2$ down 2 units.



Graph $f(x)=(x+2)2-3$ using transformations.



Graph $f(x)=(x-3)2+1$ using transformations.



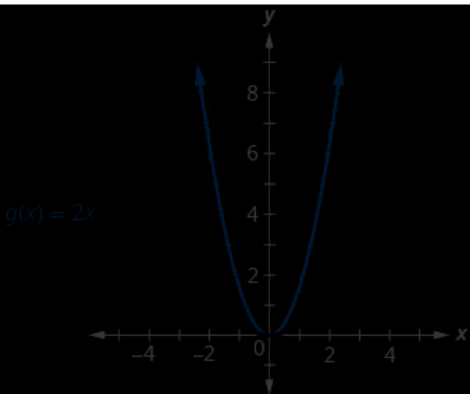
Graph Quadratic Functions of the Form $f(x)=ax2$

So far we graphed the quadratic function $f(x)=x2$ and then saw the effect of including a constant *h* or *k* in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient *a* on the resulting graph of the new function $f(x)=ax2$.

Let's look at the quadratic functions , $g(x) = 2x^2$ and .

x			$g(x) = 2x^2$	$(x, g(x))$		
–2	4	(–2, 4)	$2 \cdot 4$	(–2, 8)	$\frac{1}{2} \cdot 4$	(–2, 2)
–1	1	(–1, 1)	$2 \cdot 1$	(–1, 2)	$\frac{1}{2} \cdot 1$	$\left(-1, \frac{1}{2}\right)$
0	0	(0, 0)	$2 \cdot 0$	(0, 0)	$\frac{1}{2} \cdot 0$	(0, 0)
1	1	(1, 1)	$2 \cdot 1$	(1, 2)	$\frac{1}{2} \cdot 1$	$\left(1, \frac{1}{2}\right)$
2	4	(2, 4)	$2 \cdot 4$	(2, 8)	$\frac{1}{2} \cdot 4$	(2, 2)

If we graph these functions, we can see the effect of the constant a, assuming $a > 0$.



The graph of the function $g(x) = 2x^2$ is "skinnier" than the graph of

The graph of the function is "wider" than the graph of

To graph a function with constant a it is easiest to choose a few points on f(x)=x2 and multiply the y-values by a.

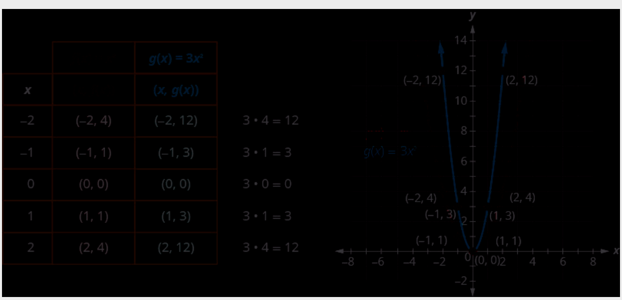
Graph of a Quadratic Function of the form f(x)=ax2

The coefficient *a* in the function f(x) = ax2 affects the graph of f(x) = x2 by stretching or compressing it.

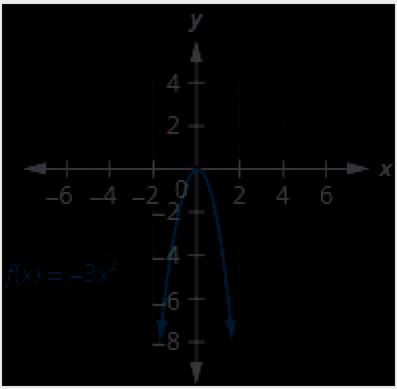
- If 0 < |a| < 1, the graph of f(x) = ax2 will be "wider" than the graph of f(x) = x2.
- If |a| > 1, the graph of f(x) = ax2 will be "skinnier" than the graph of f(x) = x2.

Graph f(x) = 3x2.

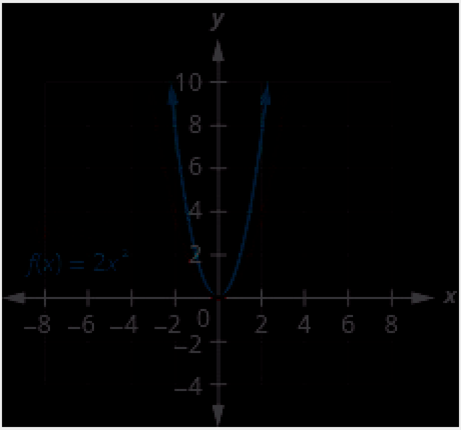
We will graph the functions f(x) = x2 and g(x) = 3x2 on the same grid. We will choose a few points on f(x) = x2 and then multiply the *y*-values by 3 to get the points for g(x) = 3x2.



Graph $f(x) = -3x2$.



Graph $f(x) = 2x2$.



Graph Quadratic Functions Using Transformations

We have learned how the constants *a*, *h*, and *k* in the functions, $f(x)=x^2+k, f(x)=(x-h)^2$, and $f(x)=ax^2$ affect their graphs. We can now put this together and graph quadratic functions $f(x)=ax^2+bx+c$ by first putting them into the form $f(x)=a(x-h)^2+k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to

both sides as we did when we completed the square with quadratic equations.

Quadratic Equation	Quadratic Function
$x^2+8x+6=0$	$f(x)=x^2+8x+6$
$x^2+8x=-6$	$f(x)=x^2+8x+6$
$x^2+8x+16=-6+16$	$f(x)=x^2+8x+16+6-16$
$(x+4)^2=10$	$f(x)=(x+4)^2-10$
Add 16 to both sides	Add and subtract 16 from the same side

When we complete the square in a function with a coefficient of x^2 that is not one, we have to factor that coefficient from just the *x*-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the *x*-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

Rewrite $f(x)=-3x^2-6x-1$ in the $f(x)=a(x-h)^2+k$ form by completing the square.

[redacted]

Separate the x terms from the constant.

[redacted]

Factor the coefficient of x^2, -3.

[redacted]

Prepare to complete the square.

[redacted]

Take half of 2 and then square it to complete the square. $(\frac{1}{2}\cdot 2)^2 = 1$

The constant 1 completes the square in pa [redacted] pa [redacted] multiplied by -3. So we are really adding -3 We must then add 3 to not change the value of the function.

Rewrite the trinomial as a square and su [redacted]

constants.
The function is now in the $f(x)=a(x-h)2+k$ for

$f(x) = a(x-h)^2 + k$

$f(x) = -3(x+1)^2 + 2$

Rewrite $f(x)=-4x2-8x+1$ in the $f(x)=a(x-h)2+k$ form by completing the square.
$f(x)=-4(x+1)2+5$

Rewrite $f(x)=2x2-8x+3$ in the $f(x)=a(x-h)2+k$ form by completing the square.
$f(x)=2(x-2)2-5$

Once we put the function into the $f(x)=(x-h)2+k$ form, we can then use the transformations as we did

in the last few problems. The next example will show us how to do this.

Graph $f(x)=x^2+6x+5$ by using transformations.

Step 1. Rewrite the function in $f(x)=a(x-h)^2+k$ vertex form by completing the square.

$f(x)=x^2+6x+5$

Separate the x terms from the constant.

$f(x)=x^2+6x+5$

Take half of 6 and then square it to complete the square.
$(\frac{1}{2}\cdot 6)^2=9$

We both add 9 and subtract 9 to not change the value of the

$f(x)=x^2+6x+9+5-9$

function.

Rewrite the trinomial as a square and su[redacted]

The function is now in the f(x)=(x−h)2+k fo[redacted]

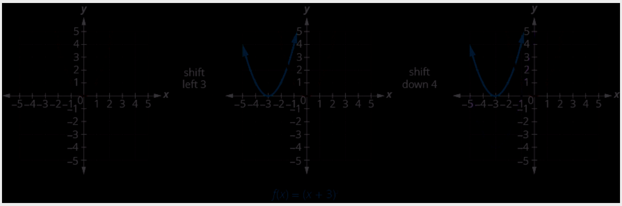
Step 2: Graph the function using transformations.

Looking at the *h*, *k* values, we see the graph will take the graph of f(x)=x2 and shift it to the left 3 units and down 4 units.

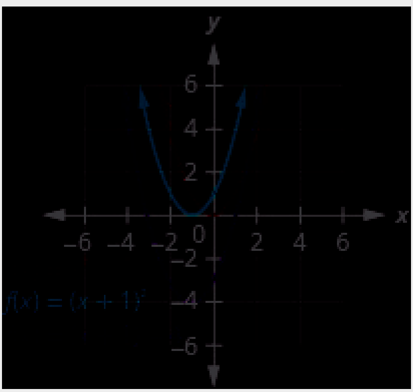
[redacted]

We first draw the graph of f(x)=x2 on the grid.

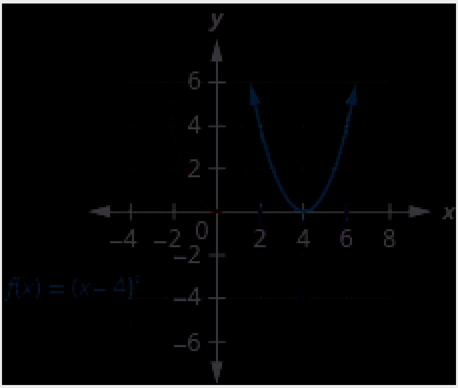
[redacted]



Graph $f(x)=x2+2x-3$ by using transformations.



Graph $f(x) = x2 - 8x + 12$ by using transformations.



We list the steps to take to graph a quadratic function using transformations here.

Graph a quadratic function using transformations.

Rewrite the function in $f(x) = a(x - h)2 + k$ form by completing the square. Graph the function using transformations.

Graph $f(x) = -2x2 - 4x + 2$ by using transformations.

Step 1. Rewrite the function in $f(x) = a(x - h)2 + k$ vertex form by completing the square.

[redacted]

Separate the *x* terms from the constant.

[redacted]

We need the coefficient of x2 to be one.

We[redacted]

x terms.

Take half of 2 and then square it to complete the square.

$(12 \cdot 2)2 = 1$

We add 1 to complete the square in the

pa[redacted]

parentheses is multiplied by −2. Se we are really adding −2. To not change the value of the function we add 2.

Rewrite the trinomial as a square and su [redacted]

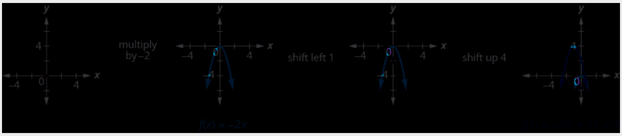
The function is now in the f(x)=a(x−h)2+k for [redacted]

Step 2. Graph the function using transformations.

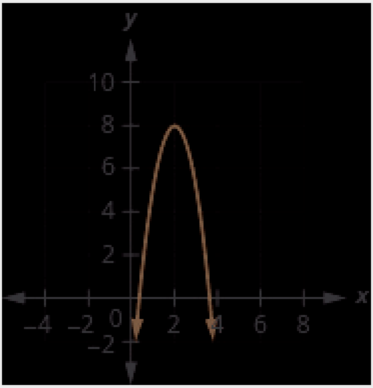
[redacted]

We first draw the graph of f(x)=x2 on the grid.

[redacted]

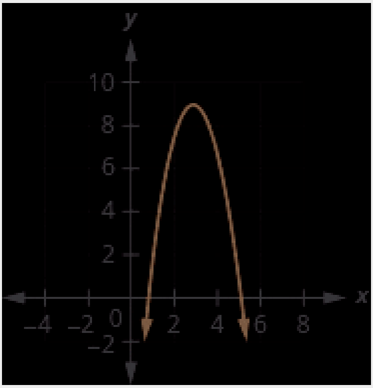


Graph $f(x) = -3x^2 + 12x - 4$ by using transformations.



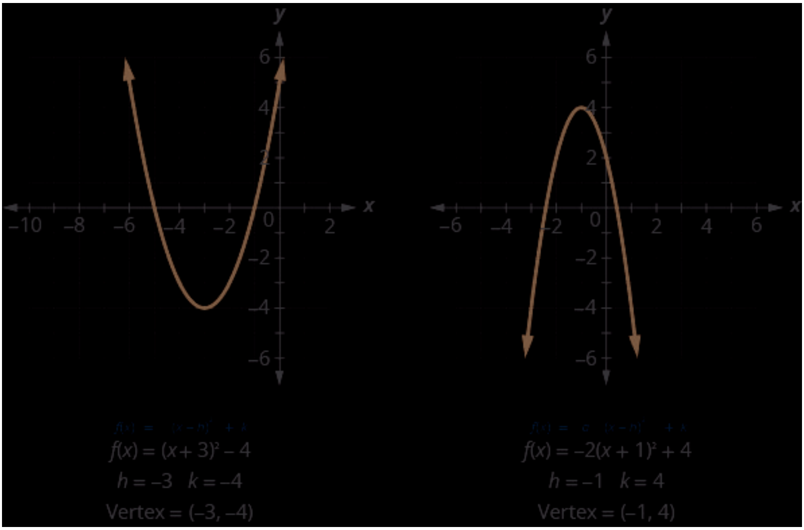
Graph $f(x) = -2x^2 + 12x - 9$ by using

transformations.



Now that we have completed the square to put a quadratic function into $f(x)=a(x-h)2+k$ form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants *h* and *k*.



In each case, the vertex is (h, k). Also the axis of symmetry is the line $x = h$.

We rewrite our steps for graphing a quadratic function using properties for when the function is in f(x)=a(x−h)2+k form.

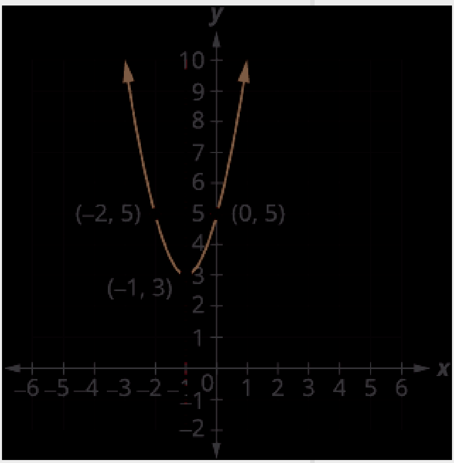
Graph a quadratic function in the form f(x)=a(x−h)2+k using properties.

Rewrite the function in f(x)=a(x−h)2+k form. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$. Find the axis of symmetry, $x = h$. Find the vertex, (h, k). Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry. Find the x-intercepts. Graph the parabola.

ⓐ Rewrite $f(x)=2x2+4x+5$ in $f(x)=a(x-h)2+k$ form and ⓑ graph the function using properties.

Rewrite the function in $f(x)=a(x-h)2+k$ form by completing the square.	$f(x)=2x2+4x+5$
	$f(x)=2(x2+2x)+5$
	$f(x)=2(x2+2x+1)+5-2$
	$f(x)=2(x+1)2+3$
Identify the constants a,h,k.	$a=2h=-1k=3$
Since $a=2$, the parabola opens up. 	
The axis of symmetry is $x=h$.	The axis of symmetry is $x=-1$.
The vertex is (h,k).	The vertex is $(-1,3)$.
Find the *y*-intercept by finding $f(0)$.	$f(0)=2\cdot02+4\cdot0+5$

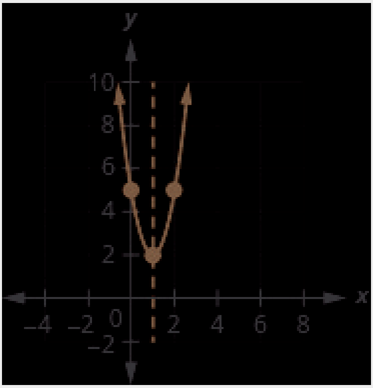
	$f(0)=5$ y intercept $(0,5)$
Find the point symmetric to $(0,5)$ across the axis of symmetry.	$(-2,5)$
Find the x-intercepts.	The discriminant negative, so there are no x-intercepts. Graph the parabola.



ⓐ Rewrite $f(x)=3x2-6x+5$ in $f(x)=a(x-h)2+k$ form and ⓑ graph the function using properties.

ⓐ $f(x)=3(x-1)2+2$

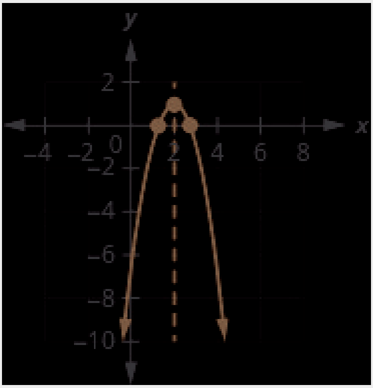
ⓑ



ⓐ Rewrite $f(x)=-2x2+8x-7$ in $f(x)=a(x-h)2+k$ form and ⓑ graph the function using properties.

ⓐ $f(x)=-2(x-2)2+1$

ⓑ

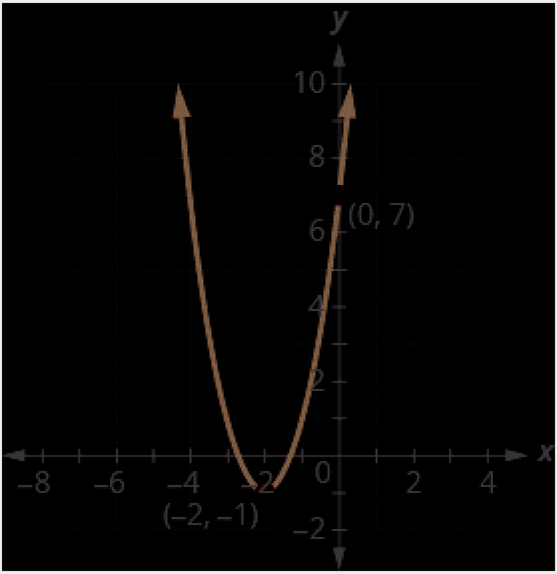


Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

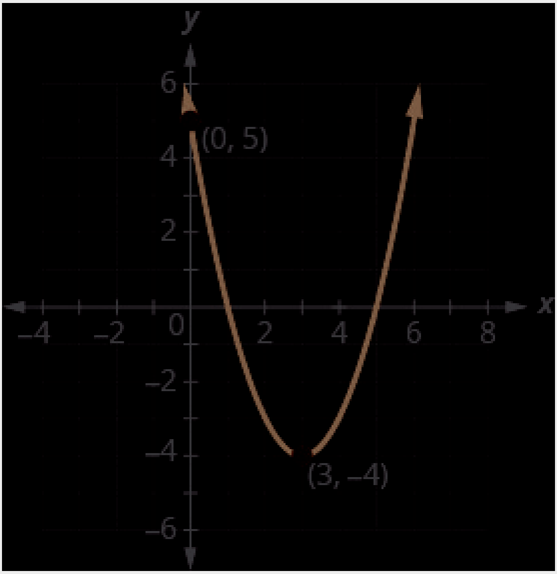
Determine the quadratic function whose graph is shown.



Since it is quadratic, we start with the $f(x)=a(x-h)2+k$form.	
The vertex, (h,k), is $(-2,-1)$ so $h=-2$ and $k=-1$.	$f(x)=a(x-(-2))2-1$
To find a, we use the y-intercept, $(0,7)$.	
So $f(0)=7$.	$7=a(0+2)2-1$
Solve for a.	$7=4a-1$
	$8=4a$
	$2=a$

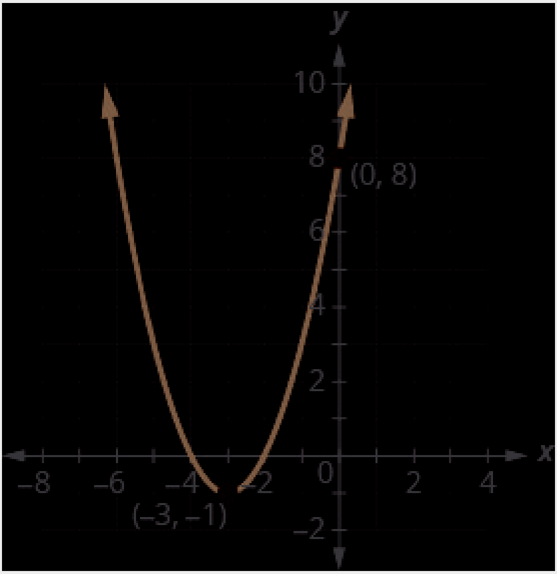
Write the function.	$f(x)=a(x-h)2+k$
Substitute in $h=-2, k=-1$ and $a=2$.	$f(x)=2(x+2)2-1$

Write the quadratic function in $f(x)=a(x-h)2+k$ form whose graph is shown.



$f(x)=(x-3)2-4$

Determine the quadratic function whose graph is shown.



$f(x) = (x + 3)2 - 1$

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- Function Shift Rules Applied to Quadratic Functions
- Changing a Quadratic from Standard Form to

Key Concepts

- Graph a Quadratic Function of the form $f(x)=x2+k$ Using a Vertical Shift
    - The graph of $f(x)=x2+k$ shifts the graph of $f(x)=x2$ vertically k units.
        - If $k > 0$, shift the parabola vertically up k units.
        - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x)=(x-h)2$ Using a Horizontal Shift
    - The graph of $f(x)=(x-h)2$ shifts the graph of $f(x)=x2$ horizontally h units.
        - If $h > 0$, shift the parabola horizontally left h units.

- If $h < 0$, shift the parabola horizontally right $|h|$ units.

- Graph of a Quadratic Function of the form $f(x)=ax2$
    - The coefficient a in the function $f(x)=ax2$ affects the graph of $f(x)=x2$ by stretching or compressing it.
      If $0<|a|<1$, then the graph of $f(x)=ax2$ will be “wider” than the graph of $f(x)=x2$.
      If $|a|>1$, then the graph of $f(x)=ax2$ will be “skinnier” than the graph of $f(x)=x2$.

- How to graph a quadratic function using transformations

  Rewrite the function in $f(x)=a(x-h)2+k$ form by completing the square. Graph the function using transformations.

- Graph a quadratic function in the vertex form $f(x)=a(x-h)2+k$ using properties

  Rewrite the function in $f(x)=a(x-h)2+k$ form. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$. Find the axis of symmetry, $x = h$. Find the vertex, (h, k). Find the$y$-intercept. Find the point symmetric to the y-intercept across the axis of symmetry. Find the x-intercepts, if possible. Graph the parabola.

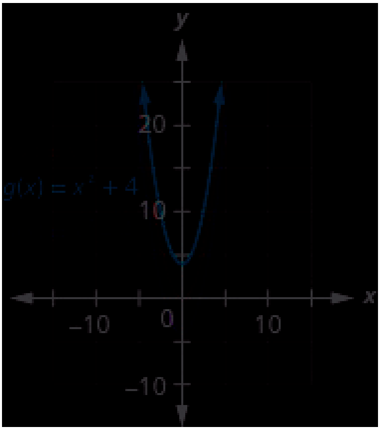
Practice Makes Perfect

Graph Quadratic Functions of the form $f(x)=x2+k$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, *k*, to the function has on the basic parabola.

$f(x)=x2, g(x)=x2+4$, and $h(x)=x2-4$.

ⓐ

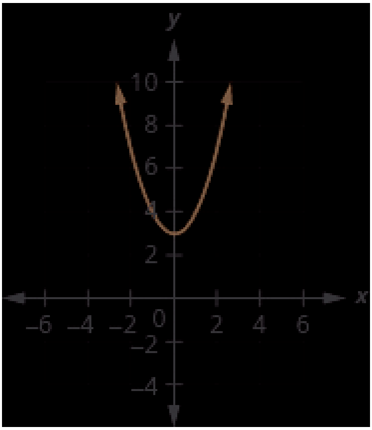


ⓑ The graph of $g(x)=x2+4$ is the same as the graph of $f(x)=x2$ but shifted up 4 units. The graph of $h(x)=x2-4$ is the same as the graph of $f(x)=x2$ but shift down 4 units.

$f(x)=x2, g(x)=x2+7$, and $h(x)=x2-7$.

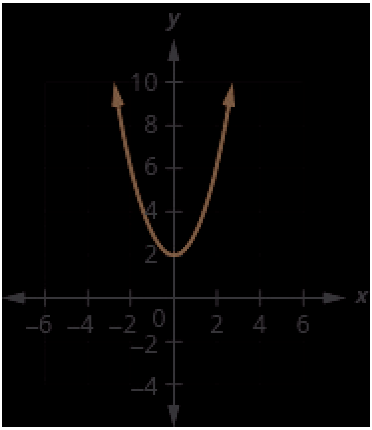
In the following exercises, graph each function using a vertical shift.

$f(x)=x2+3$



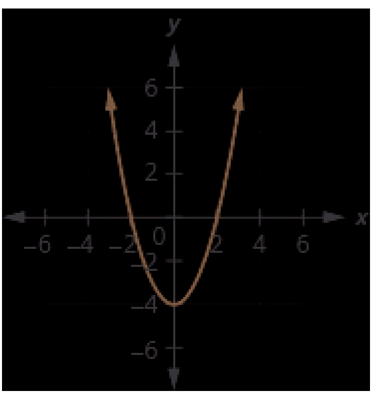
$f(x)=x2-7$

$g(x)=x2+2$



$g(x) = x2 + 5$

$h(x) = x2 - 4$



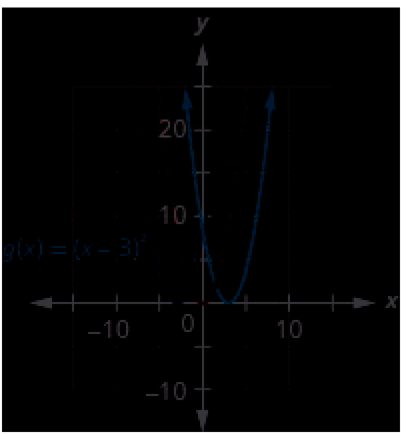
$h(x)=x2-5$

Graph Quadratic Functions of the form $f(x)=(x-h)2$

In the following exercises, ⓐ graph the quadratic functions on the same rectangular coordinate system and ⓑ describe what effect adding a constant, h, inside the parentheses has

$f(x)=x2, g(x)=(x-3)2$, and $h(x)=(x+3)2$.

ⓐ



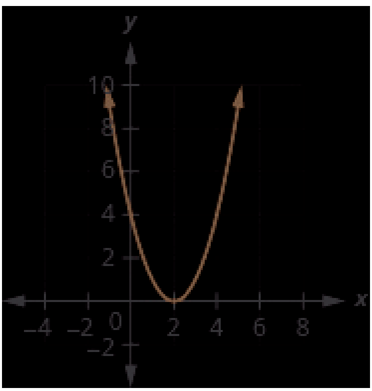
ⓑ The graph of $g(x)=(x-3)2$ is the same as the graph of $f(x)=x2$ but shifted right 3 units.

The graph of $h(x) = (x + 3)2$ is the same as the graph of $f(x) = x2$ but shifted left 3 units.

$f(x) = x2, g(x) = (x + 4)2$, and $h(x) = (x - 4)2$.

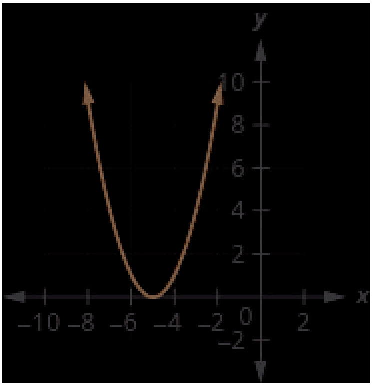
In the following exercises, graph each function using a horizontal shift.

$f(x) = (x - 2)2$



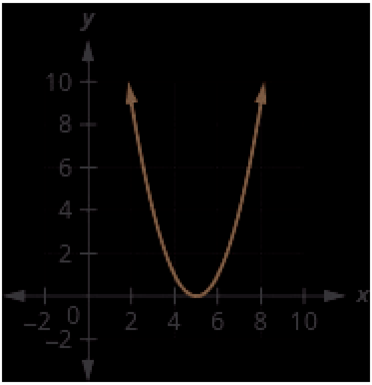
$f(x) = (x - 1)2$

$f(x) = (x + 5)2$



$f(x) = (x + 3)2$

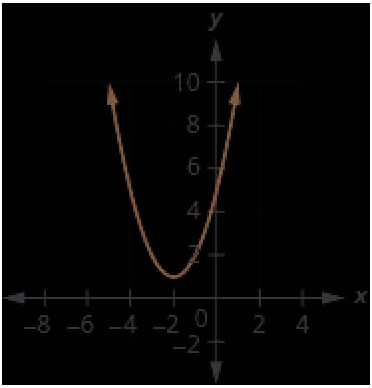
$f(x) = (x - 5)2$



$f(x) = (x+2)2$

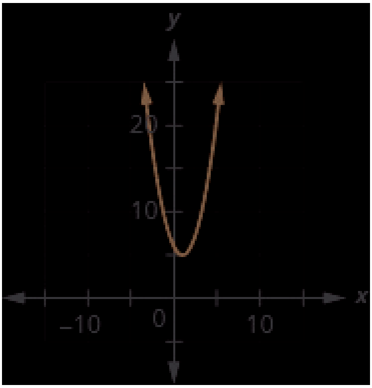
In the following exercises, graph each function using transformations.

$f(x) = (x+2)2+1$



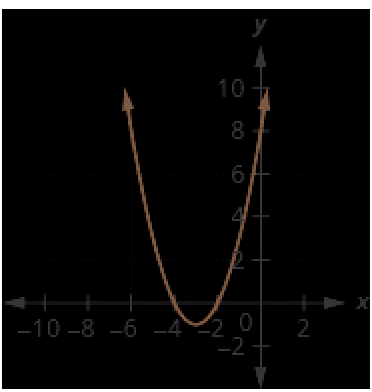
$f(x) = (x+4)2+2$

$f(x) = (x-1)2+5$



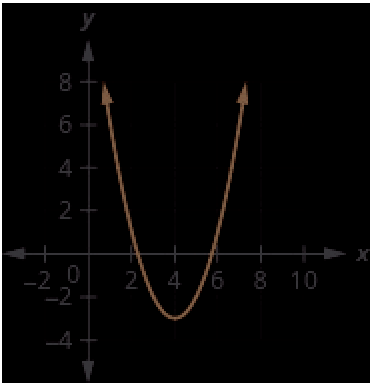
$f(x) = (x - 3)2 + 4$

$f(x) = (x + 3)2 - 1$



$f(x) = (x + 5)2 - 2$

$f(x) = (x - 4)2 - 3$

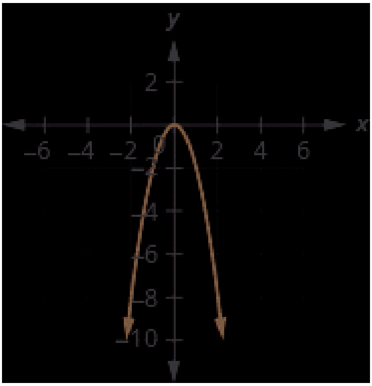


$f(x) = (x - 6)2 - 2$

Graph Quadratic Functions of the form $f(x) = ax2$

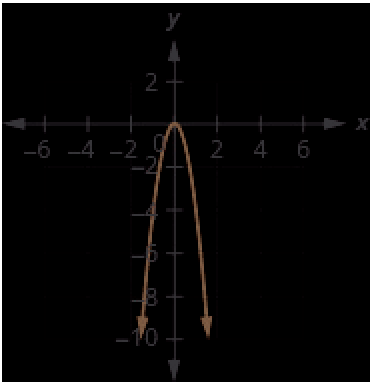
In the following exercises, graph each function.

$f(x) = -2x2$



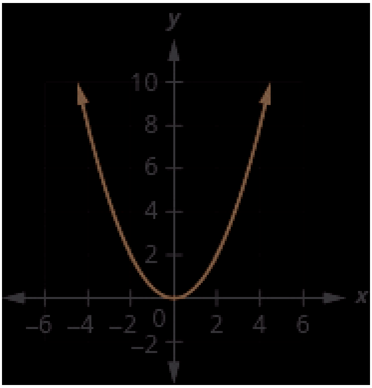
$f(x)=4x2$

$f(x)=-4x2$



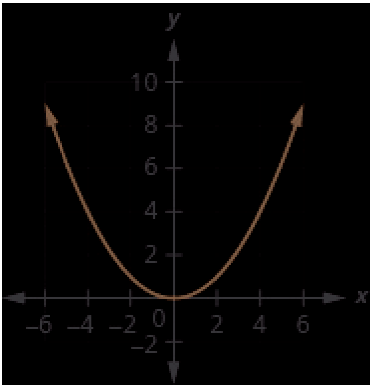
$f(x)=-x2$

$f(x) = 12x2$



$f(x) = 13x2$

$f(x) = 14x2$



$f(x) = -12x2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x-h)2 + k$ form by completing the square.

$f(x) = -3x2 - 12x - 5$

$f(x) = -3(x+2)2 + 7$

$f(x) = 2x2 - 12x + 7$

$f(x) = 3x2 + 6x - 1$

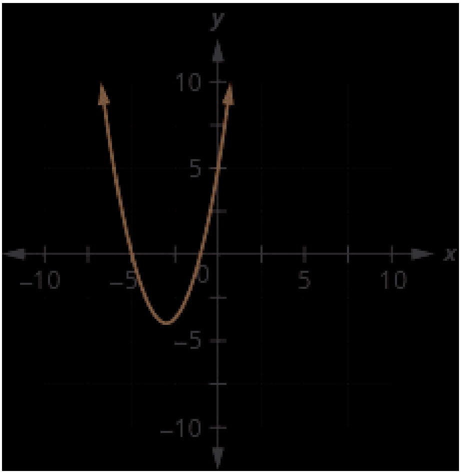
$f(x) = 3(x+1)2 - 4$

$f(x) = -4x2 - 16x - 9$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x-h)2 + k$ form and ⓑ graph it by using transformations.

$f(x) = x2 + 6x + 5$

ⓐ $f(x) = (x+3)2 - 4$
ⓑ

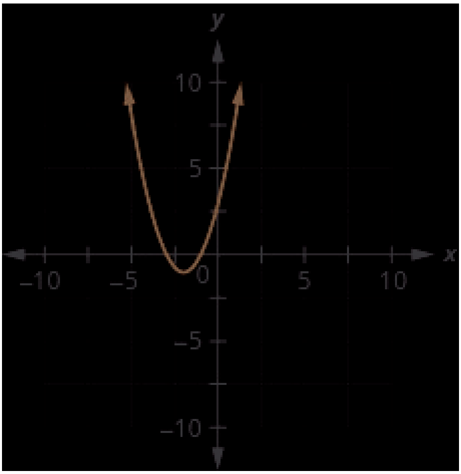


$f(x) = x2 + 4x - 12$

$f(x) = x2 + 4x + 3$

ⓐ $f(x) = (x + 2)2 - 1$

ⓑ

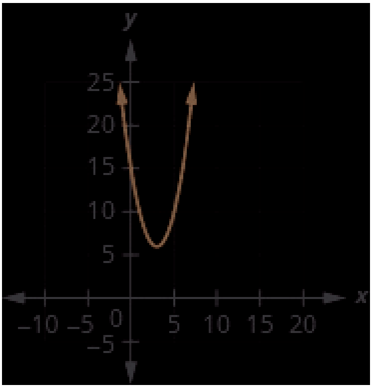


$f(x) = x2 - 6x + 8$

$f(x) = x2 - 6x + 15$

ⓐ $f(x) = (x - 3)2 + 6$

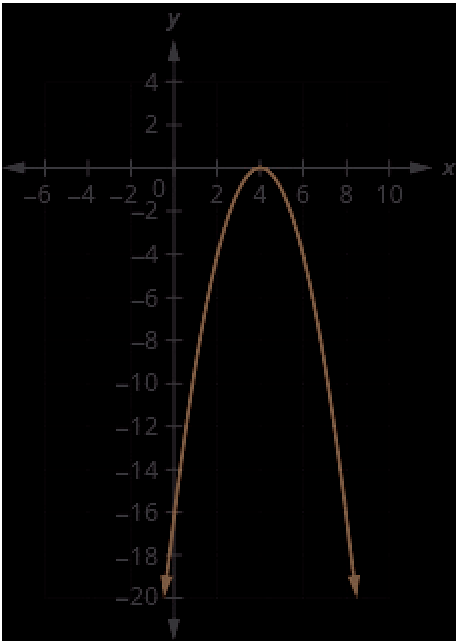
ⓑ



$f(x) = x2 + 8x + 10$

$f(x) = -x2 + 8x - 16$

ⓐ $f(x) = -(x - 4)2 + 0$
ⓑ

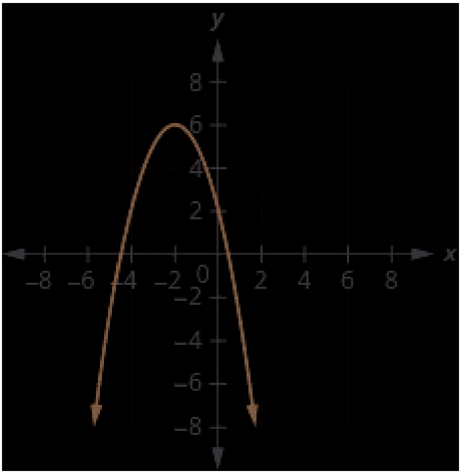


$f(x) = -x2 + 2x - 7$

$f(x) = -x2 - 4x + 2$

ⓐ $f(x) = -(x + 2)2 + 6$

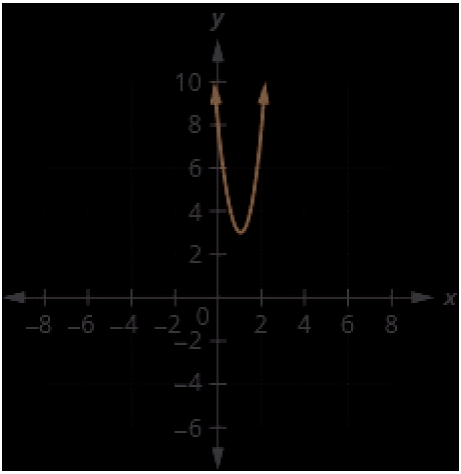
ⓑ



$f(x) = -x2 + 4x - 5$

$f(x) = 5x2 - 10x + 8$

ⓐ $f(x) = 5(x - 1)2 + 3$
ⓑ

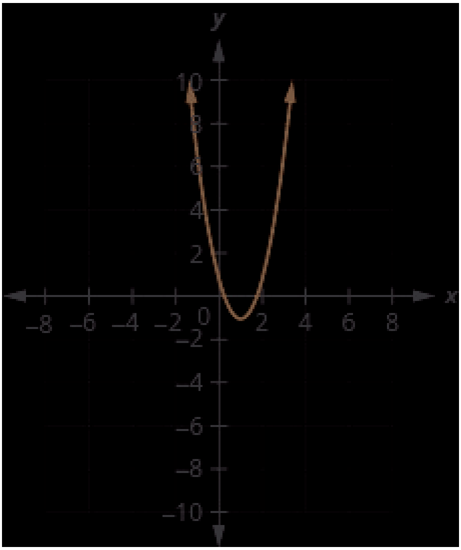


$f(x) = 3x2 + 18x + 20$

$f(x) = 2x2 - 4x + 1$

ⓐ $f(x) = 2(x - 1)2 - 1$

ⓑ

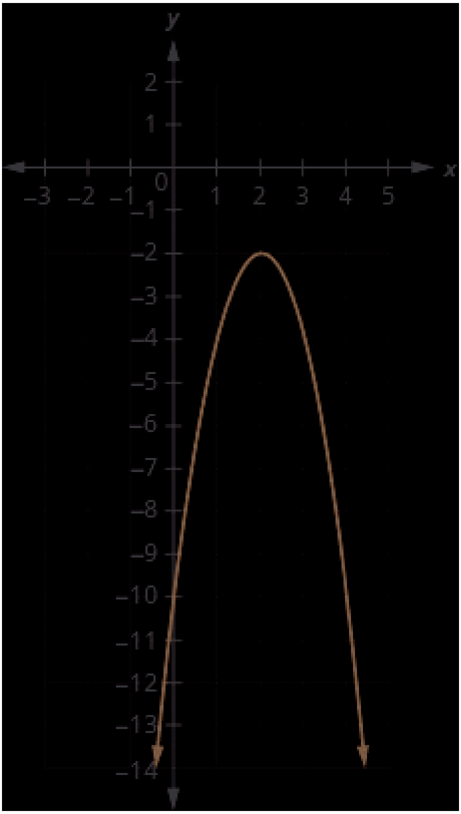


$$f(x) = 3x2 - 6x - 1$$

$$f(x) = -2x2 + 8x - 10$$

ⓐ $f(x) = -2(x-2)2 - 2$

ⓑ



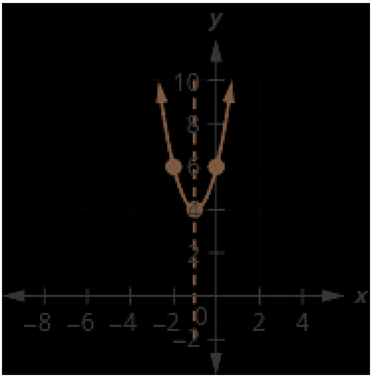
$f(x) = -3x2 + 6x + 1$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)2 + k$ form and ⓑ graph it using properties.

$f(x) = 2x2 + 4x + 6$

ⓐ $f(x) = 2(x + 1)2 + 4$

ⓑ

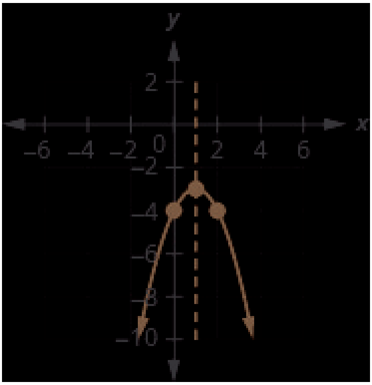


$f(x) = 3x2 - 12x + 7$

$f(x) = -x2 + 2x - 4$

ⓐ $f(x) = -(x - 1)2 - 3$

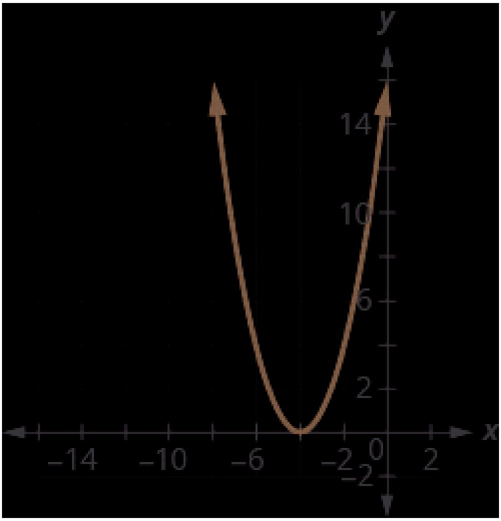
ⓑ



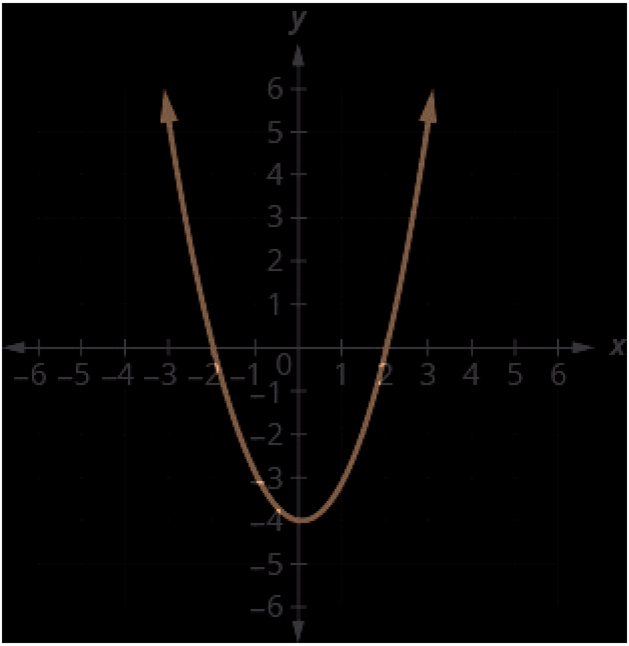
$f(x) = -2x2 - 4x - 5$

Matching

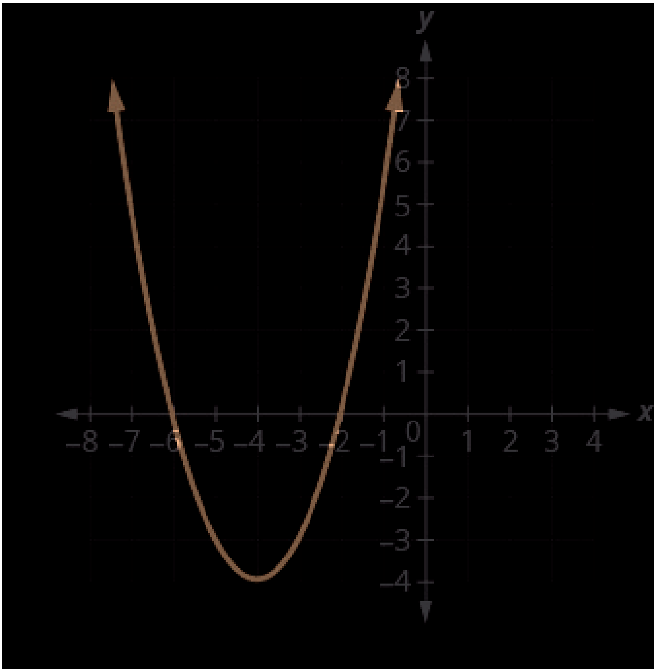
In the following exercises, match the graphs to one of the following functions: ⓐ $f(x) = x2 + 4$ ⓑ $f(x) = x2 - 4$ ⓒ $f(x) = (x + 4)2$ ⓓ $f(x) = (x - 4)2$ ⓔ $f(x) = (x + 4)2 - 4$ ⓕ $f(x) = (x + 4)2 + 4$ ⓖ $f(x) = (x - 4)2 - 4$ ⓗ $f(x) = (x - 4)2 + 4$



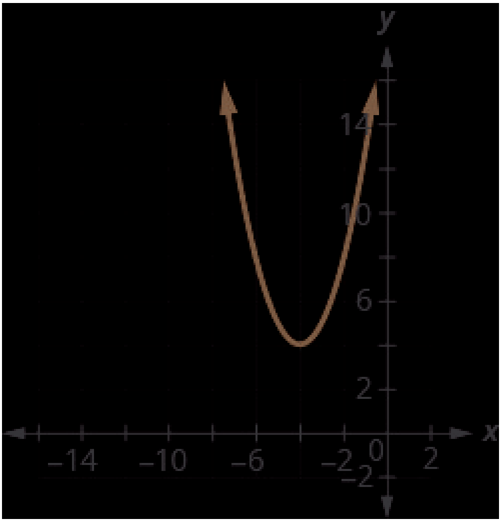
ⓒ



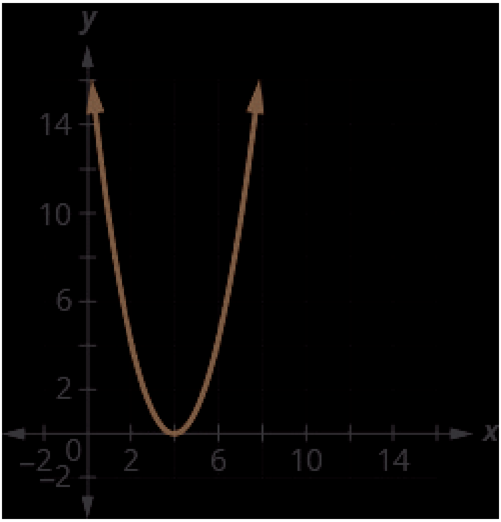
y
x
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6



e

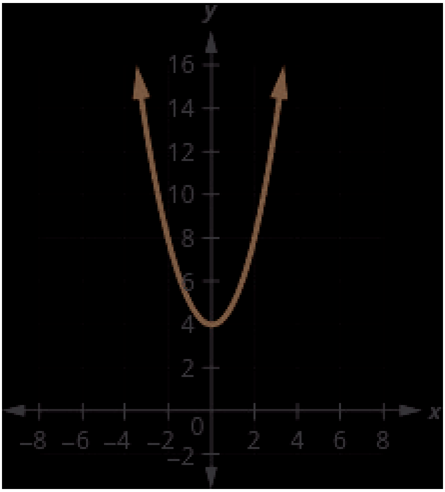


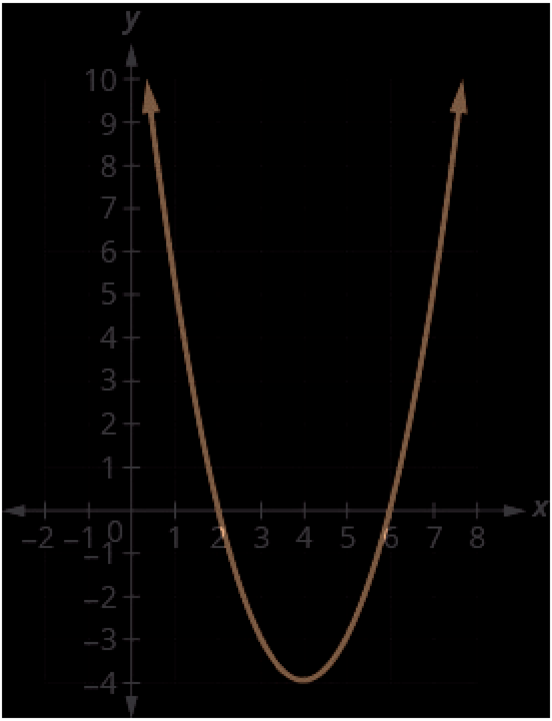
y
14
10
6
2
x
−14
−10
−6
−2
0
2
−2



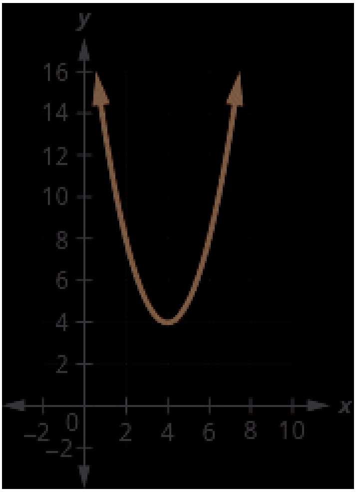
y
14
10
6
2
x
−2
0
2
6
10
14
−2

ⓓ


y
16
14
12
10
8
6
4
2
0
−2
x
−8 −6 −4 −2 2 4 6 8

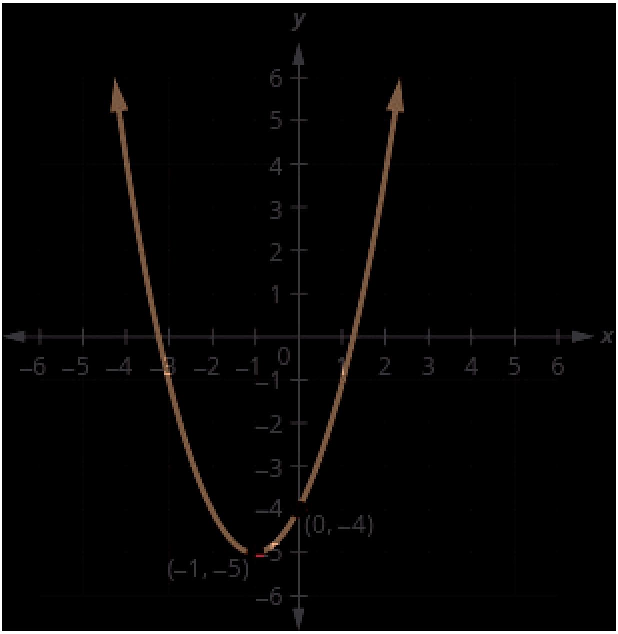

y
10
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
x
−2 −1 1 2 3 4 5 6 7 8

ⓖ

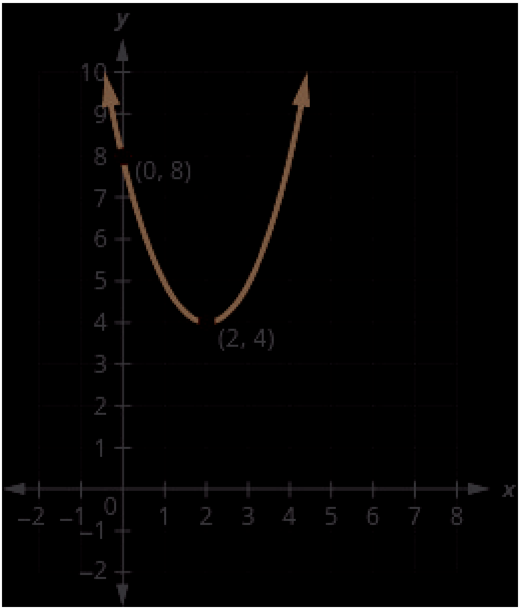


Find a Quadratic Function from its Graph

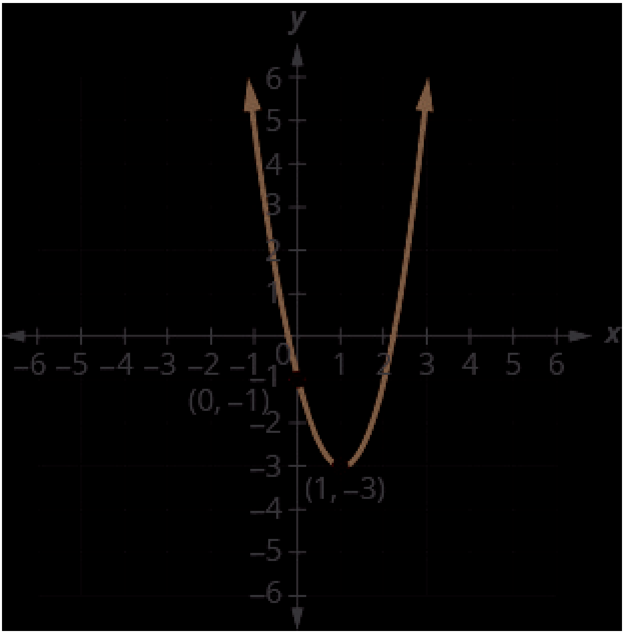
In the following exercises, write the quadratic function in $f(x)=a(x-h)2+k$ form whose graph is shown.



$f(x) = (x + 1)2 - 5$

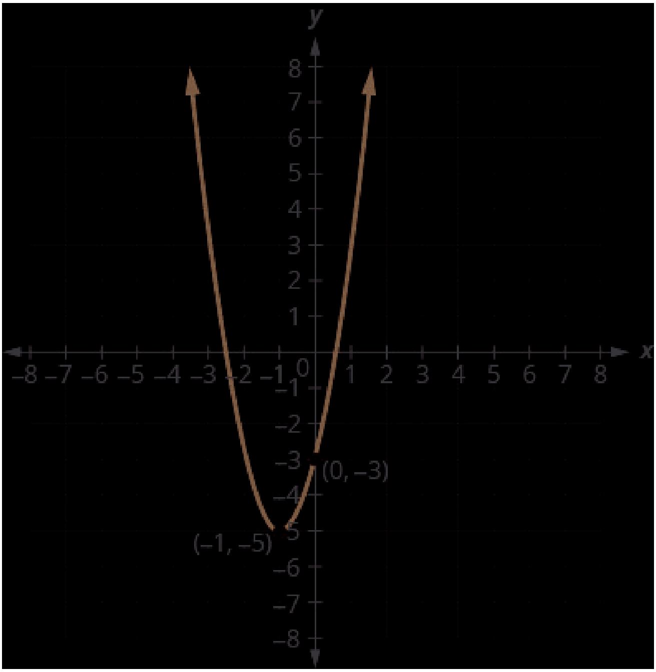


y
x
(0, 8)
(2, 4)



y
x
(0, –1)
(1, –3)

$f(x) = 2(x-1)2-3$



Writing Exercise

Graph the quadratic function $f(x) = x2 + 4x + 5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Answers will vary.

Graph the quadratic function $f(x) = 2x2 - 4x - 3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
graph Quadratic Functions of the form $f(x) = x^2 + k$.			
graph Quadratic Functions of the form $f(x) = (x - h)^2$.			
graph Quadratic Functions of the form $f(x) = ax^2$.			
graph Quadratic Functions Using Transformations.			
find a Quadratic Function from its Graph.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Solve Systems of Nonlinear Equations
By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

1. Solve the system by graphing: $\{x-3y=-3x+y=5$.
   If you missed this problem, review [link].
2. Solve the system by substitution: $\{x-4y=-4-3x+4y=0$.
   If you missed this problem, review [link].
3. Solve the system by elimination: $\{3x-4y=-95x+3y=14$.
   If you missed this problem, review [link].

Solve a System of Nonlinear Equations Using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.

$$\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases} \quad \begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases} \quad \begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$$

> System of Nonlinear Equations
>
> A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear

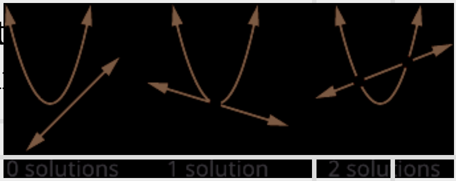
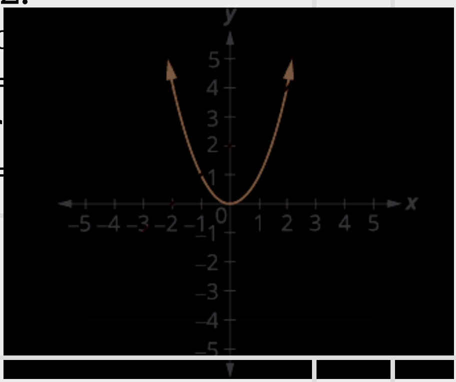
equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.

Solve a system of nonlinear equations by graphing.

Identify the graph of each equation. Sketch the possible options for intersection. Graph the first equation. Graph the second equation on the same rectangular coordinate system. Determine whether the graphs intersect. Identify the points of intersection. Check that each ordered pair is a solution to both original equations.

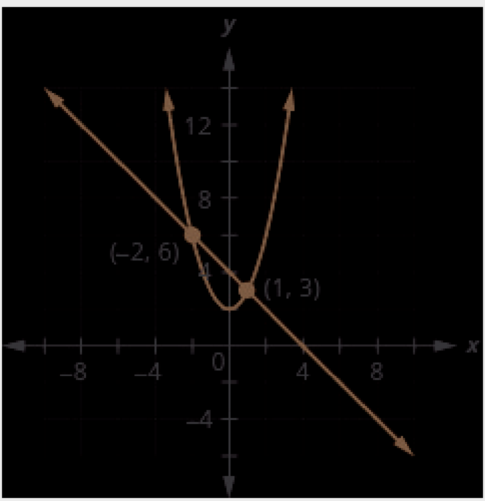
Solve the system by graphing: $\{x-y=-2\ y=x2$.

Identify each graph.	$\begin{cases} x - y = -2 & \text{line} \\ y = x^2 & \text{parabola} \end{cases}$
Sketch the possible options for int… pa…	
Graph the line, $x - y = -2$. Slo… $y =$ … Gr… $y =$ …	
Identify the points of intersection.	The points of intersection appear to be $(2,4)$ and $(-1,1)$.
Check to make sure each solution makes both equations true. $(2,4)$ $x - y = -2$ $y = x^2$ $2 - 4 \stackrel{?}{=} -2$ $4 \stackrel{?}{=} 2^2$ $-2 = -2 ✓$ $4 = 4 ✓$ $(-1,1)$	

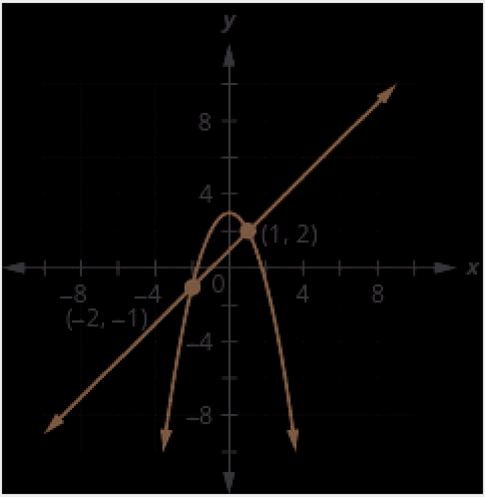
$x - y =$
$-2y = x2 - 1 - 1 = ?$
$-21 = ?(-1)2 - 2 =$
$-2✓1 = 1✓$

The solutions are $(2,4)$ and $(-1,1)$.

Solve the system by graphing: $\{x + y = 4 y = x2 + 2$.



Solve the system by graphing: $\{x - y = -1 \; y = -x2 + 3.$



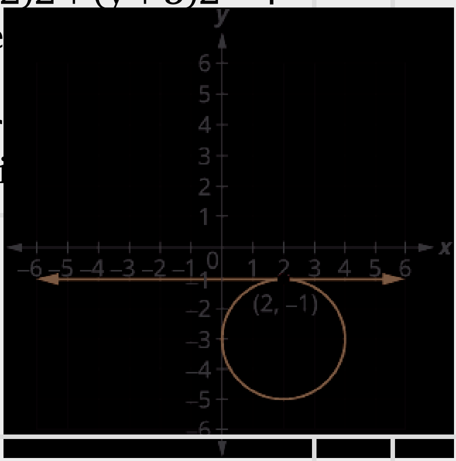
To identify the graph of each equation, keep in mind the characteristics of the x2 and y2 terms of each conic.

Solve the system by graphing: $\{y = -1(x - 2)2 + (y + 3)2 = 4.$

Identify each graph.	$\{y=-1$ line $(x-2)2+(y+3)2=4$ circle

Sketch the possible options for the int
an

Graph the circle, $(x-2)2+(y+3)2=4$
Ce
2
Gr
It i



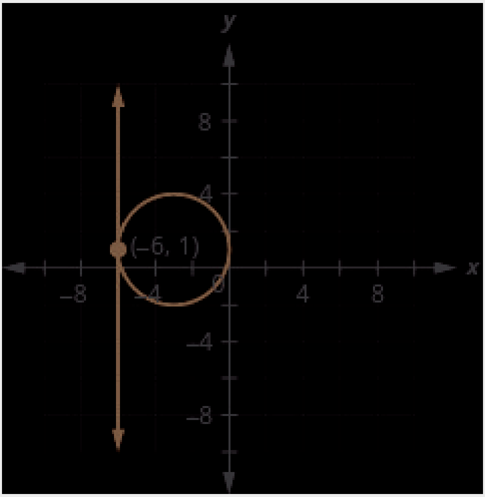
Identify the points of intersection.	The point of intersection appears to be $(2,-1)$.

Check to make sure the solution makes both equations true.
$(2,-1)$
$(x-2)2+(y+3)2=4$ $y=-1$
$(2-2)2+(-1+3)2=?$
4 $-1=$

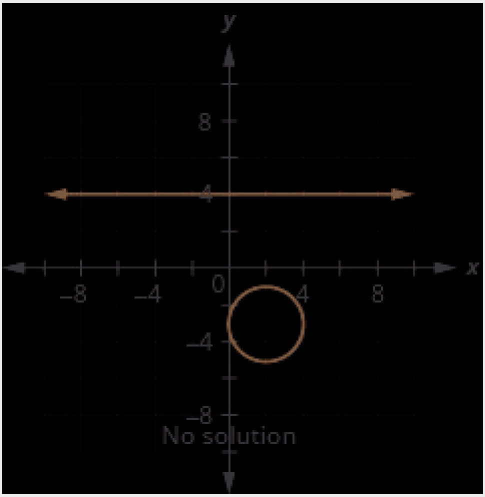
$-1✓(0)2+(2)2=?$
$44=4✓$

The solution is $(2,-1)$.

Solve the system by graphing: $\{x=-6(x+3)2+(y-1)2=9.$



Solve the system by graphing: {y = 4(x − 2)2 + (y + 3)2 = 4.



Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially

well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.

Solve a system of nonlinear equations by substitution.

Identify the graph of each equation. Sketch the possible options for intersection. Solve one of the equations for either variable. Substitute the expression from Step 2 into the other equation. Solve the resulting equation. Substitute each solution in Step 4 into one of the original equations to find the other variable. Write each solution as an ordered pair. Check that each ordered pair is a solution to **both** original equations.

Solve the system by using substitution: $\{9x2+y2=9y=3x-3.$

Identify each graph. $\begin{cases} 9x^2+y^2=9 & \text{ellipse} \\ y=3x-3 & \text{line} \end{cases}$

Sketch the possible options for intersection of ell

The equation $y=3x-3$ is solved for y.

$$y = 3x - 3$$

$$9x^2 + y^2 = 9$$

Substitute $3x-3$ for y in the first equation.

$$9x^2 + (3x - 3)^2 = 9$$

Solve the equation for x.

$$9x^2 + 9x^2 - 18x + 9 = 9$$

$$18x^2 - 18x = 0$$
$$18x(x-1) = 0$$
$$x = 0 \qquad x = 1$$

Substitute $x=0$ and $x=1$ into $y=3x-3$ to find y.

$$y = 3x - 3 \qquad y = 3x - 3$$

$$y = 3 \cdot 0 - 3 \qquad y = 3 \cdot 1 - 3$$
$$y = -3 \qquad y = 0$$

The ordered pairs are $(0,-3)$, $(1,0)$.

Check **both** ordered

pairs in **both** equations.

$(0,-3)$

$9x2+y2=9y=3x$

$-39\cdot 02+(-3)2=?$

$9-3=?3\cdot 0-30+9=?$

$9-3=?0-39=9✓$

$-3=-3✓$

$(1,0)$

$9x2+y2=9y=3x$

$-39\cdot 12+02=?90=?$

$3\cdot 1-39+0=?90=?$

$3-39=9✓0=0✓$

The solutions are $(0,-3),(1,0)$.

Solve the system by using substitution: $\{x2+9y2=9y=13x-3$.

No solution

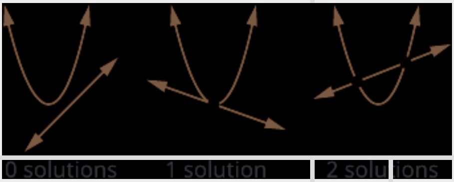
Solve the system by using substitution:
$\begin{cases} 4x^2+y^2=4 \\ y=x+2. \end{cases}$

$\left(-\frac{4}{5},\frac{6}{5}\right),(0,2)$

So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

Solve the system by using substitution:
$\begin{cases} x^2-y=0 \\ y=x-2. \end{cases}$

Identify each graph.	$\begin{cases} x^2-y=0 & \text{parabola} \\ y=x-2 & \text{line} \end{cases}$
Sketch the possible options for intersection of a parabola and a line	



The equation $y = x - 2$ is solved for y.

[redacted]

[redacted]

Substitute $x - 2$ for y in the first equation.

[redacted]

Solve the equation for x.

[redacted]

This doesn't factor easily, so we can check the discriminant.

$b^2 - 4ac$
$(-1)^2 - 4\cdot 1\cdot 2$
-7

The discriminant is negative, so there is no real solution.
The system has no solution.

Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = 2x - 3 \end{cases}$.

No solution

Solve the system by using substitution: $\{y2 - x = 0y = 3x - 2.$

$(49, -23),(1,1)$

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of

linear equations. The steps are listed for reference.

Solve a system of equations by elimination.

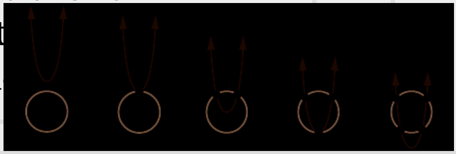
Identify the graph of each equation. Sketch the possible options for intersection. Write both equations in standard form. Make the coefficients of one variable opposites.
Decide which variable you will eliminate.
Multiply one or both equations so that the coefficients of that variable are opposites. Add the equations resulting from Step 3 to eliminate one variable. Solve for the remaining variable.
Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable. Write each solution as an ordered pair.
Check that each ordered pair is a solution to **both** original equations.

Solve the system by elimination:
{x2 + y2 = 4x2 − y = 4.

Identify each graph.

[redacted]

Sketch the possible options for int



an

[redacted]

Both equations are in standard form.

[redacted]

To get opposite coefficients of x2, we

[redacted]

second equation by -1.

Simplify.

[redacted]

Add the two equations to eliminate x2.

[redacted]

Solve for y.

[redacted]

[redacted]

Substitute $y = 0$ and $y = -1$ into one of the

[redacted]

Then solve for x.

$$\begin{aligned} x^2 - y &= 4 \\ x^2 - 0 &= 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned} \qquad \begin{aligned} x^2 - y &= 4 \\ x^2 - (-1) &= 4 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Write each solution as an ordered pair.	The ordered pairs are $(-2,0)$ $(2,0)$. $(3,-1)(-3,-1)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,0)$, $(2,0)$, $(3,-1)$, and $(-3,-1)$.

Solve the system by elimination: $\{x2+y2=9x2-y=9$.

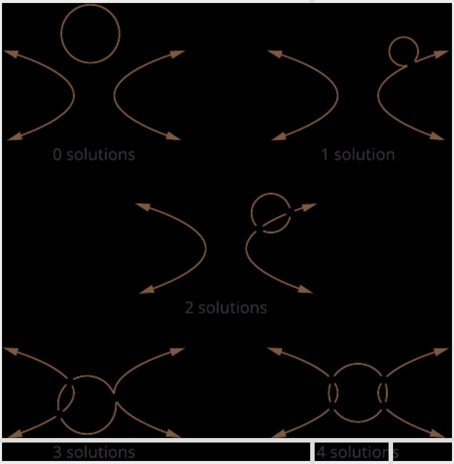
$(-3,0),(3,0),(-22,-1),(22,-1)$

Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$.
$(-1,0),(0,1),(0,-1)$

There are also four options when we consider a circle and a hyperbola.

Solve the system by elimination: $\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$.

Identify each graph. Sketch the possible options for intersection of a circle and hyperbola.	$\begin{cases} x^2+y^2=7 & \text{circle} \\ x^2-y^2=1 & \text{hype} \end{cases}$



Both equations are in standard form.	$\begin{cases}x^2+y^2=7\\x^2-y^2=1\end{cases}$
The coefficients of y^2 are opposite, so we will add the equations.	$\begin{cases}x^2+y^2=7\\x^2-y^2=1\end{cases}$ ______ 2x
Simplify.	$x^2=4$ $x=\pm2$ $x=2$ $x=-2$
Substitute $x=2$ and $x=-2$ into one of the original equations. Then solve for *y*.	$x^2+y^2=7$ $x^2+y^2=7$ $2^2+y^2=7$ ±3 $y=\pm3$
Write each solution as an ordered pair.	The ordered pairs are $(-2,3)$, $(-2,-3)$, $(2,3)$, and $(2,-3)$.
Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four	The solutions are $(-2,3)$, $(-2,-3)$, $(2,3)$,

solutions to you.		and $(2, -3)$.

Solve the system by elimination: $\begin{cases} x^2+y^2=25 \\ y^2-x^2=7 \end{cases}$.

$(-3,-4),(-3,4),(3,-4),(3,4)$

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y^2=4 \end{cases}$.

$(-2,0),(2,0)$

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Identify what we are looking for.	Two different numbers.
Define the variables.	$x =$ first number $y =$ second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.

$$x^2 - y^2 = 15$$

Second sentence.	The sum of the numbers is 5.

[redacted]

Solve the system by substitution

[redacted]

Solve the second equation for x.

[redacted]

Substitute x into the first equation.

[redacted]

Expand and simplify.

[redacted]

Solve for y.

[redacted]

Substitute back into the second equation.

[redacted]

[redacted]

The numbers are 1 and 4.

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

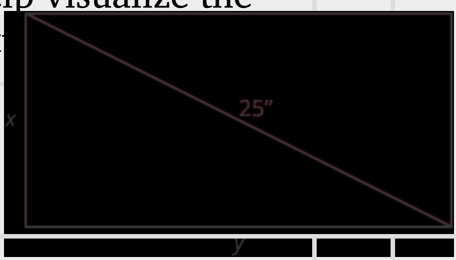
4 and 6

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

-18 and 17

Myra purchased a small 25” TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Identify what we are looking for.	The length and width of the rectangle
Define the variables.	Let $x =$ width of the rectangle $y =$ length of the rectangle
Draw a diagram to help visualize the sit	
	Area is 300 square inches. The diagonal of the right triangle is 25 inches.
Translate the information into a system of	



equations.

[redacted]

The area of the rectangle is 300 square inches.

[redacted]

Solve the system using substitution.

[redacted]

Solve the second equation for x.

[redacted]

Substitute x into the first equation.

[redacted]

[redacted]

Simplify.

[redacted]

Multiply by y2 to clear the fractions.

[redacted]

Put in standard form.

[redacted]

Solve by factoring.

[redacted]

[redacted]

[redacted]

Since y is a side of the rectangle, we discard the negative values.

[redacted]

Substitute back into the second equation.

[redacted]

[redacted]

If the length is 15 inches, the width is 20 inches.

If the length is 20 inches, the width is 15 inches.

Edgar purchased a small 20” TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and

width of the TV screen?

If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

Access these online resources for additional instructions and practice with solving nonlinear equations.

- Nonlinear Systems of Equations
- Solve a System of Nonlinear Equations
- Solve a System of Nonlinear Equations by

Key Concepts

- **How to solve a system of nonlinear equations by graphing.**

  Identify the graph of each equation. Sketch the possible options for intersection. Graph the first equation. Graph the second equation on the same rectangular coordinate system. Determine whether the graphs intersect. Identify the points of intersection. Check that each ordered pair is a solution to both original equations.

- **How to solve a system of nonlinear equations by substitution.**

  Identify the graph of each equation. Sketch the possible options for intersection.

  Solve one of the equations for either variable. Substitute the expression from Step 2 into the other equation. Solve the resulting equation.

Substitute each solution in Step 4 into one of the original equations to find the other variable. Write each solution as an ordered pair. Check that each ordered pair is a solution to **both** original equations.

- **How to solve a system of equations by elimination.**

  Identify the graph of each equation. Sketch the possible options for intersection. Write both equations in standard form. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites. Add the equations resulting from Step 3 to eliminate one variable. Solve for the remaining variable. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable. Write each solution as an ordered pair. Check that each ordered pair is a solution to **both** original equations.

Section Exercises

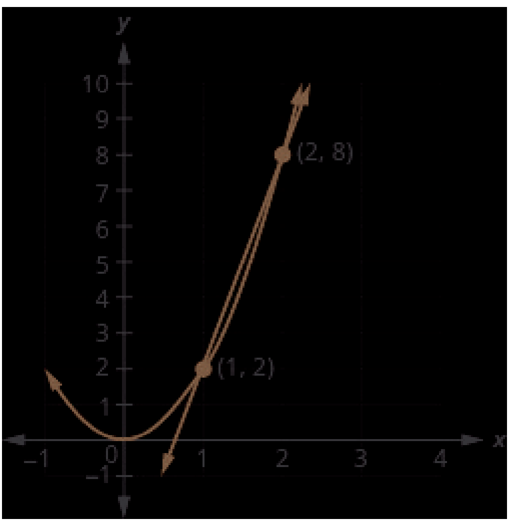
Practice Makes Perfect

Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

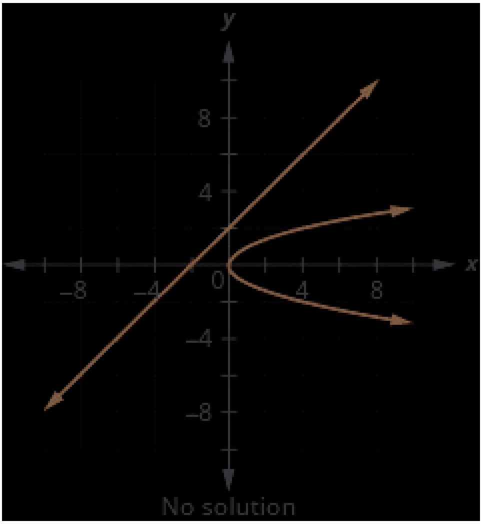
$\{y=2x+2y=-x2+2$

$\{y=6x-4y=2x2$



$\{x+y=2x=y2$

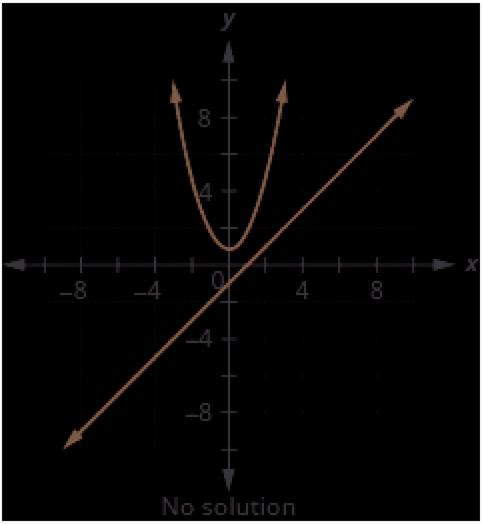
$\{x-y=-2x=y2$



No solution

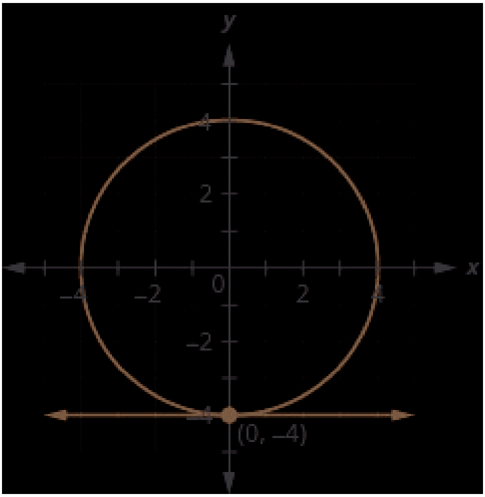
$\{y = 32x + 3y = -x2 + 2$

$\{y = x - 1y = x2 + 1$



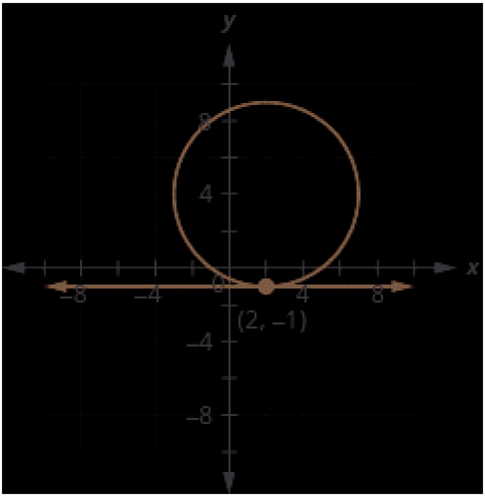
$\{x = -2x2 + y2 = 4$

$\{y = -4x2 + y2 = 16$



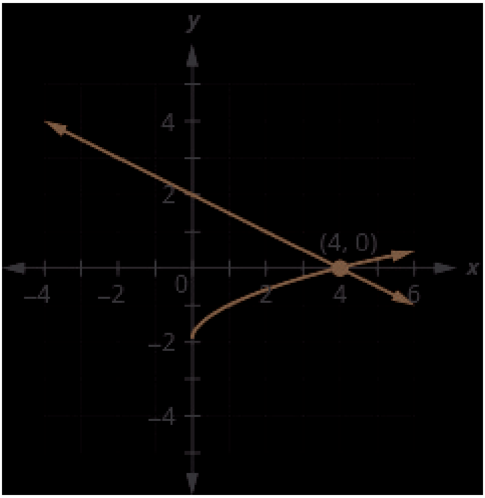
$\{x = 2(x + 2)2 + (y + 3)2 = 16$

$\{y = -1(x - 2)2 + (y - 4)2 = 25$



$\{y = -2x + 4y = x + 1$

$\{y = -12x + 2y = x - 2$



Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

$\{x2+4y2=4y=12x-1$

$\{9x2+y2=9y=3x+3$

$(-1,0),(0,3)$

$\{9x2+y2=9y=x+3$

$\{9x2+4y2=36x=2$

$(2,0)$

$\{4x2+y2=4y=4$

$\{x2+y2=169x=12$

$(12,-5),(12,5)$

$\{3x2-y=0y=2x-1$

$\{2y2-x=0y=x+1$

No solution

$\{y=x2+3y=x+3$

$\{y=x2-4y=x-4$

$(0,-4),(1,-3)$

$\{x2+y2=25x-y=1$

$\{x2+y2=252x+y=10$

$(3,4),(5,0)$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

$\{x2+y2=16x2-2y=8$

$\{x2+y2=16x2-y=4$

$(0,-4),(-7,3),(7,3)$

$\{x2+y2=4x2+2y=1$

$\{x2+y2=4x2-y=2$

$(0,-2),(-3,1),(3,1)$

$\{x2+y2=9x2-y=3$

$\begin{cases} x^2+y^2=4 \\ y^2-x=2 \end{cases}$

$(-2,0),(1,-3),(1,3)$

$\begin{cases} x^2+y^2=25 \\ 2x^2-3y^2=5 \end{cases}$

$\begin{cases} x^2+y^2=20 \\ x^2-y^2=-12 \end{cases}$

$(-2,-4),(-2,4),(2,-4),(2,4)$

$\begin{cases} x^2+y^2=13 \\ x^2-y^2=5 \end{cases}$

$\begin{cases} x^2+y^2=16 \\ x^2-y^2=16 \end{cases}$

$(-4,0),(4,0)$

$\begin{cases} 4x^2+9y^2=36 \\ 2x^2-9y^2=18 \end{cases}$

$\begin{cases} x^2-y^2=3 \\ 2x^2+y^2=6 \end{cases}$

$(-3,0),(3,0)$

$\{4x2-y2=44x2+y2=4$

$\{x2-y2=-53x2+2y2=30$

$(-2,-3),(-2,3),(2,-3),(2,3)$

$\{x2-y2=1x2-2y=4$

$\{2x2+y2=11x2+3y2=28$

$(-1,-3),(-1,3),(1,-3),(1,3)$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

The sum of two numbers is -6 and the product is 8. Find the numbers.

The sum of two numbers is 11 and the product is -42. Find the numbers.

-3 and 14

The sum of the squares of two numbers is 65. The difference of the number is 3. Find the numbers.

The sum of the squares of two numbers is 113. The difference of the number is 1. Find the numbers.

-7 and -8 or 8 and 7

The difference of the squares of two numbers is 15. The difference of twice the square of the first number and the square of the second number is 30. Find the numbers.

The difference of the squares of two numbers is 20. The difference of the square of the first number and twice the square of the second number is 4. Find the numbers.

-6 and -4 or -6 and 4 or 6 and -4 or 6 and 4

The perimeter of a rectangle is 32 inches and its area is 63 square inches. Find the length and

width of the rectangle.

The perimeter of a rectangle is 52 cm and its area is 165 cm2. Find the length and width of the rectangle.

If the length is 11 cm, the width is 15 cm. If the length is 15 cm, the width is 11 cm.

Dion purchased a new microwave. The diagonal of the door measures 17 inches. The door also has an area of 120 square inches. What are the length and width of the microwave door?

Jules purchased a microwave for his kitchen. The diagonal of the front of the microwave measures 26 inches. The front also has an area of 240 square inches. What are the length and width of the microwave?

If the length is 10 inches, the width is 24 inches. If the length is 24 inches, the width is 10 inches.

Roman found a widescreen TV on sale, but isn't sure if it will fit his entertainment center. The

TV is 60”. The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1728 square inches. His entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Roman’s entertainment center?

Donnette found a widescreen TV at a garage sale, but isn’t sure if it will fit her entertainment center. The TV is 50”. The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1200 square inches. Her entertainment center has an insert for the TV with a length of 38 inches and width of 27 inches. What are the length and width of the TV screen and will it fit Donnette’s entertainment center?

The length is 40 inches and the width is 30 inches. The TV will not fit Donnette’s entertainment center.

Writing Exercises

In your own words, explain the advantages and disadvantages of solving a system of equations by graphing.

Explain in your own words how to solve a system of equations using substitution.

Answers will vary.

Explain in your own words how to solve a system of equations using elimination.

A circle and a parabola can intersect in ways that would result in 0, 1, 2, 3, or 4 solutions. Draw a sketch of each of the possibilities.

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
solve a system of nonlinear equations using graphing.			
solve a system of nonlinear equations using substitution.			
solve a system of nonlinear equations using elimination.			
use a system of nonlinear equations to solve applications.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

Distance and Midpoint Formulas; Circles

Use the Distance Formula

In the following exercises, find the distance between the points. Round to the nearest tenth if needed.

$(-5,1)$ and $(-1,4)$

$(-2,5)$ and $(1,5)$

$d=3$

$(8,2)$ and $(-7,-3)$

$(1,-4)$ and $(5,-5)$

$d=17, d\approx 4.1$

Use the Midpoint Formula

In the following exercises, find the midpoint of the line segments whose endpoints are given.

$(-2,-6)$ and $(-4,-2)$

$(3,7)$ and $(5,1)$

$(4,4)$

$(-8,-10)$ and $(9,5)$

$(-3,2)$ and $(6,-9)$

$(32,-72)$

Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given information.

radius is 15 and center is $(0,0)$

radius is 7 and center is $(0,0)$

$x2+y2=7$

radius is 9 and center is $(-3,5)$

radius is 7 and center is $(-2,-5)$

$(x+2)2+(y+5)2=49$

center is $(3,6)$ and a point on the circle is $(3,-2)$

center is $(2,2)$ and a point on the circle is $(4,4)$

$(x-2)2+(y-2)2=8$

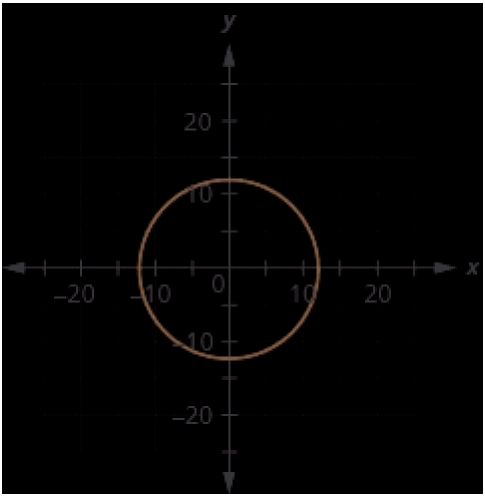
Graph a Circle

In the following exercises, ⓐ find the center and radius, then ⓑ graph each circle.

$2x2+2y2=450$

$3x2+3y2=432$

ⓐ radius: 12, center: (0,0)
ⓑ

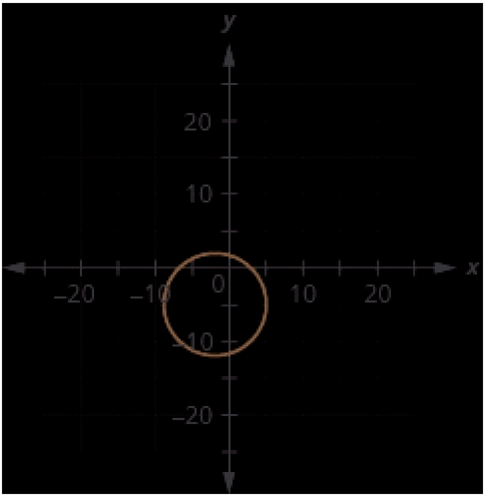


$(x+3)2+(y-5)2=81$

$(x+2)2+(y+5)2=49$

ⓐ radius: 7, center: $(-2,-5)$

ⓑ

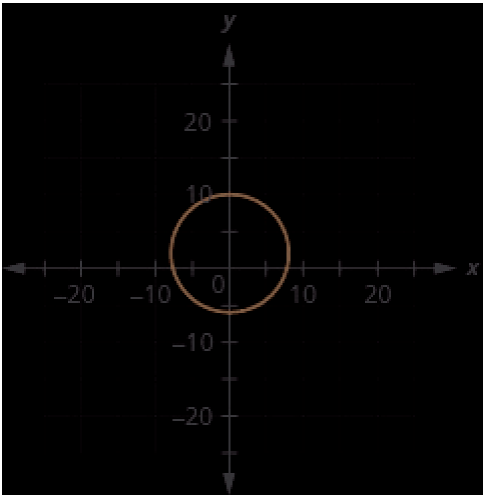


$$x2+y2-6x-12y-19=0$$

$$x2+y2-4y-60=0$$

ⓐ radius: 8, center: $(0,2)$

ⓑ



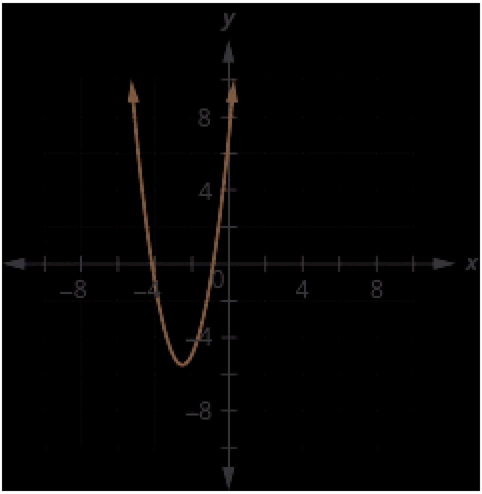
Parabolas

Graph Vertical Parabolas

In the following exercises, graph each equation by using its properties.

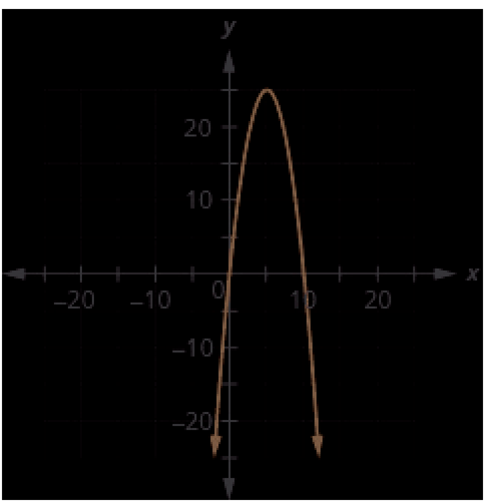
$y = x2 + 4x - 3$

$y = 2x2 + 10x + 7$



$y = -6x2 + 12x - 1$

$y = -x2 + 10x$

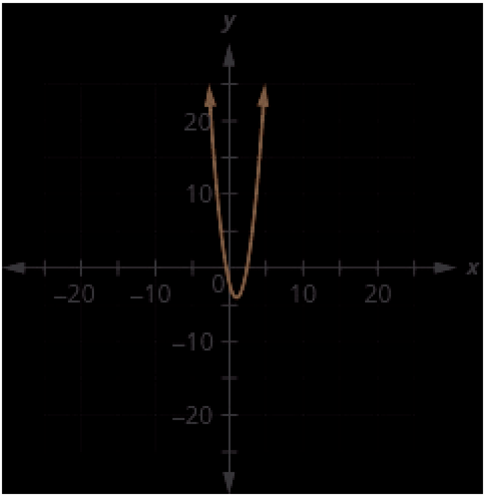


In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

$y=x2+4x+7$

$y=2x2-4x-2$

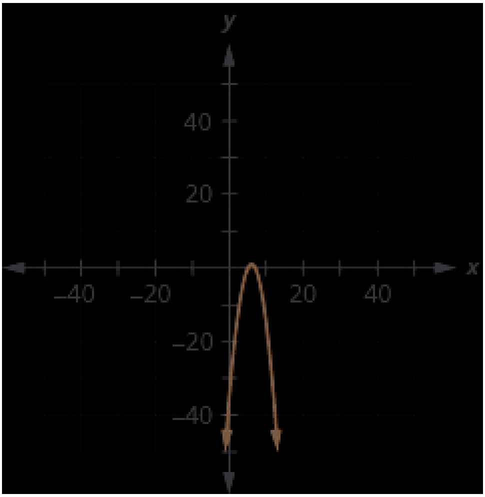
ⓐ $y=2(x-1)2-4$
ⓑ



$y=-3x2-18x-29$

$y = -x^2 + 12x - 35$

ⓐ $y = -(x-6)^2 + 1$
ⓑ

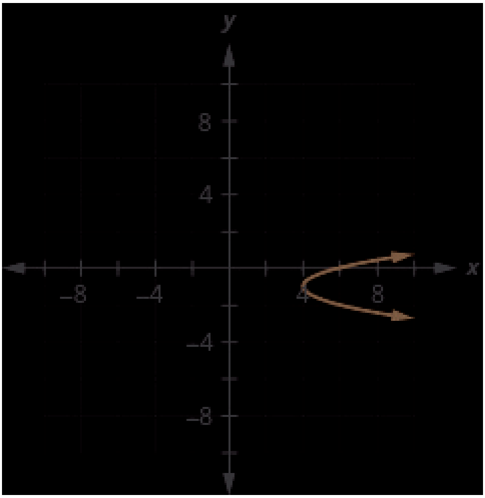


Graph Horizontal Parabolas

In the following exercises, graph each equation by using its properties.

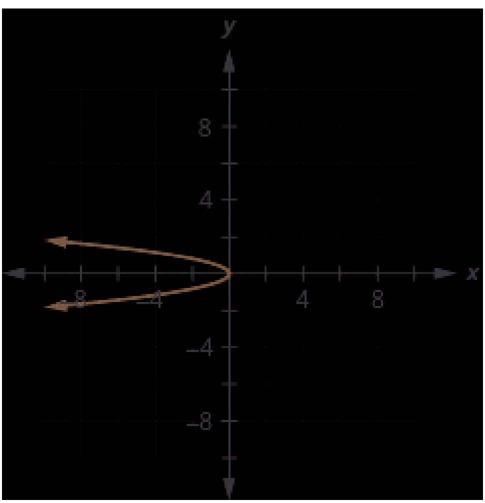
$x = 2y^2$

$x = 2y^2 + 4y + 6$



$x = -y2 + 2y - 4$

$x = -3y2$

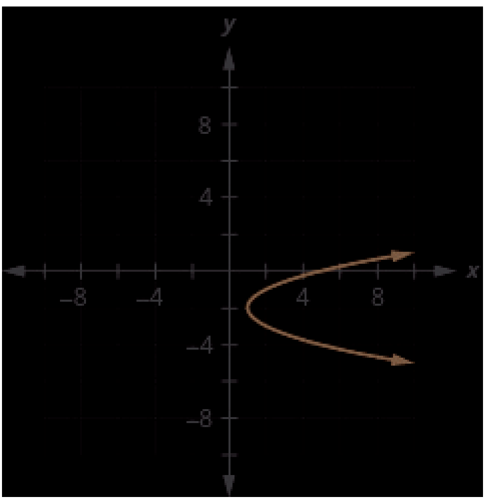


In the following exercises, ⓐ write the equation in standard form, then ⓑ use properties of the standard form to graph the equation.

$x = 4y2 + 8y$

$x = y2 + 4y + 5$

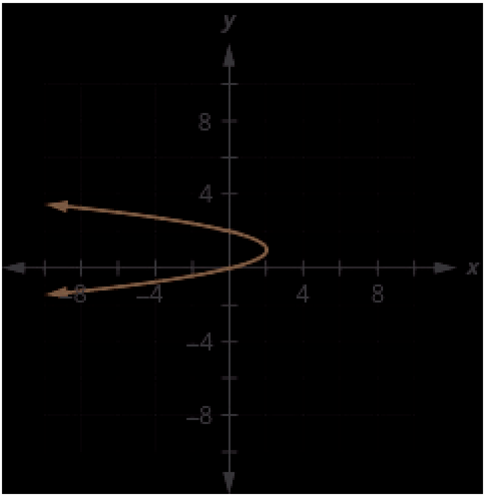
ⓐ $x = (y + 2)2 + 1$
ⓑ



$x = -y2 - 6y - 7$

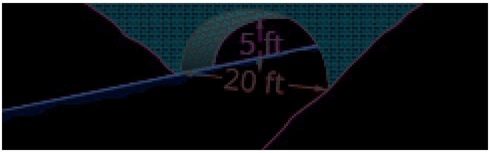
$x=-2y2+4y$

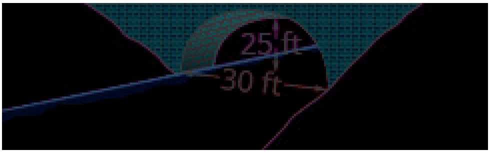
ⓐ $x=-2(y-1)2+2$
ⓑ



Solve Applications with Parabolas

In the following exercises, create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in standard form.





$y = -19x2 + 103x$

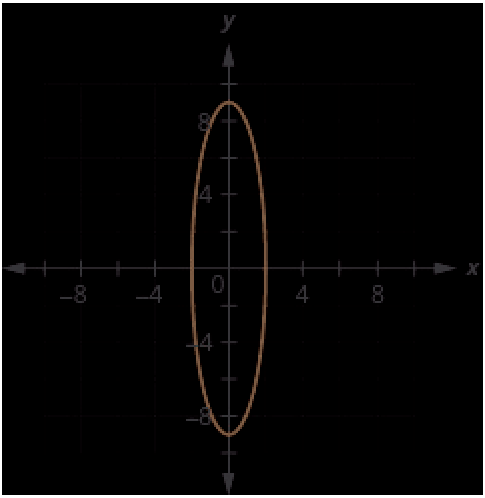
Ellipses

Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

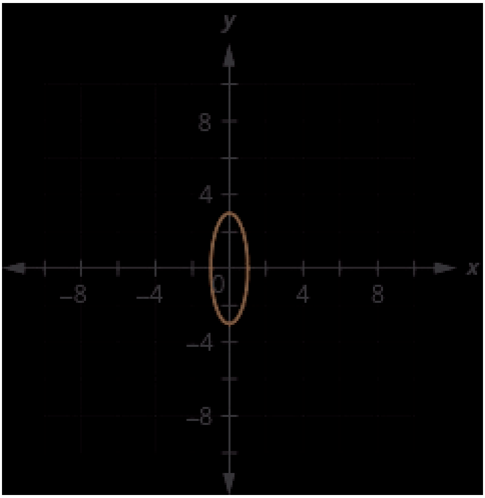
$x236 + y225 = 1$

$x24 + y281 = 1$



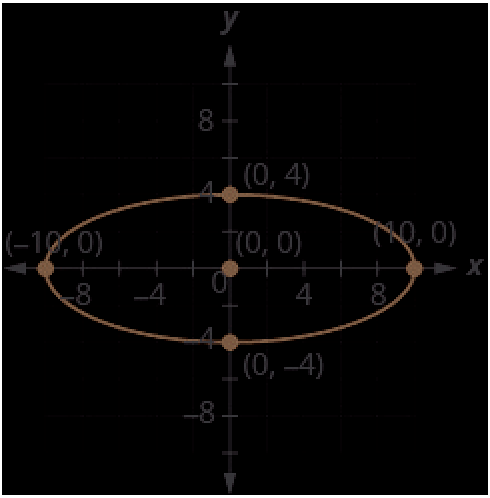
49x2 + 64y2 = 3136

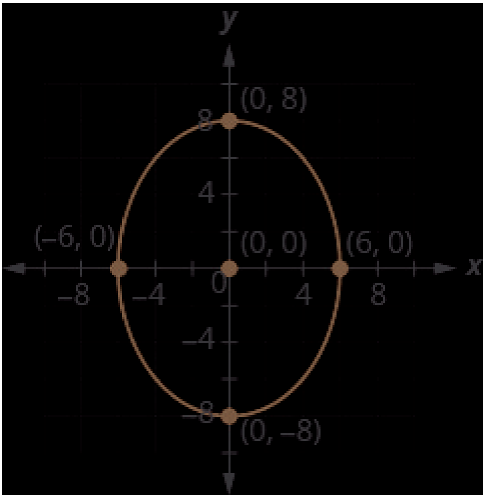
9x2 + y2 = 9



Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.





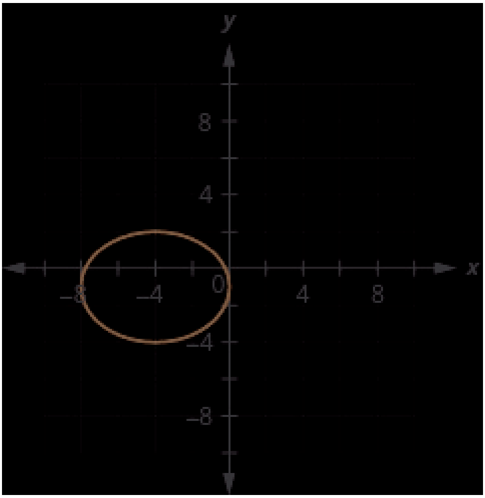
$x236 + y264 = 1$

Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

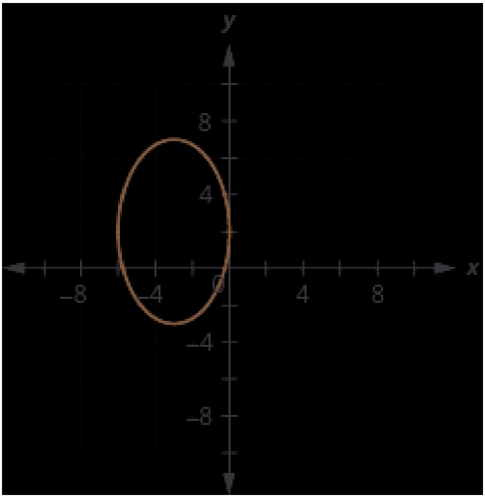
$(x - 1)225 + (y - 6)24 = 1$

$(x + 4)216 + (y + 1)29 = 1$



$(x-5)216+(y+3)236=1$

$(x+3)29+(y-2)225=1$

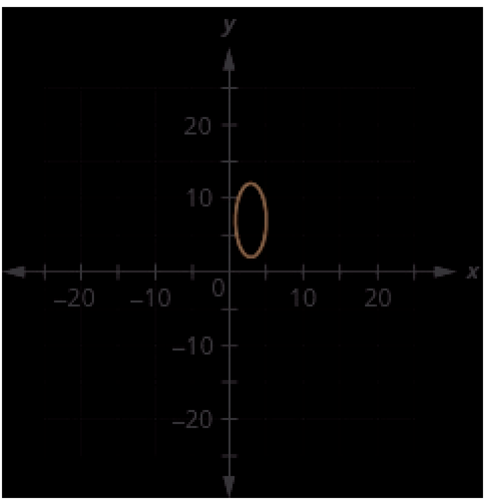


In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

$$x2 + y2 + 12x + 40y + 120 = 0$$

$$25x2 + 4y2 - 150x - 56y + 321 = 0$$

ⓐ $(x - 3)24 + (y - 7)225 = 1$
ⓑ

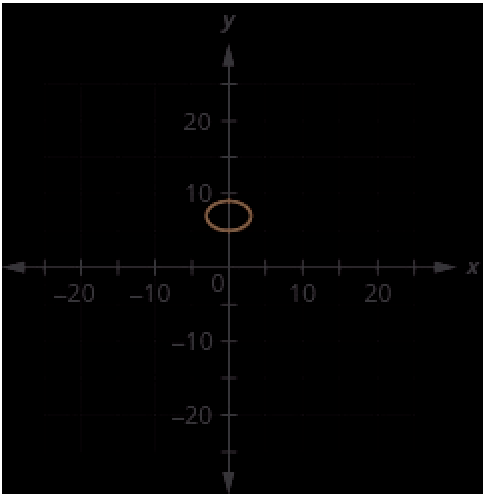


$$25x2 + 4y2 + 150x + 125 = 0$$

$$4x2 + 9y2 - 126x + 405 = 0$$

ⓐ $x29+(y-7)24=1$

ⓑ

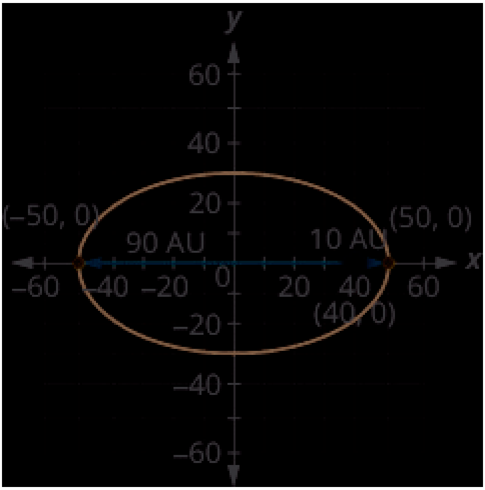


Solve Applications with Ellipses

In the following exercises, write the equation of the ellipse described.

A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 10 AU and the furthest is approximately 90 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical

orbit of the comet.

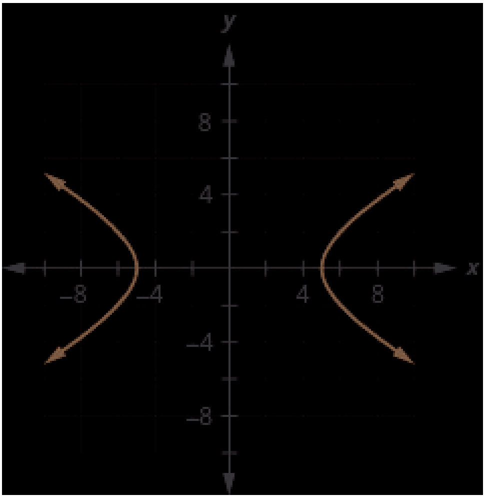


Hyperbolas

Graph a Hyperbola with Center at (0,0)

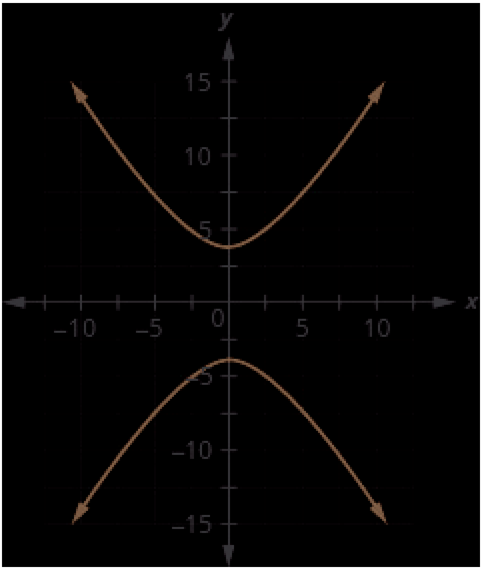
In the following exercises, graph.

$\frac{x^2}{25}-\frac{y^2}{9}=1$



$y249 - x216 = 1$

$9y2 - 16x2 = 144$

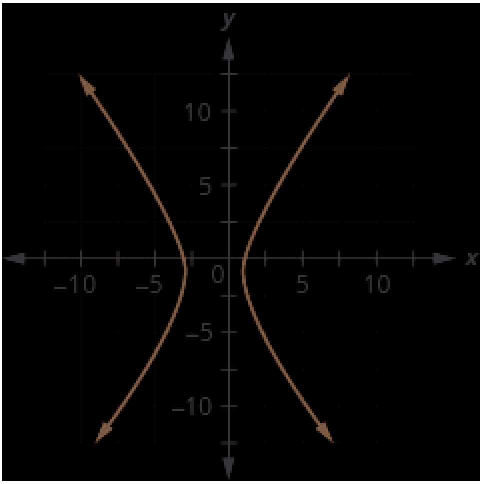


$16x2-4y2=64$

Graph a Hyperbola with Center at (h,k)

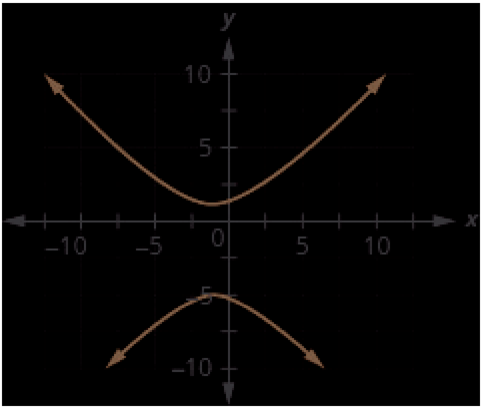
In the following exercises, graph.

$(x+1)24-(y+1)29=1$



$(x-2)24-(y-3)216=1$

$(y+2)29-(x+1)29=1$

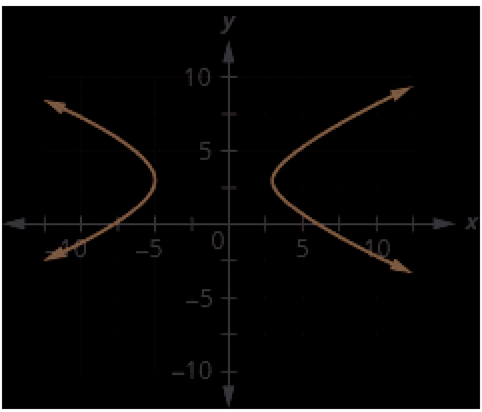


$(y-1)225-(x-2)29=1$

In the following exercises, ⓐ write the equation in standard form and ⓑ graph.

$4x2-16y2+8x+96y-204=0$

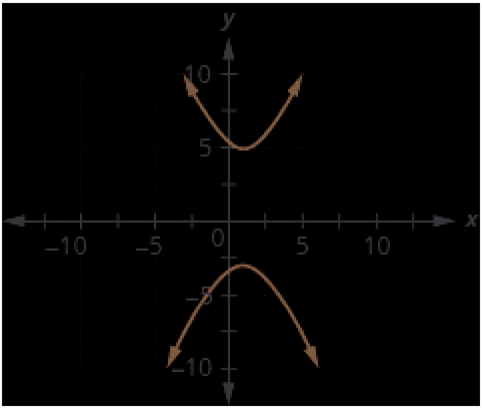
ⓐ $(x+1)216-(y-3)24=1$
ⓑ



$16x2-4y2-64x-24y-36=0$

$4y2-16x2+32x-8y-76=0$

ⓐ $(y-1)216-(x-1)24=1$
ⓑ



$36y2 - 16x2 - 96x + 216y - 396 = 0$

Identify the Graph of each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

ⓐ $16y2 - 9x2 - 36x - 96y - 36 = 0$
ⓑ $x2 + y2 - 4x + 10y - 7 = 0$
ⓒ $y = x2 - 2x + 3$
ⓓ $25x2 + 9y2 = 225$

ⓐ hyperbola ⓑ circle ⓒ parabola ⓓ ellipse

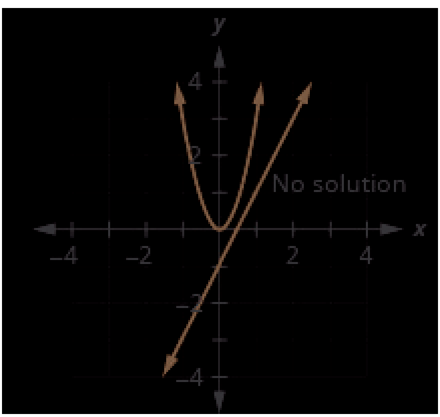
ⓐ $x2+y2+4x-10y+25=0$
ⓑ $y2-x2-4y+2x-6=0$
ⓒ $x=-y2-2y+3$
ⓓ $16x2+9y2=144$

Solve Systems of Nonlinear Equations

Solve a System of Nonlinear Equations Using Graphing

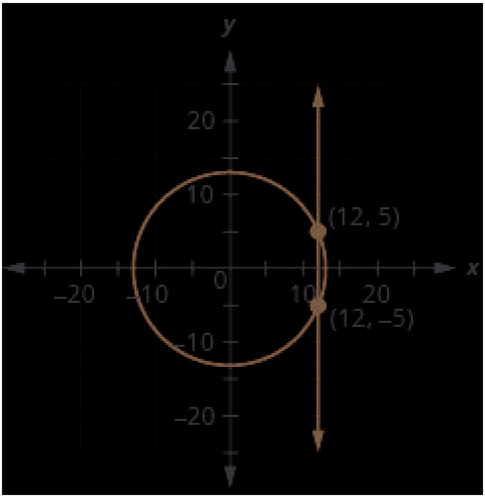
In the following exercises, solve the system of equations by using graphing.

$\{3x2-y=0y=2x-1$



$\{y=x2-4y=x-4$

{x2 + y2 = 169x = 12



{x2 + y2 = 25y = − 5

Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

{y = x2 + 3y = − 2x + 2

(− 1,4)

$\{x2+y2=4x-y=4$

$\{9x2+4y2=36y-x=5$

No solution

$\{x2+4y2=42x-y=1$

Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

$\{x2+y2=16x2-2y-1=0$

$(-7,3),(7,3)$

$\{x2-y2=5-2x2-3y2=-30$

$\{4x2+9y2=363y2-4x=12$

$(-3,0),(0,-2),(0,2)$

$\{x^2+y^2=14\; x^2-y^2=16$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

The sum of the squares of two numbers is 25. The difference of the numbers is 1. Find the numbers.

-3 and -4 or 4 and 3

The difference of the squares of two numbers is 45. The difference of the square of the first number and twice the square of the second number is 9. Find the numbers.

The perimeter of a rectangle is 58 meters and its area is 210 square meters. Find the length and width of the rectangle.

If the length is 14 inches, the width is 15 inches. If the length is 15 inches, the width is 14 inches.

Colton purchased a larger microwave for his kitchen. The diagonal of the front of the microwave measures 34 inches. The front also has an area of 480 square inches. What are the length and width of the microwave?

Practice Test

In the following exercises, find the distance between the points and the midpoint of the line segment with the given endpoints. Round to the nearest tenth as needed.

$(-4,-3)$ and $(-10,-11)$

---

distance: 10, midpoint: $(-7,-7)$

$(6,8)$ and $(-5,-3)$

In the following exercises, write the standard form of the equation of the circle with the given information.

radius is 11 and center is $(0,0)$

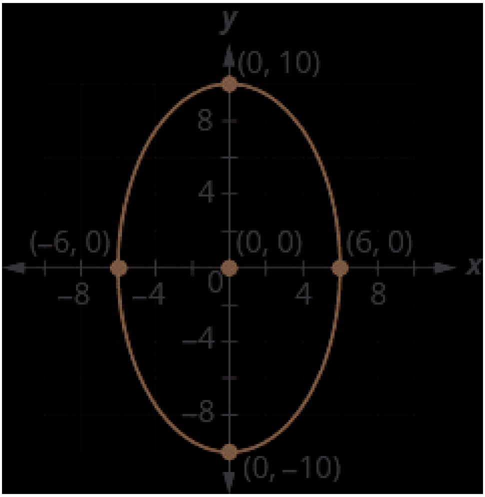
$x2+y2=121$

radius is 12 and center is $(10, -2)$

center is $(-2, 3)$ and a point on the circle is $(2, -3)$

$(x+2)2+(y-3)2=52$

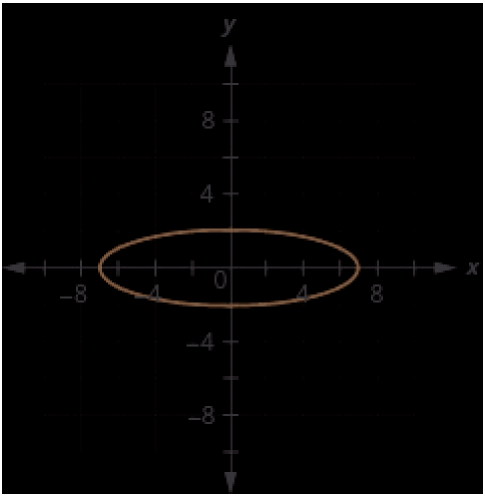
Find the equation of the ellipse shown in the graph.



In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, and ⓑ graph the equation.

$4x2 + 49y2 = 196$

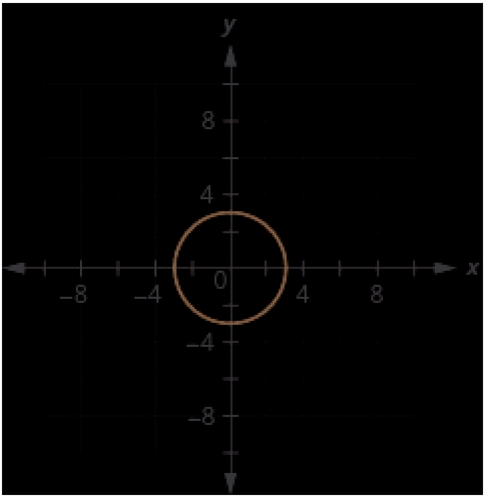
ⓐ ellipse
ⓑ



$y = 3(x - 2)2 - 2$

$3x2 + 3y2 = 27$

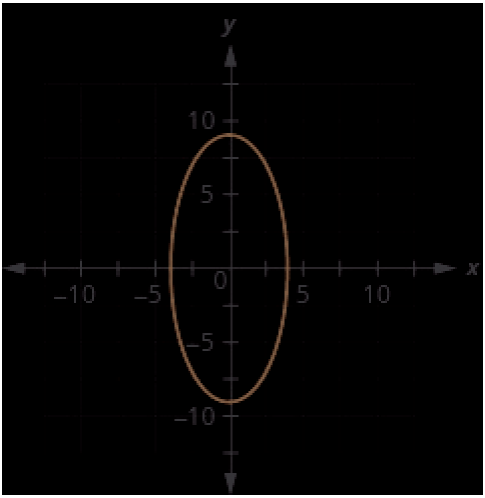
ⓐ circle
ⓑ



$y2100 - x236 = 1$

$x216 + y281 = 1$

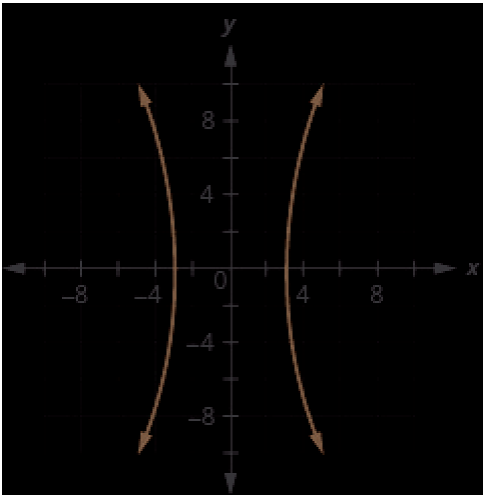
ⓐ ellipse
ⓑ



$x = 2y2 + 10y + 7$

$64x2 - 9y2 = 576$

ⓐ hyperbola
ⓑ

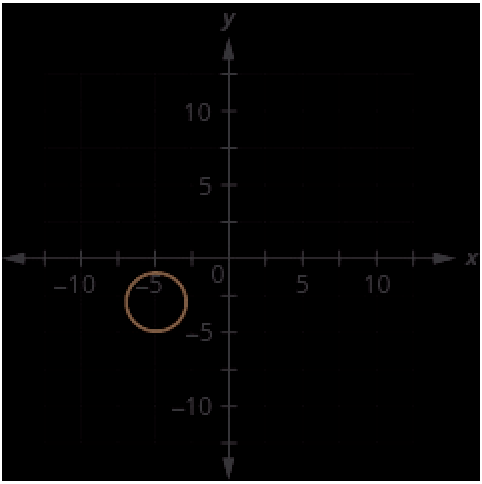


In the following exercises, ⓐ identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, ⓑ write the equation in standard form, and ⓒ graph the equation.

$$25x2+64y2+200x-256y-944=0$$

$$x2+y2+10x+6y+30=0$$

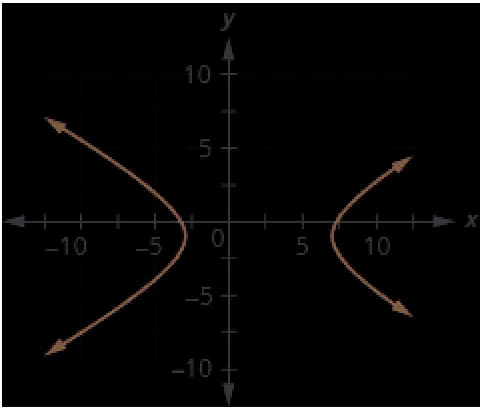
ⓐ circle
ⓑ $(x+5)2+(y+3)2=4$
ⓒ



$x = -y2 + 2y - 4$

$9x2 - 25y2 - 36x - 50y - 214 = 0$

ⓐ hyperbola
ⓑ $(x-2)225-(y+1)29=1$
ⓒ



$y=x2+6x+8$

Solve the nonlinear system of equations by graphing:
$\{3y2-x=0y=-2x-1.$

No solution

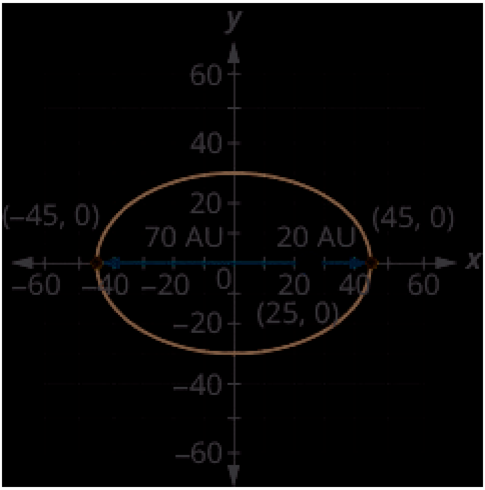
Solve the nonlinear system of equations using substitution:
$\{x2+y2=8y=-x-4.$

Solve the nonlinear system of equations using elimination:
$\{x2+9y2=92x2-9y2=18.$

(0, − 3),(0,3)

Create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in y = ax2 + bx + c form.

A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 20 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.



$x22025 + y21400 = 1$

The sum of two numbers is 22 and the product is -240. Find the numbers.

For her birthday, Olive’s grandparents bought her a new widescreen TV. Before opening it she wants to make sure it will fit her entertainment center. The TV is 55”. The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1452 square inches. Her entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Olive’s entertainment center?

The length is 44 inches and the width is 33 inches. The TV will fit Olive's entertainment center.

Glossary

system of nonlinear equations
: A system of nonlinear equations is a system where at least one of the equations is not linear.

Exponential Functions
In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2% each year[footnote]. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.
http://www.worldometers.info/world-population/. Accessed February 24, 2014.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x)=3x+4$, the

slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over

equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See [link].

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

From [link] we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original

value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

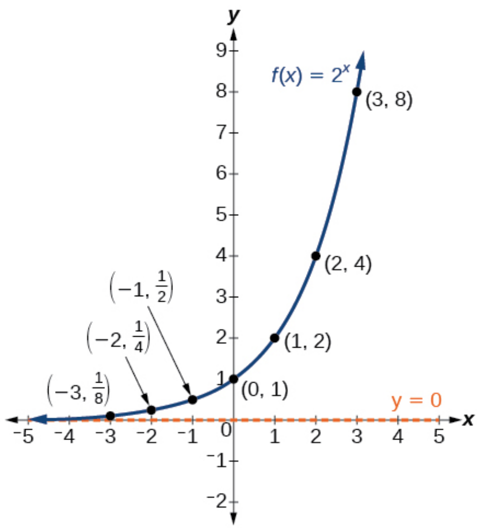
The general form of the exponential function is $f(x)=ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b>1$, the function grows at a rate proportional to its size.
- If $0<b<1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x)=2^x$ from our example. We will create a table ([link]) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs we observe on the table in [link], and then make a few observations.



Let’s define the behavior of the graph of the exponential function $f(x)=2^x$ and highlight some its key characteristics.

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty, f(x)\to\infty$,
- as $x\to-\infty, f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never

has the result of zero.

- $y=0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Exponential Function

For any real number x, an exponential function is a function with the form

$$f(x)=ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b\neq1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a>0$.
- The range of f is all negative real numbers if $a<0$.
- The *y*-intercept is $(0,a)$, and the horizontal asymptote is $y=0$.

Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x)=4^{3(x-2)}$

- $g(x)=x^3$
- $h(x)=\left(\frac{1}{3}\right)^x$
- $j(x)=(-2)^x$

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x)=x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x)=x^3$ is a power function.

Recall that the base *b* of an exponential function is always a positive constant, and $b\neq 1$. Thus, $j(x)=(-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Which of the following equations represent exponential functions?

- $f(x)=2x^2-3x+1$
- $g(x)=0.875^x$
- $h(x)=1.75x+2$
- $j(x)=1095.6^{-2x}$

$g(x)=0.875^x$ and $j(x)=1095.6^{-2x}$

represent exponential functions.

The graph shows the numbers of stores Companies A and B opened over a five-year period.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b=-9$ and $x=\frac{1}{2}$. Then $f(x)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

- Let $b=1$. Then $f(x)=1^x=1$ for any value of x.

To evaluate an exponential function with the form $f(x)=b^x$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x)=2^x$. What is $f(3)$?

$f(x) = 2 x$ $f(3) = 2 3$ Substitute $x=3$. $=8$ Evaluate the power.

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x)=30 (2) x$. What is $f(3)$?
$f(x) =30 (2) x$ $f(3) =30 (2) 3$ Substitute $x=3$. $=30(8)$ Simplify the power first. $=240$ Multiply.

Note that if the order of operations were not followed, the result would be incorrect:
$f(3)=30 (2) 3 \neq 60 3 =216,000$

Evaluating Exponential Functions

Let $f(x)=5 (3) x+1$. Evaluate $f(2)$ without using a calculator.

Follow the order of operations. Be sure to pay attention to the parentheses.
$f(x) =5 (3) x+1$ $f(2) =5 (3) 2+1$ Substitute $x=2$. $=5 (3) 3$ Add the exponents. $=5(27)$ Simplify the power. $=135$ Multiply.

Let $f(x)=8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Exponential Growth

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x)=ab^x$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

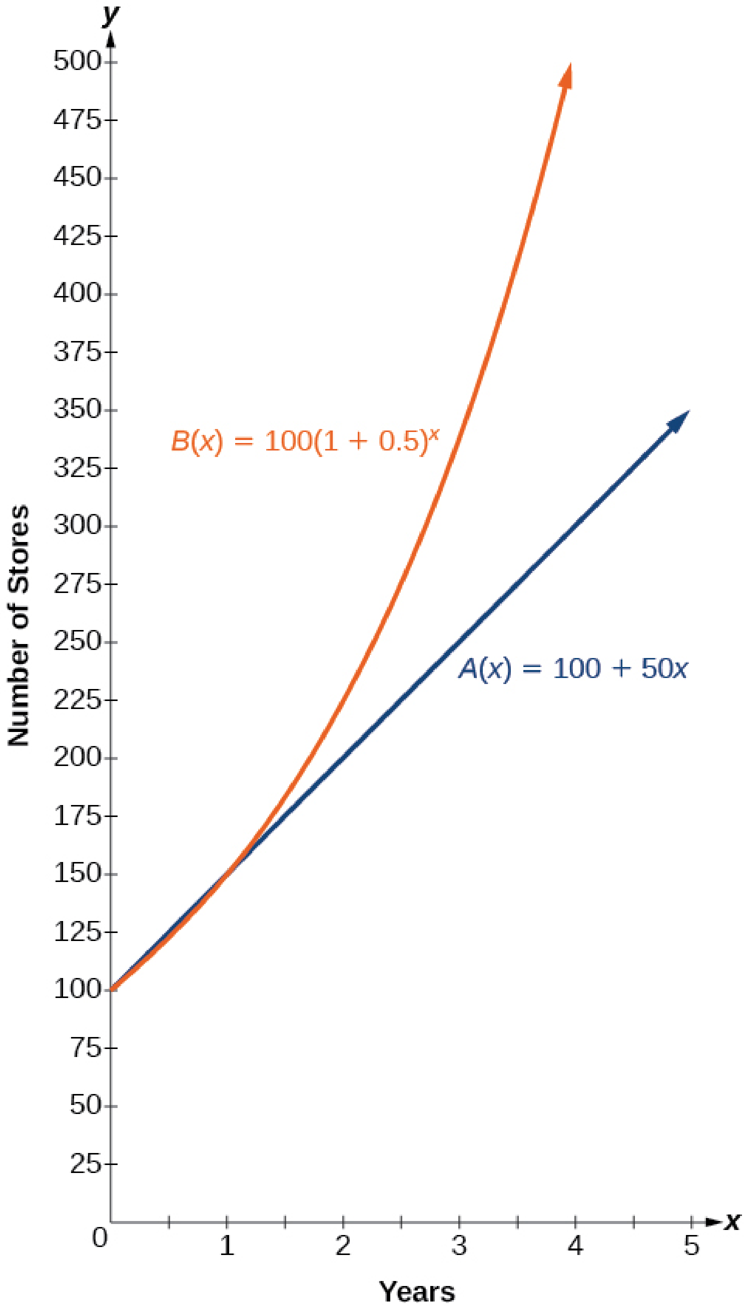
In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x)=100+50x.$ Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x)=100(1+0.5)^x$.

A few years of growth for these companies are illustrated in [link].

Year, x	Stores, Company A	Stores, Company B
0	$100+50(0)=100$	$100(1+0.5)^0=100$
1	$100+50(1$	$100(1+0.5)^1$

	$)=150$	$=150$
2	$100+50(2)=200$	$100(1+0.5)^2=225$
3	$100+50(3)=250$	$100(1+0.5)^3=337.5$
x	$A(x)=100+50x$	$B(x)=100(1+0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in [link]. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.



Notice that the domain for both functions is $[0, \infty)$,

and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest

thousandth,
$P(18) = 1.25(1.012)^{18} \approx 1.549$

There will be about 1.549 billion people in India in the year 2031.

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in [link]?

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

Given two data points, write an exponential model.

1. If one of the data points has the form $(0,a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x)=a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0,a)$, substitute both points into two equations with the form $f(x)=a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x)=a(b)^x$.

Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

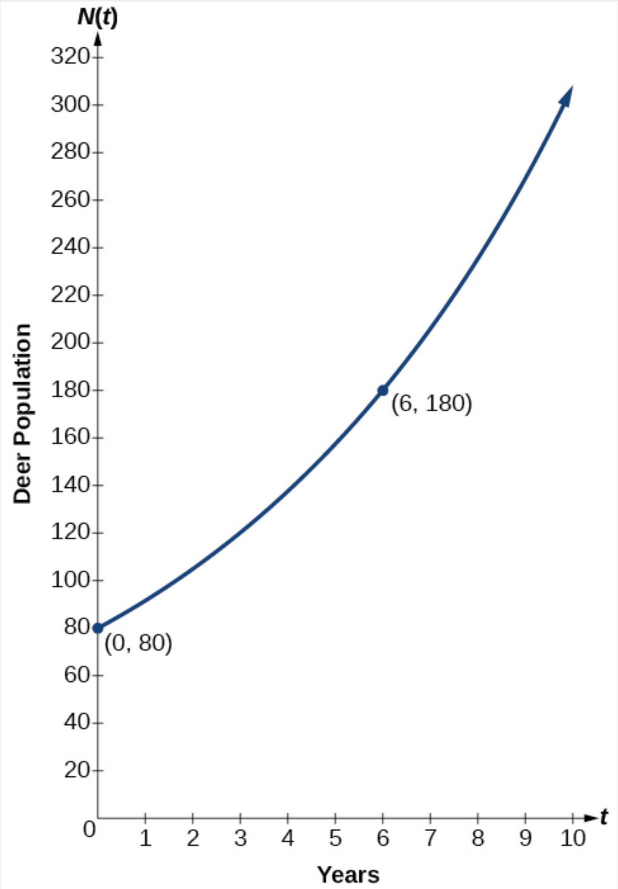
We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a=80$. We can now substitute the second point into the equation $N(t)=80b^t$ to find b:

$N(t)=80b^t$ $\quad 180=80b^6$
Substitute using point (6, 180). $\quad \frac{9}{4}=b^6$
Divide and write in lowest terms. $\quad b=\left(\frac{9}{4}\right)^{\frac{1}{6}}$ Isolate b using properties of exponents.
$\quad b\approx 1.1447$ Round to 4 decimal places.

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t)=80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in [link] passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0,\infty)$, and the range for the function is $[80,\infty)$.
Graph showing the population of deer over time, $N(t)=80(1.1447)^t$, t years after 2006



A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population

N of wolves over time t.

$(0,129)$ and $(2,236)$; $N(t)=129(1.3526)^t$

Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2,6)$ and $(2,1)$.

Because we don’t have the initial value, we substitute both points into an equation of the form $f(x)=ab^x$, and then solve the system for a and b.

- Substituting $(-2,6)$ gives $6=ab^{-2}$
- Substituting $(2,1)$ gives $1=ab^2$

Use the first equation to solve for a in terms of b:

$$6 = ab^{-2}$$

$$\frac{6}{b^{-2}} = a \qquad \text{Divide.}$$

$$a = 6b^2 \qquad \text{Use properties of exponents to rewrite the denominator.}$$

Substitute a in the second equation, and solve for b:

$$
\begin{array}{ll}
1 = ab^2 & \\
1 = 6b^2b^2 = 6b^4 & \text{Substitute } a. \\
b = \left(\frac{1}{6}\right)^{\frac{1}{4}} & \text{Use properties of exponents to isolate } b. \\
b \approx 0.6389 & \text{Round 4 decimal places.}
\end{array}
$$

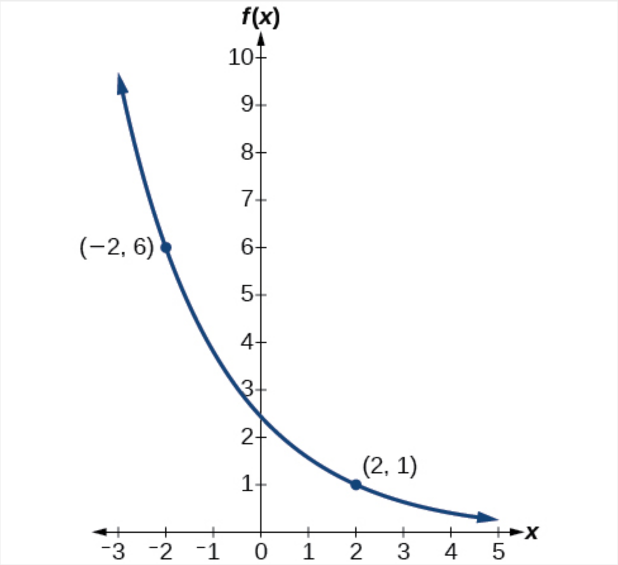
Use the value of b in the first equation to solve for the value of a:

$$a = 6b^2 \approx 6(0.6389)^2 \approx 2.4492$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in [link] passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.



Given the two points $(1,3)$ and $(2,4.5)$, find the equation of the exponential function that passes through these two points.

$f(x) = 2(1.5)^x$

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-

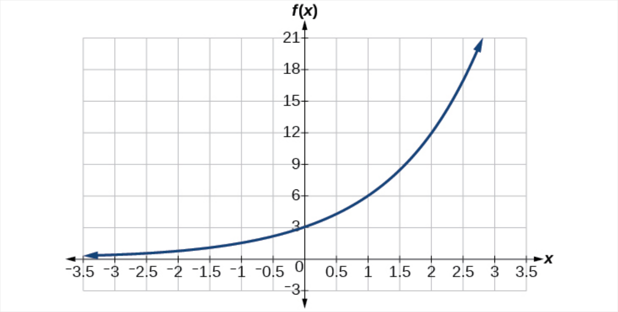
coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x, which in many real world cases involves time.

Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the *y*-intercept $(0,a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x)=a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0,a)$, substitute both points into two equations with the form $f(x)=a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x)=a(b)^x$.

Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in [link].



We can choose the y-intercept of the graph, $(0,3)$, as our first point. This gives us the initial value, $a=3$. Next, choose a point on the curve some distance away from $(0,3)$ that has integer coordinates. One such point is $(2,12)$.

$y=ab^x$ Write the general form of an exponential equation.

$y=3b^x$ Substitute the initial value 3 for a.

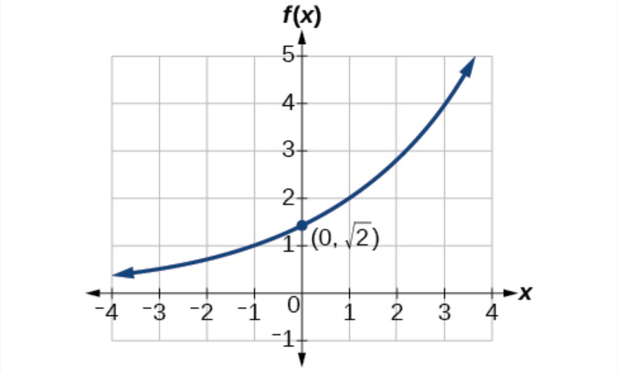
$12=3b^2$ Substitute in 12 for y and 2 for x.

$4=b^2$ Divide by 3.

$b=\pm 2$ Take the square root.

Because we restrict ourselves to positive values of b, we will use $b=2$. Substitute a and b into the standard form to yield the equation $f(x)=3(2)^x$.

Find an equation for the exponential function graphed in [link].



f(x)= 2 (2) x . Answers may vary due to round-off error. The answer should be very close to 1.4142 (1.4142) x .

Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT]** .
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential**

Regression), and press **[ENTER]**.

6. The screen displays the values of *a* and *b* in the exponential equation $y=a\cdot b^x$.

Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2,24.8)$ and $(5,198.4)$.

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…]**, and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a=6.2$ and $b=2$ will be displayed. The exponential equation is $y=6.2\cdot 2^x$.

Use a graphing calculator to find the

exponential equation that includes the points (3, 75.98) and (6, 481.07).
$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the

compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n:

$$A(t)=P\left(1+\frac{r}{n}\right)^{nt}$$

For example, observe [link], which shows the result of investing \$1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

The Compound Interest Formula

Compound interest can be calculated using the formula

$$A(t)=P\left(1+\frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,

- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ Use the compound interest formula.

$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 10}$ Substitute using given values.

$\approx \$4045.05$ Round to two decimal places.

The account will be worth about $4,045.05 in 10 years.

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

about $3,644,675.88

Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

The nominal interest rate is 6%, so $r=0.06$. Interest is compounded twice a year, so $k=2$.

We want to find the initial investment, P, needed so that the value of the account will be worth \$40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$
Use the compound interest formula.

$40,000 = P\left(1+\frac{0.06}{2}\right)^{2(18)}$
Substitute using given values A, r, n, and t.

$40,000 = P(1.03)^{36}$ Simplify.

$\frac{40,000}{(1.03)^{36}} = P$ Isolate P.

$P \approx \$13,801$
Divide and round to the nearest dollar.

Lily will need to invest \$13,801 to have \$40,000 in 18 years.

Refer to [link]. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

\$13,693

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. [link] shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in [link].

Frequency	$A(t)=\left(1+\frac{1}{n}\right)^n$	Value
Annually	$\left(1+\frac{1}{1}\right)^1$	$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279

Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

The Number *e*

The letter *e* represents the irrational number

$\left(1+\frac{1}{n}\right)^n$, as n increases without bound

The letter *e* is used as a base for many real-world exponential models. To work with base *e*, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Using a Calculator to Find Powers of *e*

Calculate $e^{3.14}$. Round to five decimal places.

On a calculator, press the button labeled $[e^x]$. The window shows $[e^\wedge(\ ]$. Type 3.14 and then close parenthesis, $[\ )\ ]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for

exponential functions. For most real-world phenomena, however, *e* is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

The Continuous Growth/Decay Formula

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t)=ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r>0$, then the formula represents continuous growth. If $r<0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t)=Pe^{rt}$$

where

- P is the principal or the initial invested,

- r is the growth or interest rate per unit time,
- and t is the period or term of the investment.

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
    1. If the problem refers to continuous growth, then $r > 0$.
    2. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at

the end of one year?

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was \$1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$A(t) = Pe^{rt}$
Use the continuous compounding formula.
$= 1000(e)^{0.1}$
Substitute known values for P, r, and t.
≈ 1105.17 Use a calculator to approximate.

The account is worth \$1,105.17 after one year.

A person invests \$100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

\$3,659,823.44

Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Since the substance is decaying, the rate, 17.3% , is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$A(t) = a e^{rt}$

Use the continuous growth formula. $= 100 e^{-0.173(3)}$

Substitute known values for a, r, and t.

≈ 59.5115 Use a calculator to approximate.

So 59.5115 mg of radon-222 will remain.

Using the data in [link], how much radon-222 will remain after one year?

3.77E-26 (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this

number is so close to zero that for all practical purposes we can accept zero as the answer.)

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x)=b^x$, where $b>0$, $b\neq 1$
definition of exponential growth	$f(x)=ab^x$, where $a>0$, $b>0$, $b\neq 1$
compound interest formula	$A(t)=P\left(1+\frac{r}{n}\right)^{nt}$, where $A(t)$ is the account value at time t is the number of years

	P is the initial investment, often r is the annual percentage rate (A n is the number of compounding
continuous growth formula	$A(t)=ae^{rt}$, where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e\approx 2.718282$

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See [link].
- A function is evaluated by solving at a specific value. See [link] and [link].
- An exponential model can be found when the growth rate and initial value are known. See [link].
- An exponential model can be found when the two data points from the model are known. See [link].
- An exponential model can be found using two data points from the graph of the model. See

[link].

- An exponential model can be found using two data points from the graph and a calculator. See [link].
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See [link].
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See [link].
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $[e^x]$ or $[\exp(x)]$ for calculating powers of e. See [link].
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See [link] and [link].

Section Exercises

Verbal

Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[footnote] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.
Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

When interest is compounded, the percentage

of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of *nominal*.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

The average annual population increase of a pack of wolves is 25.

A population of bacteria decreases by a factor of $1\ 8$ every 24 hours.

---

exponential; the population decreases by a proportional rate. .

The value of a coin collection has increased by 3.25% annually over the last 20 years.

For each training session, a personal trainer charges his clients $\$5$ less than the previous

training session.

---

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

Which forest's population is growing at a faster rate?

The forest represented by the function $B(t) = 82(1.029)^t$.

Which forest had a greater number of trees initially? By how many?

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

After $t=20$ years, forest A will have 43 more trees than forest B.

Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might

influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

$y = 300 (1 - t) 5$

$y = 220 (1.06) x$

---

exponential growth; The growth factor, 1.06, is greater than 1.

$y = 16.5 (1.025) 1 x$

$y = 11{,}701 (0.97) t$

---

exponential decay; The decay factor, 0.97, is between 0 and 1.

For the following exercises, find the formula for an exponential function that passes through the two points given.

$(0,6)$ and $(3,750)$

$(0,2000)$ and $(2,20)$

$f(x) = 2000\,(0.1)\,x$

$(-1, 3\ 2)$ and $(3,24)$

$(-2,6)$ and $(3,1)$

$f(x) = (1\ 6)\ -\ 3\ 5\ (1\ 6)\ x\ 5\ \approx 2.93\,(0.699)$
x

$(3,1)$ and $(5,4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

x	1	2	3	4
f(x)	70	40	10	-20

Linear

x	1	2	3	4
h(x)	70	49	34.3	24.01

x	1	2	3	4
m(x)	80	61	42.9	25.61

Neither

x	1	2	3	4
f(x)	10	20	40	80

x	1	2	3	4
g(x)	-3.25	2	7.25	12.5

Linear

For the following exercises, use the compound interest formula, $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$.

After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1+\frac{0.04}{12}\right)^{120}$. What is the value of the account?

What was the initial deposit made to the account in the previous exercise?

$10,250

How many years had the account from the previous exercise been accumulating interest?

An account is opened with an initial deposit of $6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

$13,268.58

How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

Solve the compound interest formula for the principal, P.

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $14,472.74 after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

How much more would the account in the

previous two exercises be worth if it were earning interest for 5 more years?

$4,572.56

Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

4%

Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $5,500, and was worth $38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

$y = 3742(e)^{0.75t}$

---

continuous growth; the growth rate is greater than 0.

$y = 150(e)^{3.25t}$

$y = 2.25(e)^{-2t}$

---

continuous decay; the growth rate is less than 0.

Suppose an investment account is opened with an initial deposit of $12,000 earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

$669.42

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

$f(x)=2(5)x$, for $f(-3)$

$f(x)=-4\,2x+3$, for $f(-1)$

$f(-1)=-4$

$f(x)=e\,x$, for $f(3)$

$f(x)=-2\,e\,x-1$, for $f(-1)$

$f(-1)\approx-0.2707$

$f(x)=2.7(4)\,-x+1\,+1.5$, for $f(-2)$

$f(x)=1.2\,e\,2x\,-0.3$, for $f(3)$

$f(3)\approx 483.8146$

$f(x)=-3\,2(3)\,-x\,+\,3\,2$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

$(0,3)$ and $(3,375)$

$y=3\cdot 5^x$

$(3,222.62)$ and $(10,77.456)$

$(20,29.495)$ and $(150,730.89)$

$y\approx 18\cdot 1.025^x$

$(5,2.909)$ and $(13,0.005)$

$(11,310.035)$ and $(25,356.3652)$

$y\approx 0.2\cdot 1.95^x$

Extensions

The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY=\left(1+\frac{r}{12}\right)^{12}-1.$

Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

$APY=\frac{A(t)-a}{a}=\frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a}=\frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a}=\left(1+\frac{r}{365}\right)^{365}-1;\ I(n)=\left(1+\frac{r}{n}\right)^{n}-1$

Recall that an exponential function is any equation written in the form $f(x)=a\cdot b^{x}$ such that a and b are positive numbers and $b\neq 1$. Any positive number b can be written as $b=e^{n}$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

In an exponential decay function, the base of

the exponent is a value between 0 and 1. Thus, for some number $b>1$, the exponential decay function can be written as $f(x)=a\cdot\left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b=e^n$, to show that an exponential decay function takes the form $f(x)=a(e)^{-nx}$ for some positive number n.

Let f be the exponential decay function $f(x)=a\cdot\left(\frac{1}{b}\right)^x$ such that $b>1$. Then for some number $n>0$, $f(x)=a\cdot\left(\frac{1}{b}\right)^x=a\left(b^{-1}\right)^x=a\left(\left(e^n\right)^{-1}\right)^x=a\left(e^{-n}\right)^x=a(e)^{-nx}$.

The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t)=a(e)^{rt}$, where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t)=e^{rt}-1$.

Real-World Applications

The fox population in a certain region has an annual growth rate of 9% per year. In the year

2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

47,622 fox

A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

In the year 1985, a house was valued at \$110,000. By the year 2005, the value had appreciated to \$145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

1.39%; \$155,368.09

A car was valued at \$38,000 in the year 2007. By 2013, the value had depreciated to \$11,000 If the car's value continues to drop by the same percentage, what will it be worth by 2017?

Jamal wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

$35,838.76

Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

$82,247.78; $449.75

An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Glossary

annual percentage rate (APR)
: the yearly interest rate earned by an investment account, also called *nominal rate*

compound interest
: interest earned on the total balance, not just the principal

exponential growth
: a model that grows by a rate proportional to the amount present

nominal rate
: the yearly interest rate earned by an investment account, also called *annual percentage rate*

Graphs of Exponential Functions

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events. Notice that the graph gets close to the *x*-axis, but never touches it.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in [link] change as the input increases by 1.

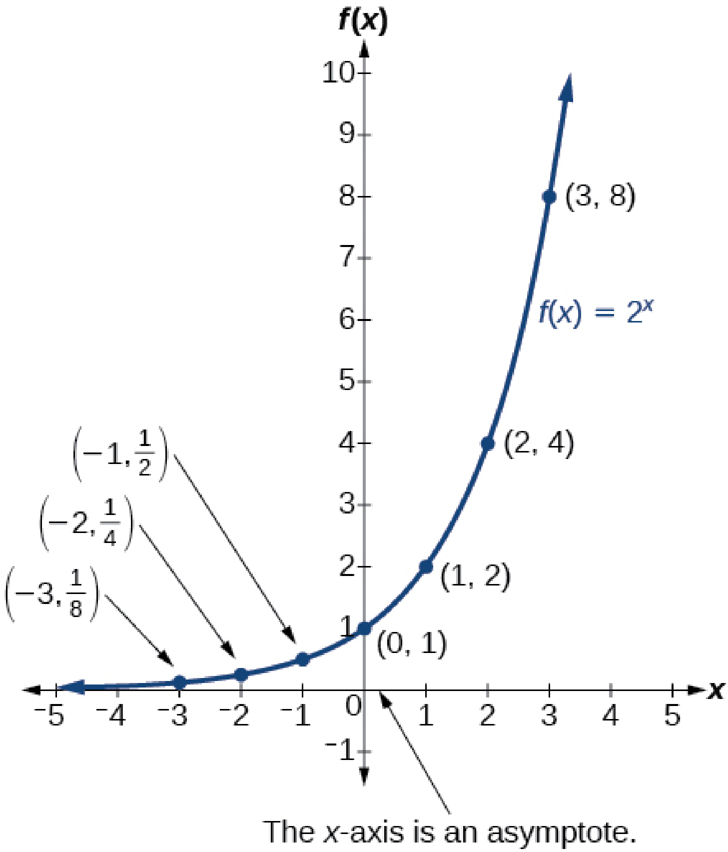
x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x)=ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

[link] shows the exponential growth function $f(x)=2^x$.



The domain of $f(x)=2^x$ is all real numbers, the range is $(0,\infty)$, and the horizontal asymptote is $y=0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x)=b^x$ whose base is between zero and one. We'll use the function $g(x)=\left(\frac{1}{2}\right)^x$. Observe how the output values in [link] change as the input increases by 1.

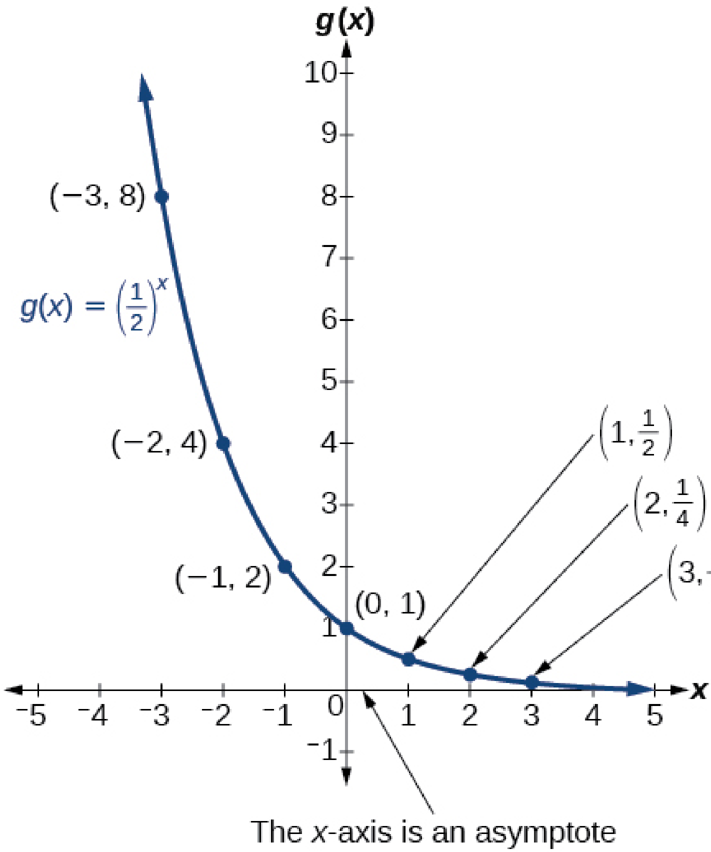
x	3	2	1	0	1	2	3
$g(x)=\left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

[link] shows the exponential decay function, $g(x)=\left(\frac{1}{2}\right)^x$.



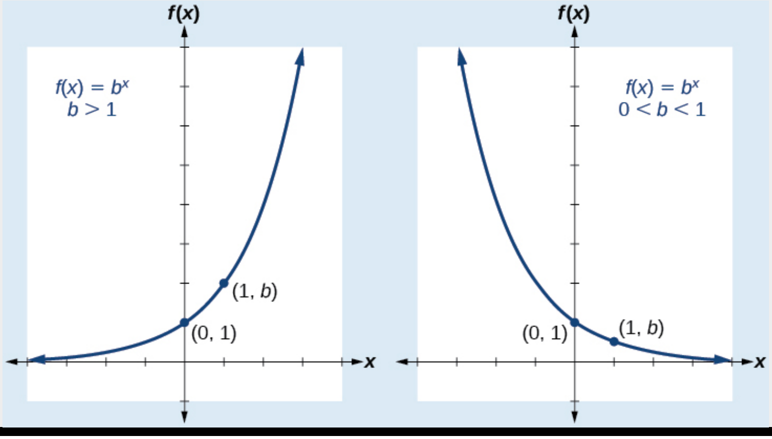
The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Characteristics of the Graph of the Parent Function $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

[link] compares the graphs of exponential growth and decay functions.



Given an exponential function of the form $f(x) = b^x$, graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.

4. State the domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y=0$.

Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x)= 0.25^x$. State the domain, range, and asymptote.

Before graphing, identify the behavior and create a table of points for the graph.

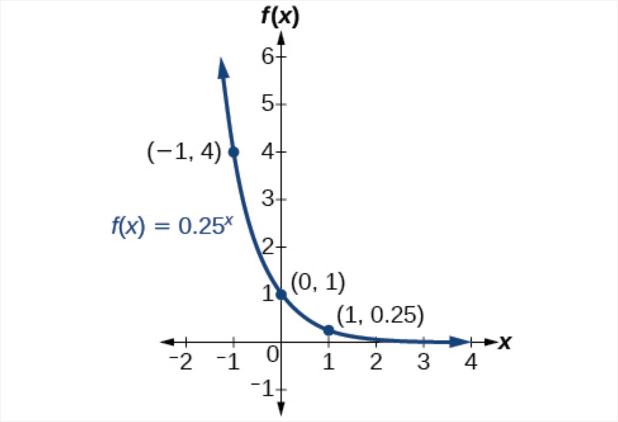
- Since $b=0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y=0$.
- Create a table of points as in [link].

x	-3	-2	-1	0	1	2	3
$f(x)=0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the *y*-intercept, $(0,1)$, along with two other points. We can use $(-1,4)$ and $(1,0.25)$.

Draw a smooth curve connecting the points as

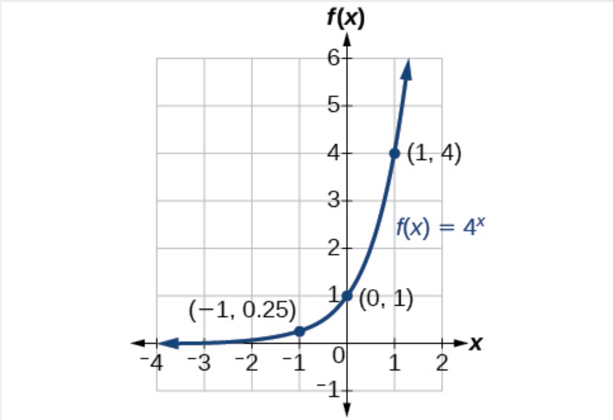
in [link].



The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y=0$.

Sketch the graph of $f(x)=4^x$. State the domain, range, and asymptote.

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y=0$.



(a) $g(x)=3(2)^x$ stretches the graph of $f(x)=2^x$ vertically by a factor of 3. (b) $h(x)=\frac{1}{3}(2)^x$ compresses the graph of $f(x)=2^x$ vertically by a factor of $\frac{1}{3}$.(a) $g(x)=-2^x$ reflects the graph of $f(x)=2^x$ about the x-axis. (b) $g(x)=2^{-x}$ reflects the graph of $f(x)=2^x$ about the *y*-axis.

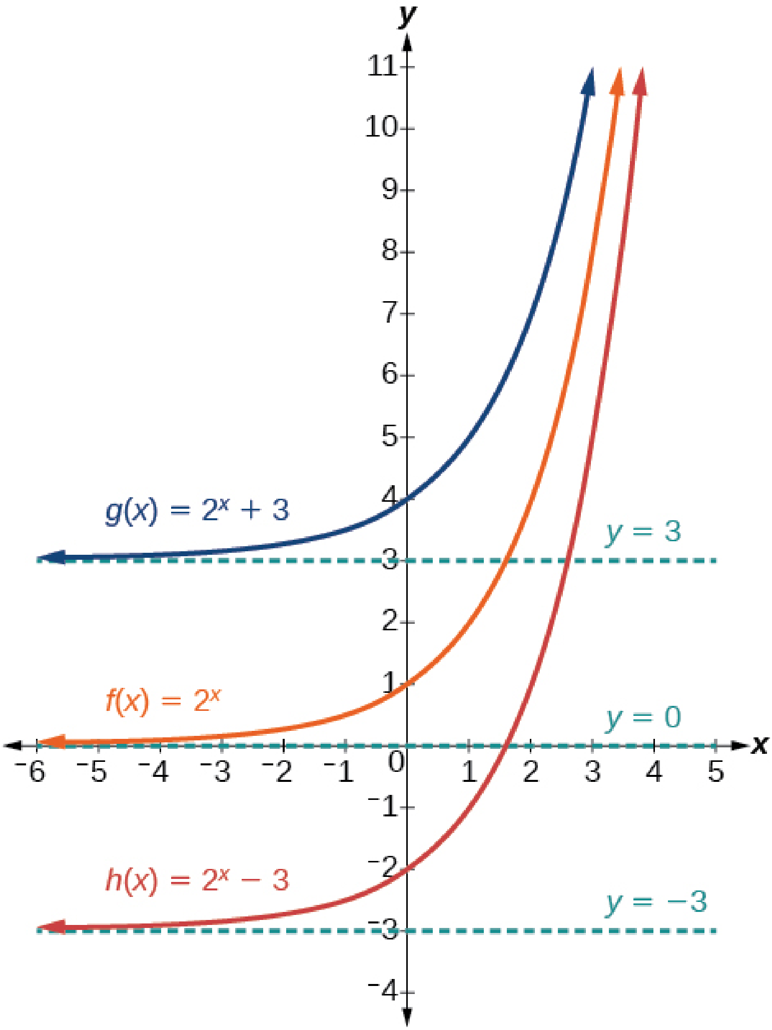
Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x)=b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape

when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x)= b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x)= 2^x$, we can then graph two vertical shifts alongside it, using $d=3$: the upward shift, $g(x)= 2^x+3$ and the downward shift, $h(x)= 2^x-3.$ Both vertical shifts are shown in [link].



Observe the results of shifting $f(x) = 2^x$ vertically:

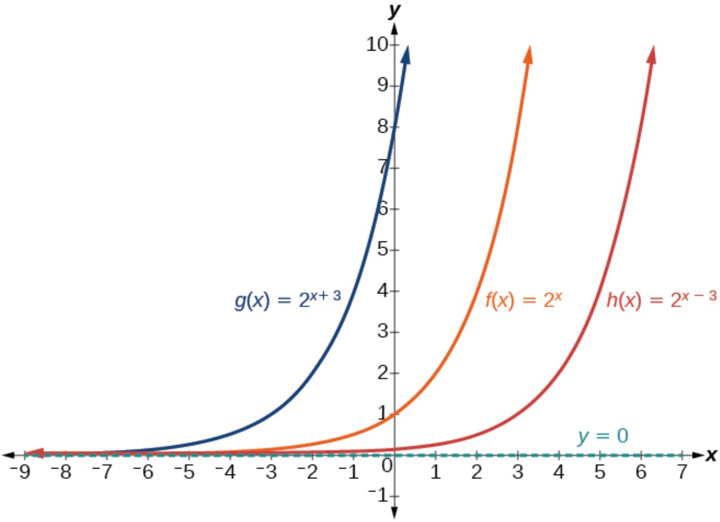
- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up 3 units to $(0,4)$.

- The asymptote shifts up 3 units to $y=3$.
- The range becomes $(3,\infty)$.

- When the function is shifted down 3 units to $h(x)=2^x-3$:
  - The *y*-intercept shifts down 3 units to $(0,-2)$.
  - The asymptote also shifts down 3 units to $y=-3$.
  - The range becomes $(-3,\infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x)=b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x)=2^x$, we can then graph two horizontal shifts alongside it, using $c=3$: the shift left, $g(x)=2^{x+3}$, and the shift right, $h(x)=2^{x-3}$. Both horizontal shifts are shown in [link].



Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The *y*-intercept shifts such that:
    - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$, so the initial value of the function is 8.
    - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the *y*-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

Shifts of the Parent Function $f(x) = b^x$

For any constants c and d, the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$, remains unchanged.

Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Graphing a Shift of an Exponential

Function

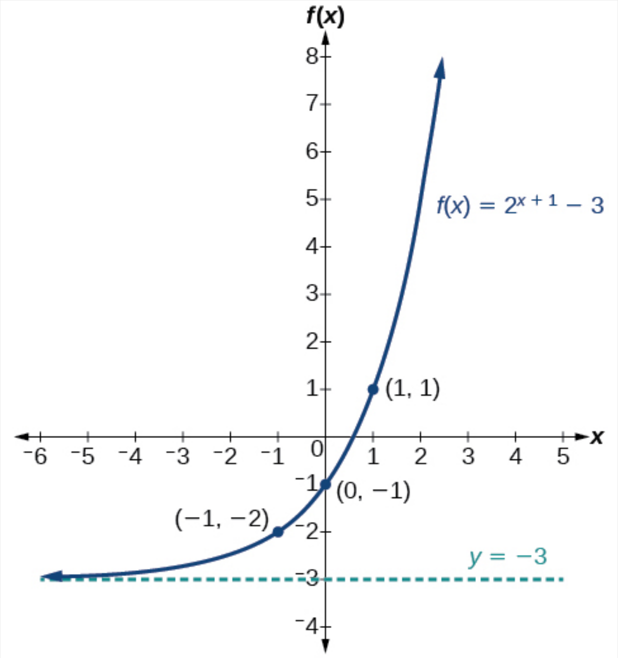
Graph $f(x)=2^{x+1}-3$. State the domain, range, and asymptote.

We have an exponential equation of the form $f(x)=b^{x+c}+d$, with $b=2$, $c=1$, and $d=-3$.

Draw the horizontal asymptote $y=d$, so draw $y=-3$.

Identify the shift as $(-c,d)$, so the shift is $(-1,-3)$.

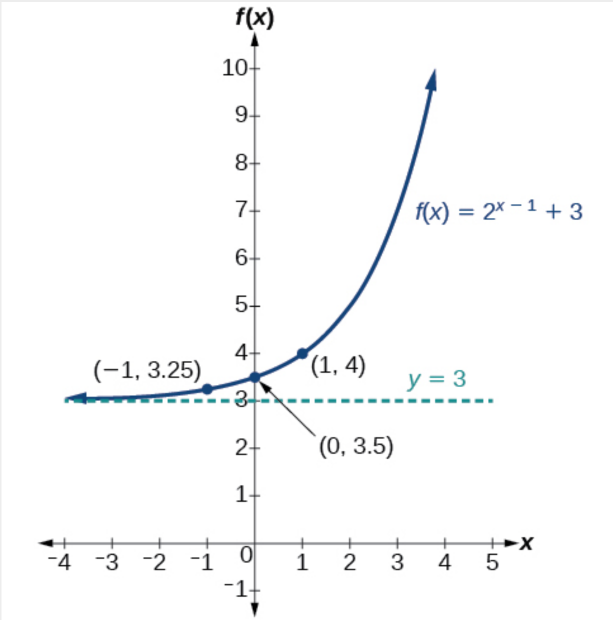
Shift the graph of $f(x)=b^{x}$ left 1 units and down 3 units.



The domain is $(-\infty,\infty)$; the range is $(-3,\infty)$; the horizontal asymptote is $y=-3$.

Graph $f(x)=2^{x-1}+3$. State domain, range, and asymptote.

The domain is $(-\infty,\infty)$; the range is $(3,\infty)$; the horizontal asymptote is $y=3$.



Given an equation of the form $f(x) = b\ x + c\ + d$ for x, use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y1=**".
- Enter the given value for $f(x)$ in the line headed "**Y2=**".
- Press **[WINDOW]**. Adjust the *y*-axis so that it includes the value entered for "**Y2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.

- To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select “intersect” and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

Approximating the Solution of an Exponential Equation

Solve $42=1.2(5)^x+2.8$ graphically. Round to the nearest thousandth.

Press **[Y=]** and enter $1.2(5)^x+2.8$ next to **Y1=**. Then enter 42 next to **Y2=**. For a window, use the values –3 to 3 for x and –5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x=2$.

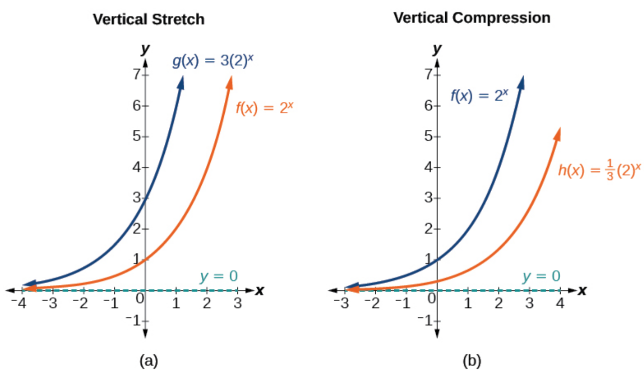
For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x\approx2.166$.

Solve $4=7.85(1.15)^{x}-2.27$ graphically. Round to the nearest thousandth.

$x\approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f(x)=b^{x}$ by a constant $|a|>0$. For example, if we begin by graphing the parent function $f(x)=2^{x}$, we can then graph the stretch, using $a=3$, to get $g(x)=3(2)^{x}$ as shown on the left in [link], and the compression, using $a=\frac{1}{3}$, to get $h(x)=\frac{1}{3}(2)^{x}$ as shown on the right in [link].



Stretches and Compressions of the Parent Function $f(x) = b^x$

For any factor $a > 0$, the function $f(x) = a\left(b\right)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a *y*-intercept of $\left(0, a\right)$.
- has a horizontal asymptote at $y = 0$, a range of $\left(0, \infty\right)$, and a domain of $\left(-\infty, \infty\right)$, which are unchanged from the parent function.

Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Before graphing, identify the behavior and key points on the graph.

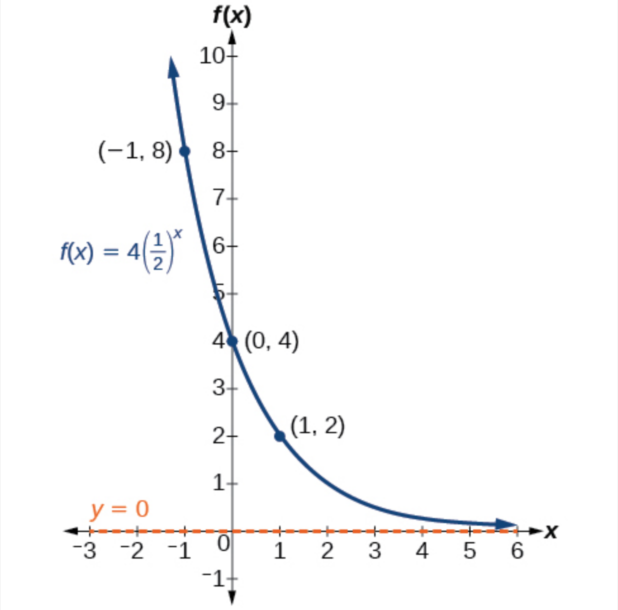
- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the *x*-axis as x increases.

- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in [link].

x	−3	−2	−1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

- Plot the *y*-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

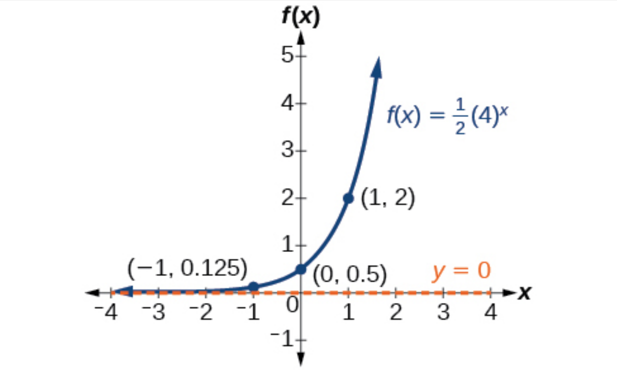
Draw a smooth curve connecting the points, as shown in [link].



The domain is $(-\infty,\infty)$; the range is $(0,\infty)$; the horizontal asymptote is $y=0$.

Sketch the graph of $f(x)=\frac{1}{2}(4)^x$. State the domain, range, and asymptote.

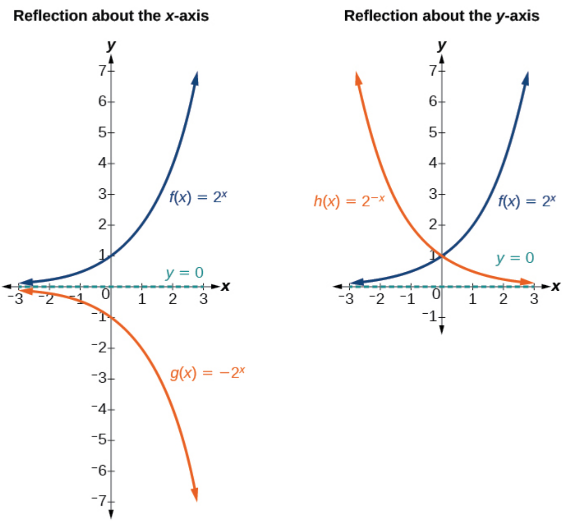
The domain is $(-\infty,\infty)$; the range is $(0,\infty)$; the horizontal asymptote is $y=0$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the

y-axis. When we multiply the parent function $f(x)=b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection about the y-axis. For example, if we begin by graphing the parent function $f(x)=2^x$, we can then graph the two reflections alongside it. The reflection about the x-axis, $g(x)=-2^x$, is shown on the left side of [link], and the reflection about the y-axis $h(x)=2^{-x}$, is shown on the right side of [link].



Reflections of the Parent Function $f(x)=b^x$

The function $f(x)=-b^x$

- reflects the parent function $f(x)=b^x$ about the x-axis.
- has a y-intercept of $(0,-1)$.
- has a range of $(-\infty,0)$.

- has a horizontal asymptote at $y=0$ and domain of $(-\infty,\infty)$, which are unchanged from the parent function.

The function $f(x)=b^{-x}$

- reflects the parent function $f(x)=b^x$ about the *y*-axis.
- has a *y*-intercept of $(0,1)$, a horizontal asymptote at $y=0$, a range of $(0,\infty)$, and a domain of $(-\infty,\infty)$, which are unchanged from the parent function.

Writing and Graphing the Reflection of an Exponential Function

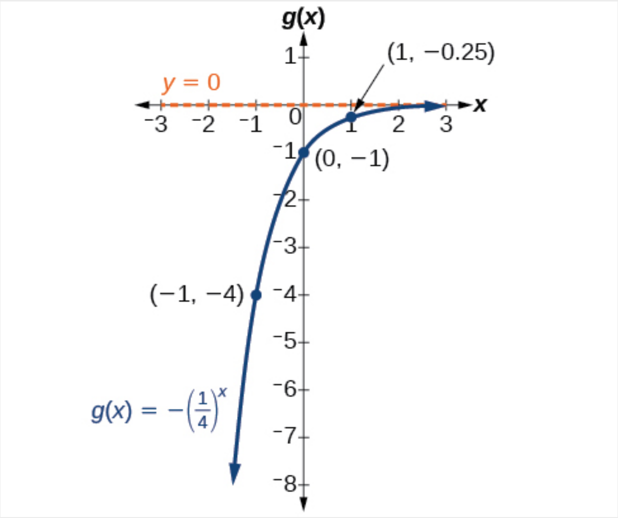
Find and graph the equation for a function, $g(x)$, that reflects $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

Since we want to reflect the parent function $f(x)=\left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x)=-\left(\frac{1}{4}\right)^x$. Next we create a table of points as in [link].

x	−3	−2	−1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	−64	−16	−4	−1	−0.25	−0.0625	−0.0156

Plot the *y*-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

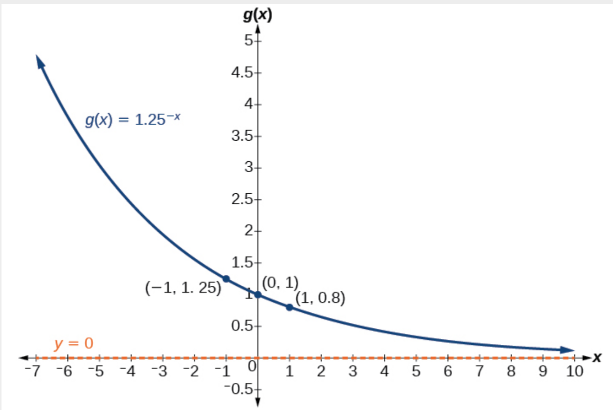


The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Find and graph the equation for a function,

$g(x)$, that reflects $f(x)=1.25^x$ about the *y*-axis. State its domain, range, and asymptote.

The domain is $(-\infty,\infty)$; the range is $(0,\infty)$; the horizontal asymptote is $y=0$.



Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $f(x)=b^x$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$f(x)=b^{x+c}+d$
Stretch and Compress • Stretch if $\lvert a \rvert > 1$ • Compression if $0 < \lvert a \rvert < 1$	$f(x)=ab^x$
Reflect about the *x* axis	$f(x)=-b^x$
Reflect about the *y* axis	$f(x)=b^{-x}=\left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x)=ab^{x+c}+d$

Translations of Exponential Functions

A translation of an exponential function has the form

$f(x)=ab^{x+c}+d$

Where the parent function, $y=b^x$, $b>1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $\lvert a \rvert$ if $\lvert a \rvert > 0$.
- compressed vertically by a factor of $\lvert a \rvert$ if $0 < \lvert a \rvert < 1$.
- shifted vertically d units.

- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the *y*-axis, and then shifted up 4 units.

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched vertically by a factor of 2, so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$f(x) = ab^{x+c} + d = 2e^{-x+0} + 4 = 2e^{-x} + 4$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

Key Equations

General Form for the Translation of the Parent Function $f(x)=b^x$	$f(x)=ab^{x+c}+d$

Key Concepts

- The graph of the function $f(x)=b^x$ has a *y*-intercept at $(0, 1)$, domain $(-\infty, \infty)$, range $(0, \infty)$, and horizontal asymptote $y=0$. See [link].
- If $b>1$, the function is increasing. The left tail of the graph will approach the asymptote $y=0$, and the right tail will increase without bound.
- If $0<b<1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y=0$.
- The equation $f(x)=b^x+d$ represents a vertical shift of the parent function $f(x)=b^x$.

- The equation $f(x)=b^{x+c}$ represents a horizontal shift of the parent function $f(x)=b^{x}$. See [link].
- Approximate solutions of the equation $f(x)=b^{x+c}+d$ can be found using a graphing calculator. See [link].
- The equation $f(x)=ab^{x}$, where $a>0$, represents a vertical stretch if $|a|>1$ or compression if $0<|a|<1$ of the parent function $f(x)=b^{x}$. See [link].
- When the parent function $f(x)=b^{x}$ is multiplied by -1, the result, $f(x)=-b^{x}$, is a reflection about the *x*-axis. When the input is multiplied by -1, the result, $f(x)=b^{-x}$, is a reflection about the *y*-axis. See [link].
- All translations of the exponential function can be summarized by the general equation $f(x)=ab^{x+c}+d$. See [link].
- Using the general equation $f(x)=ab^{x+c}+d$, we can write the equation of a function given its description. See [link].

Section Exercises

Verbal

What role does the horizontal asymptote of an

exponential function play in telling us about the end behavior of the graph?

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

The graph of $f(x)=3^x$ is reflected about the *y*-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

$g(x)=4(3)^{-x}$; *y*-intercept: $(0,4)$; Domain: all real numbers; Range: all real numbers greater than 0.

The graph of $f(x)=\ (\ 1\ 2\ )\ -x$ is reflected about the *y*-axis and compressed vertically by a factor of $1\ 5$. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

The graph of $f(x)=\ 10\ x$ is reflected about the *x*-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

$g(x)=-\ 10\ x\ +7$; *y*-intercept: $(\ 0,6\ )$; Domain: all real numbers; Range: all real numbers less than 7.

The graph of $f(x)=\ (\ 1.68\ )\ x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the *x*-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its *y*-intercept (to the nearest thousandth), domain, and range.

The graph of $f(x)=\ 2\ (\ 1\ 4\ )\ x-20$ is shifted left 2 units, stretched vertically by a factor of 4, reflected about the *x*-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

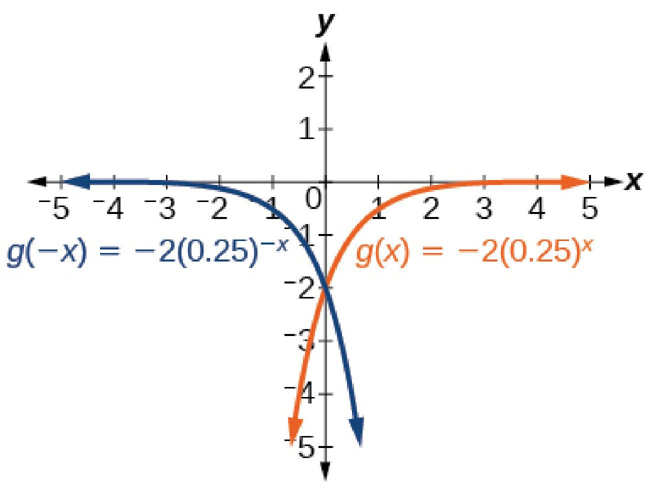
$g(x)=2\left(\frac{1}{4}\right)^x$; y-intercept: $(0, 2)$; Domain: all real numbers; Range: all real numbers greater than 0.

Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

$f(x)=3\left(\frac{1}{2}\right)^x$

$g(x)=-2(0.25)^x$

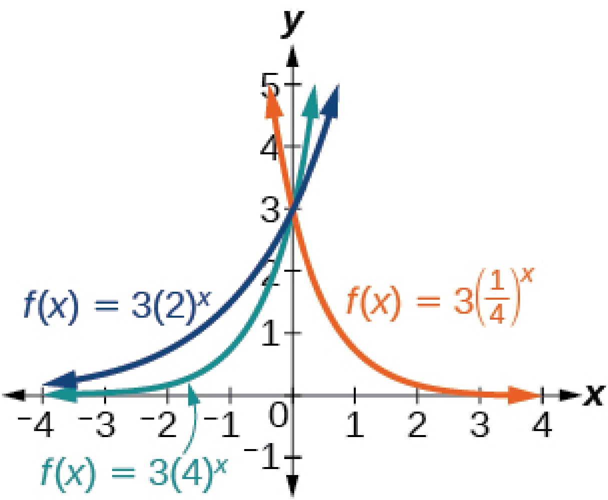


y-intercept: $(0,-2)$

$h(x)=6(1.75)^{-x}$

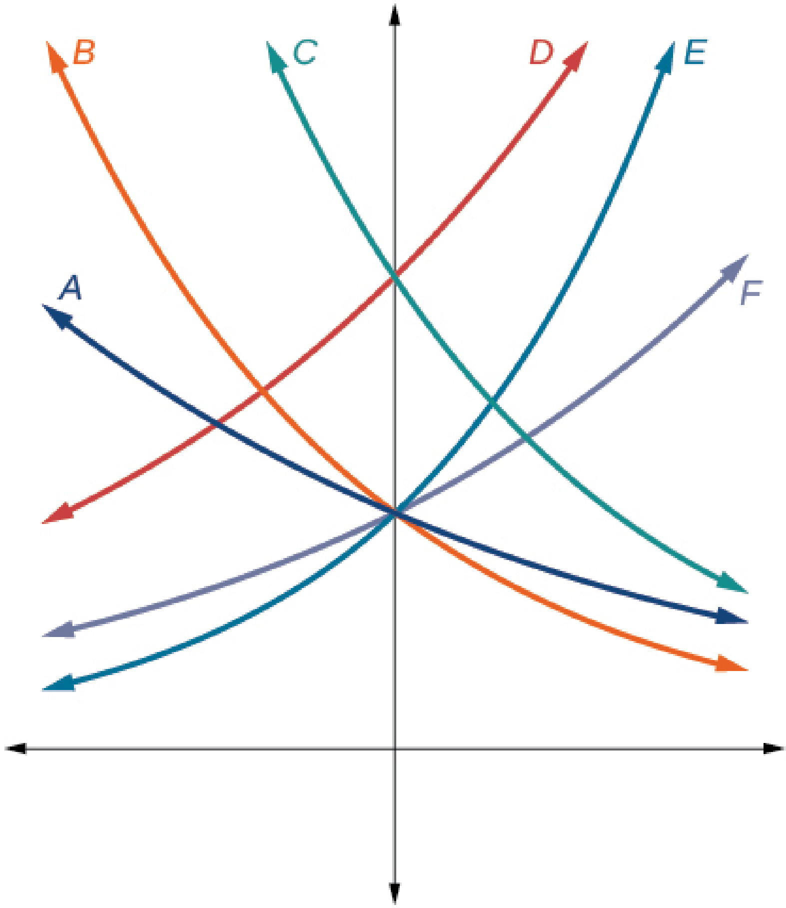
For the following exercises, graph each set of functions on the same axes.

$f(x)=3\left(\frac{1}{4}\right)^x$, $g(x)=3(2)^x$, and $h(x)=3(4)^x$



$f(x)=\frac{1}{4}(3)^x$, $g(x)=2(3)^x$, and $h(x)=4(3)^x$

For the following exercises, match each function with one of the graphs in [link].



$f(x) = 2(0.69)^x$

B

$f(x) = 2(1.28)^x$

$f(x) = 2(0.81)^x$

A

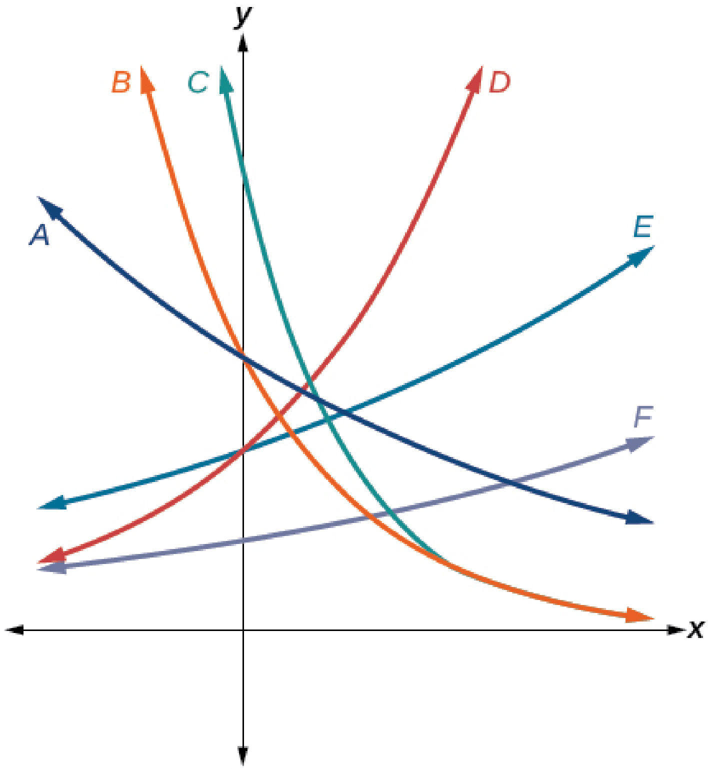
$f(x)=4(1.28)^x$

$f(x)=2(1.59)^x$

E

$f(x)=4(0.69)^x$

For the following exercises, use the graphs shown in [link]. All have the form $f(x)=ab^x$.



Which graph has the largest value for b?

D

Which graph has the smallest value for b?

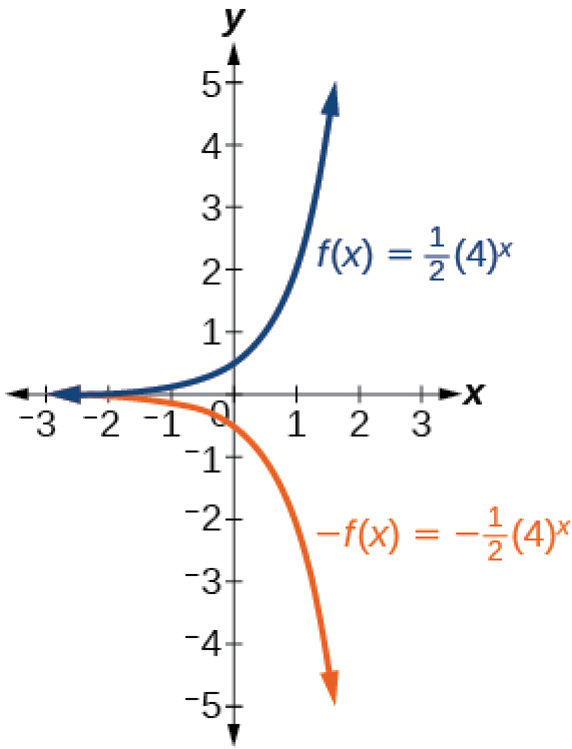
Which graph has the largest value for a?

C

Which graph has the smallest value for a?

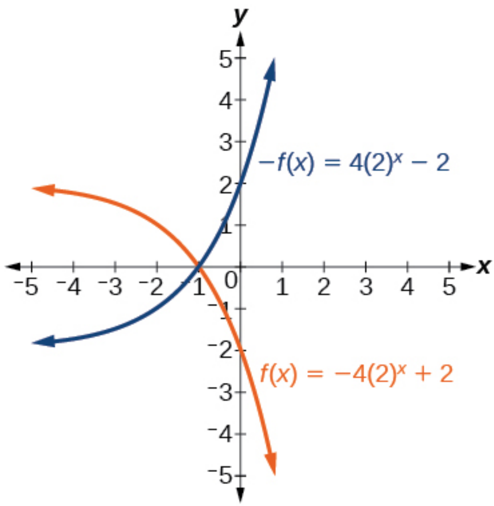
For the following exercises, graph the function and its reflection about the *x*-axis on the same axes.

$f(x)=\frac{1}{2}(4)^x$



$f(x)=3(0.75)^x-1$

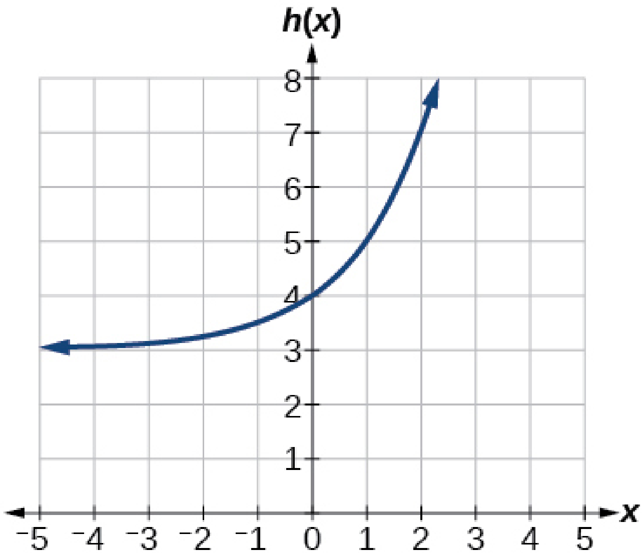
$f(x) = -4\,(2)\,x\ +2$



For the following exercises, graph the transformation of $f(x) = 2\,x$. Give the horizontal asymptote, the domain, and the range.

$f(\,x\,) = 2\ -x$

$h(\,x\,) = 2\,x\ +3$



Horizontal asymptote: $h(x) = 3$; Domain: all real numbers; Range: all real numbers strictly greater than 3.

$f(x) = 2x - 2$

For the following exercises, describe the end behavior of the graphs of the functions.

$f(x) = -5(4)x - 1$

As $x \to \infty$, $f(x) \to -\infty$;
As $x \to -\infty$, $f(x) \to -1$

$f(x) = 3(1\ 2)x - 2$

$f(x)=3 (4) -x +2$

As $x\to\infty$, $f(x)\to 2$;
As $x\to-\infty$, $f(x)\to\infty$

For the following exercises, start with the graph of $f(x)= 4 x$. Then write a function that results from the given transformation.

Shift f(x) 4 units upward

Shift f(x) 3 units downward

$f(x)= 4 x -3$

Shift f(x) 2 units left

Shift f(x) 5 units right

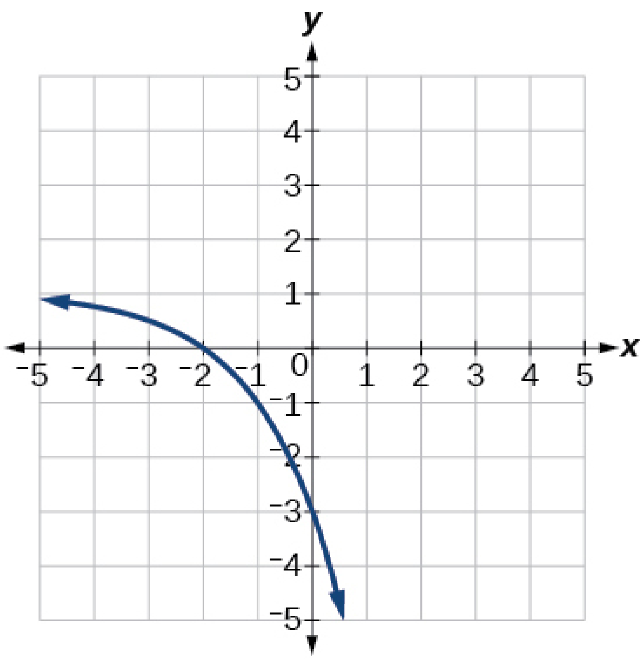
$f(x)= 4 x-5$

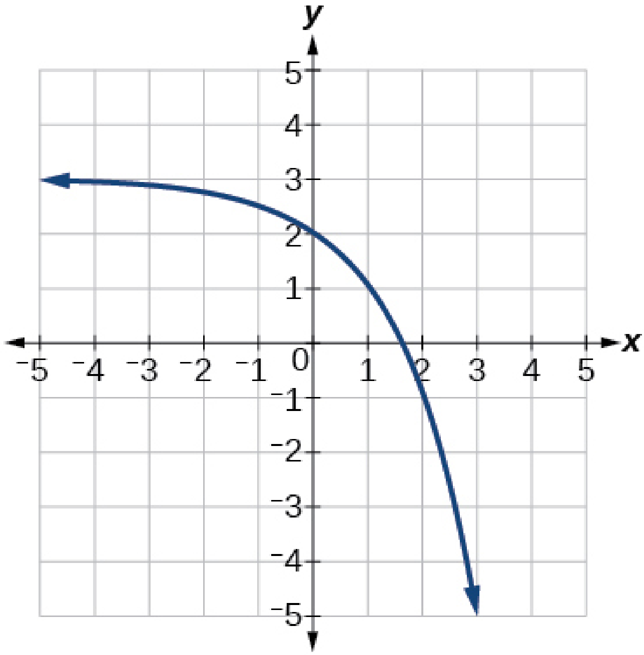
Reflect f(x) about the *x*-axis

Reflect $f(x)$ about the *y*-axis

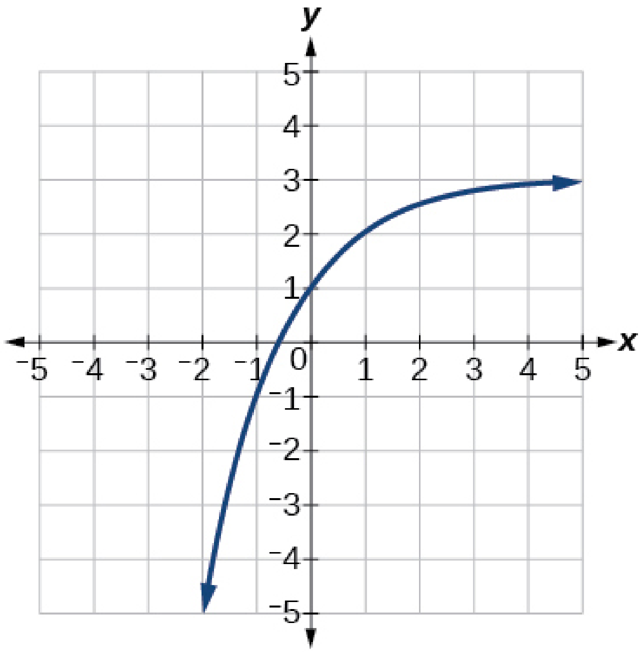
$f(x) = 4^{-x}$

For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

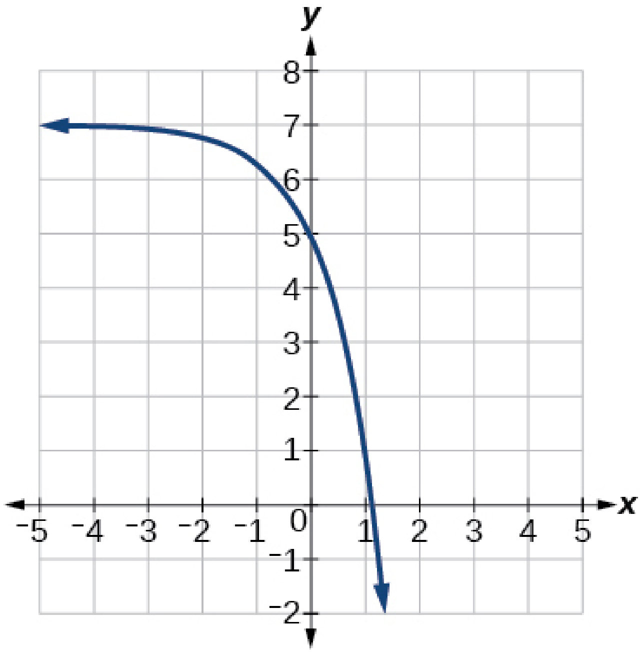




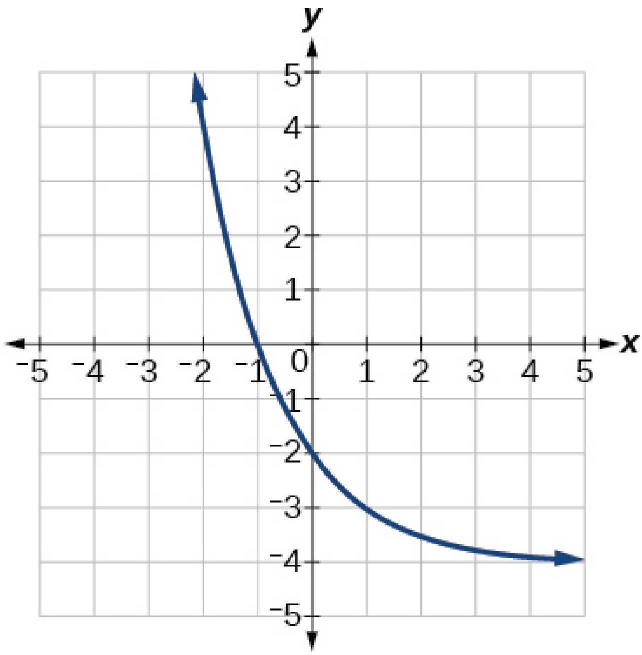
$y = -\ 2x + 3$



For the following exercises, find an exponential equation for the graph.



$y = -2(3)x + 7$



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

$g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

$f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

$h(x) = -\ 1\ 2\ (\ 1\ 2\ )\ x\ +6$ for $h(-7)$.

$h(-7) = -58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

$-50 = -\ (\ 1\ 2\ )\ -x$

$116 =\ 1\ 4\ (\ 1\ 8\ )\ x$

$x \approx -2.953$

$12 = 2\ (\ 3\ )\ x\ +1$

$5 = 3\ (\ 1\ 2\ )\ x-1\ -2$

$x \approx -0.222$

$-30 = -4\ (\ 2\ )\ x+2\ +2$

Extensions

Explore and discuss the graphs of $F(x) = (b) x$ and $G(x) = (1 b) x$. Then make a conjecture about the relationship between the graphs of the functions $b x$ and $(1 b) x$ for any real number $b > 0$.

The graph of $G(x) = (1 b) x$ is the refelction about the *y*-axis of the graph of $F(x) = b x$; For any real number $b > 0$ and function $f(x) = b x$, the graph of $(1 b) x$ is the the reflection about the *y*-axis, $F(-x)$.

Prove the conjecture made in the previous exercise.

Explore and discuss the graphs of $f(x) = 4 x$, $g(x) = 4 x - 2$, and $h(x) = (1 16) 4 x$. Then make a conjecture about the relationship between the graphs of the functions $b x$ and $(1 b n) b x$ for any real number *n* and real number $b > 0$.

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number *n*, real number $b > 0$,

and function $f(x) = b x$, the graph of $(1 b n)$ $b x$ is the horizontal shift $f(x-n)$.

Prove the conjecture made in the previous exercise.

Logarithmic Functions
In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[footnote]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[footnote] like those shown in [link]. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than

the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[footnote] whereas the Japanese earthquake registered a 9.0. [footnote]

http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details. Accessed 3/4/2013.

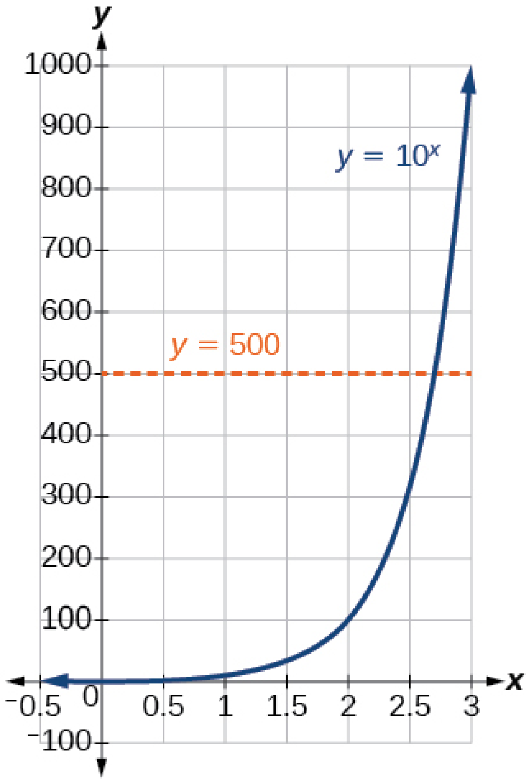
The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different

earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in [link], to better estimate the solution.



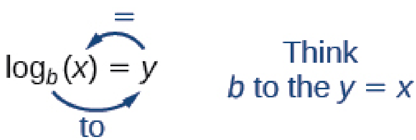
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in [link] passes the horizontal line test. The exponential function $y=b^x$ is one-to-one, so its inverse, $x=b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y=\log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, “The logarithm with base b of x is equal to y,” or, simplified, “log base b of x is y.” We can also say, “b raised to the power of y is x,” because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as “log base 2 of 32 is 5.”

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

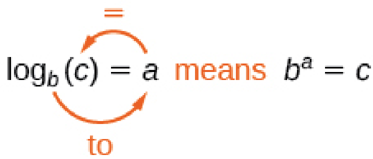
$$\log_b(x) = y \Leftrightarrow b^y = x,\ b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y=\log_b(x)$ and $y=b^x$ are inverse functions.

Definition of the Logarithmic Function

A **logarithm** base b of a positive number x satisfies the following definition.

For $x>0, b>0, b\neq 1$,

$y=\log_b(x)$ is equivalent to $b^y=x$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0,\infty)$.

- the range of the logarithm function with base b is $(-\infty, \infty)$.

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Given an equation in logarithmic form $\log_b(x)=y$, **convert it to exponential form.**

1. Examine the equation $y=\log_b x$ and identify $b, y, \text{and} x$.
2. Rewrite $\log_b x=y$ as $b^y=x$.

Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in

exponential form.

1. $\log_6(6) = \frac{1}{2}$
2. $\log_3(9) = 2$

First, identify the values of $b, y,$ and x. Then, write the equation in the form $b^y = x$.

1. $\log_6(6) = \frac{1}{2}$

   Here, $b = 6, y = \frac{1}{2},$ and $x = 6$. Therefore, the equation $\log_6(6) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = 6$.
2. $\log_3(9) = 2$

   Here, $b = 3, y = 2,$ and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Write the following logarithmic equations in exponential form.

1. $\log_{10}(1{,}000{,}000) = 6$
2. $\log_5(25) = 2$

1. $\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
2. $\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

1. $2^3 = 8$
2. $5^2 = 25$
3. $10^{-4} = \frac{1}{10,000}$

First, identify the values of $b, y,$ and x. Then, write the equation in the form $x = \log_b(y)$.

1. $2^3 = 8$

   Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.
2. $5^2 = 25$

   Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.
3. $10^{-4} = \frac{1}{10{,}000}$

   Here, $b = 10$, $x = -4$, and $y = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4$.

Write the following exponential equations in logarithmic form.

1. $3^2 = 9$
2. $5^3 = 125$
3. $2^{-1} = \frac{1}{2}$

1. $3^2=9$ is equivalent to $\log_3(9)=2$
2. $5^3=125$ is equivalent to $\log_5(125)=3$
3. $2^{-1}=\frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right)=-1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know $2^3=8$, it follows that $\log_2 8=3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2=49$. Therefore, $\log_7 49=2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3=27$. Therefore, $\log_3 27=3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? ” We know $2^2=4$ and $3^2=9$, so $\left(\frac{2}{3}\right)^2=\frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right)=2$.

Given a logarithm of the form $y=\log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of b: $b^y=x$.
2. Use previous knowledge of powers of b identify y by asking, “To what exponent should b be raised in order to get x? ”

Solving Logarithms Mentally

Solve $y=\log_4(64)$ without using a calculator.

First we rewrite the logarithm in exponential form: $4^y=64$. Next, we ask, “To what exponent must 4 be raised in order to get 64?”

We know
$4^3=64$

Therefore,
$\log_4(64)=3$

Solve $y=\log_{121}(11)$ without using a calculator.

$\log_{121}(11)=\frac{1}{2}$ (recalling that $\sqrt{121}=(121)^{\frac{1}{2}}=11$)

Evaluating the Logarithm of a Reciprocal

Evaluate $y=\log_3\left(\frac{1}{27}\right)$ without using a calculator.

First we rewrite the logarithm in exponential form: $3^y=\frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3=27$, but what must we do to

get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

$\log_2\left(\frac{1}{32}\right) = -5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the

brightness of stars and the pH of acids and bases also use common logarithms.

Definition of the Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x>0$,

$y=\log(x)$ is equivalent to $10^y=x$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

Given a common logarithm of the form $y=\log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10: $10^y=x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Finding the Value of a Common Logarithm Mentally

Evaluate $y=\log(1000)$ without using a calculator.

First we rewrite the logarithm in exponential form: $10^y=1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$10^3=1000$

Therefore, $\log(1000)=3$.

Evaluate $y=\log(1{,}000{,}000)$.

$\log(1{,}000{,}000)=6$

Given a common logarithm with the form $y=\log(x)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x, followed by **[ )]**.

3. Press **[ENTER]** .

Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

- Press **[LOG]** .
- Enter 321, followed by **[ )]** .
- Press **[ENTER]** .

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000 \quad 2 < 2.5065 < 3$$

Evaluate $y = \log(123)$ to four decimal places using a calculator.

$\log(123) \approx 2.0899$

Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500 \quad \log(500) = x$$

Use the definition of the common log.

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has

its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Definition of the Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$y = \ln(x)$ is equivalent to $e^y = x$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]** .
2. Enter the value given for x, followed by **[ )]** .
3. Press **[ENTER]** .

Evaluating a Natural Logarithm Using a Calculator

Evaluate $y=\ln(500)$ to four decimal places using a calculator.

- Press **[LN]** .
- Enter 500, followed by **[ )]** .
- Press **[ENTER]** .

Rounding to four decimal places, $\ln(500)\approx 6.2146$

Evaluate $\ln(-500)$.

It is not possible to take the logarithm of a negative number in the set of real numbers.

Finding the domain of a logarithmic related function.

Find the domain of the function, $f(x)=\log_8(2x-5)$.

Solution

Recall that we cannot take logarithm of any negative number or zero, hence to find the domain of

$f(x)=\log_8(2x-5)$, we require that $2x-5>0$. Hence $x>\frac{5}{2}$ and the domain in interval notation is

$\left(\frac{5}{2},\infty\right)$.

Find the domain of the function $f(x)=\log_4(x^2-1)$.

Solution

Since we cannot take logarithm of any negative number or zero, to find the domain of $f(x)=\log_4(x^2-1)$,

we require that $x^2-1>0$. Hence $x<-1$ or $x>1$ and the domain in interval notation is

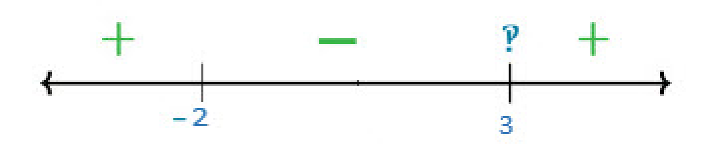
$(-\infty,-1)\cup(1,\infty)$.

Find the domain of the function $f(x)=\log_3\left(\frac{x+2}{x-3}\right)$.

Solution

Since we cannot take logarithm of any negative number or zero, to find the domain of $f(x)=\log_3\left(\frac{x+2}{x-3}\right)$, we

require that $\frac{x+2}{x-3}>0$.



Hence using a sign chart as before, the domain in interval notation is

$(-\infty,-2)\cup(3,\infty)$.

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x>0, b>0, b\neq 1$, $y=\log_b(x)$ if and only if $b^y=x$.
Definition of the common logarithm	For $x>0$, $y=\log(x)$ if and only if $10^y=x$.
Definition of the natural logarithm	For $x>0$, $y=\ln(x)$ if and only if $e^y=x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See [link].
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See [link].

- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See [link] and [link].
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See [link].
- When common logarithms cannot be evaluated mentally, a calculator can be used. See [link].
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See [link].
- Natural logarithms can be evaluated using a calculator [link].

Section Exercises

Verbal

What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given,

the base b has the same value. The exponent, y, in the expression b^y can also be written as the logarithm, $\log_b x$, and the value of x is the result of raising b to the power of y.

How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.

$\log_4(q)=m$

$\log_a(b)=c$

$a^c=b$

$\log_{16}(y)=x$

$\log_x(64)=y$

$x^y=64$

$\log_y(x)=-11$

$\log_{15}(a) = b$

$15^b = a$

$\log_y(137) = x$

$\log_{13}(142) = a$

$13^a = 142$

$\log(v) = t$

$\ln(w) = n$

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

$4^x = y$

$c^d = k$

$\log c\ (k) = d$

$m\ -7\ = n$

$19\ x\ = y$

$\log 19\ y = x$

$x\ -\ \ 10\ 13\ = y$

$n\ 4\ = 103$

$\log n\ (\ 103\ ) = 4$

$(\ 7\ 5\ )\ m\ = n$

$y\ x\ =\ 39\ 100$

$\log y\ (\ 39\ 100\ ) = x$

$10\ a\ = b$

$e^{k} = h$

$\ln(h) = k$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

$\log_3(x) = 2$

$\log_2(x) = -3$

$x = 2^{-3} = \frac{1}{8}$

$\log_5(x) = 2$

$\log_3(x) = 3$

$x = 3^3 = 27$

$\log_2(x) = 6$

$\log_9(x) = \frac{1}{2}$

$x=9^{\frac{1}{2}}=3$

$\log_{18}(x)=2$

$\log_{6}(x)=-3$

$x=6^{-3}=\frac{1}{216}$

$\log(x)=3$

$\ln(x)=2$

$x=e^{2}$

For the following exercises, use the definition of common and natural logarithms to simplify.

$\log(100^{8})$

$10^{\log(32)}$

32

$2\log(.0001)$

$e^{\ln(1.06)}$

1.06

$\ln(e^{-5.03})$

$e^{\ln(10.125)}+4$

14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

$\log_3\left(\frac{1}{27}\right)$

$\log_6(6)$

$\frac{1}{2}$

$\log_2\left(\frac{1}{8}\right)+4$

$6\log_8(4)$

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

$\log(10{,}000)$

$\log(0.001)$

-3

$\log(1)+7$

$2\log(100^{-3})$

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

$\ln(e^{\frac{1}{3}})$

$\ln(1)$

0

$\ln(e^{-0.225})-3$

$25\ln(e^{\frac{2}{5}})$

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

$\log(0.04)$

$\ln(15)$

2.708

$\ln(\frac{4}{5})$

$\log(2)$

0.151

$\ln(2)$

Extensions

Is $x=0$ in the domain of the function $f(x)=\log(x)$? If so, what is the value of the function when $x=0$? Verify the result.

No, the function has no defined value for $x=0$. To verify, suppose $x=0$ is in the domain of the function $f(x)=\log(x)$. Then there is some number n such that $n=\log(0)$. Rewriting as an exponential equation gives: $10^n=0$, which is impossible since no such real number n exists. Therefore, $x=0$ is *not* the domain of the function $f(x)=\log(x)$.

Is $f(x)=0$ in the range of the function $f(x)=\log(x)$? If so, for what value of x? Verify the result.

Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln(x) = \ln(e^2) = 2$.

Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$? Verify the result.

No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)}$ is undefined.

Real-World Applications

The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the “f-stop” setting on the camera, and t is the exposure time in seconds. Suppose the f-stop

setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

2

The intensity levels *I* of two earthquakes measured on a seismograph can be compared by the formula $\log \frac{I_1}{I_2} = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[footnote] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.
http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

Glossary

common logarithm

the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

logarithm

the exponent to which b must be raised to get x; written $y = \log_b(x)$

natural logarithm

the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

Graphs of Logarithmic Functions

In this section, you will:

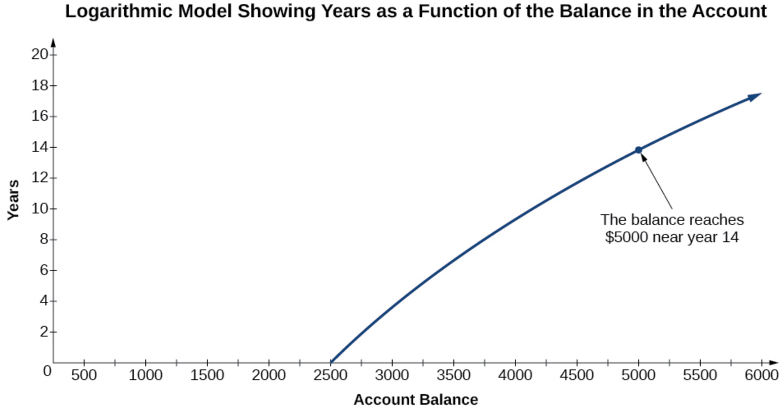
- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A=2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial

investment to double? [link] shows this point on the logarithmic graph.



In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.

- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero.

To find the domain, we set up an inequality and solve for x:

$2x-3>0$ Show the argument greater than zero.
$2x>3$ Add 3. $x>1.5$ Divide by 2.

In interval notation, the domain of $f(x)=\log_4(2x-3)$ is $(1.5,\infty)$.

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x)=\log_2(x+3)$?

The logarithmic function is defined only when the input is positive, so this function is defined when $x+3>0$. Solving this inequality,

$x+3>0$ The input must be positive. $x>-3$ Subtract 3.

The domain of $f(x) = \log_2(x+3)$ is $(-3, \infty)$.

What is the domain of $f(x) = \log_5(x-2)+1$?

$(2, \infty)$

Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5-2x)$?

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$5-2x > 0$ The input must be positive.
$-2x > -5$ Subtract 5.
$x < \frac{5}{2}$ Divide by -2 and switch the inequality.

The domain of $f(x) = \log(5-2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

What is the domain of $f(x)=\log(x-5)+2$?
$(5,\infty)$

Notice that the graphs of $f(x)=2^x$ and $g(x)=\log_2(x)$ are reflections about the line $y=x$.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y=\log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

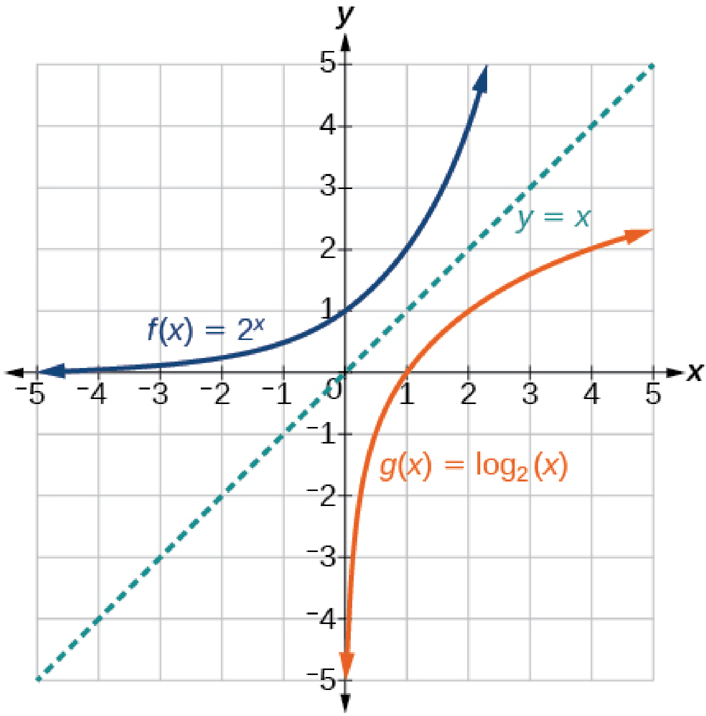
We begin with the parent function $y=\log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y=b^x$, their graphs will be reflections of each other across the line $y=x$. To illustrate this, we can observe the relationship between the input and output values of $y=2^x$ and its equivalent $x=\log_2(y)$ in [link].

x	-3	-2	-1	0	1	2	3
$2^x=y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y)=x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from [link], we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x)=2^x$ and $g(x)=\log_2(x)$. See [link].

$f(x)=2^x$	$\left(-3,\frac{1}{8}\right)$	$\left(-2,\frac{1}{4}\right)$	$\left(-1,\frac{1}{2}\right)$	$(0,1)$	$(1,2)$	$(2,4)$	$(3,8)$
$g(x)=\log_2(x)$	$\left(\frac{1}{8},-3\right)$	$\left(\frac{1}{4},-2\right)$	$\left(\frac{1}{2},-1\right)$	$(1,0)$	$(2,1)$	$(4,2)$	$(8,3)$

As we'd expect, the *x*- and *y*-coordinates are reversed for the inverse functions. [link] shows the graph of f and g.



Observe the following from the graph:

- $f(x) = 2^x$ has a *y*-intercept at $(0,1)$ and $g(x) = \log_2(x)$ has an *x*- intercept at $(1,0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

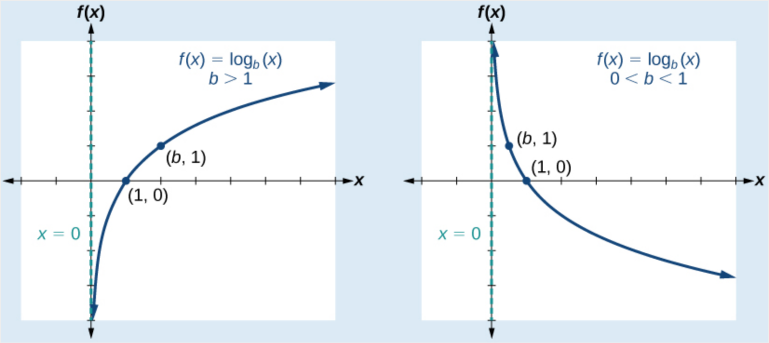
Characteristics of the Graph of the Parent Function, $f(x) = \log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$,

we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

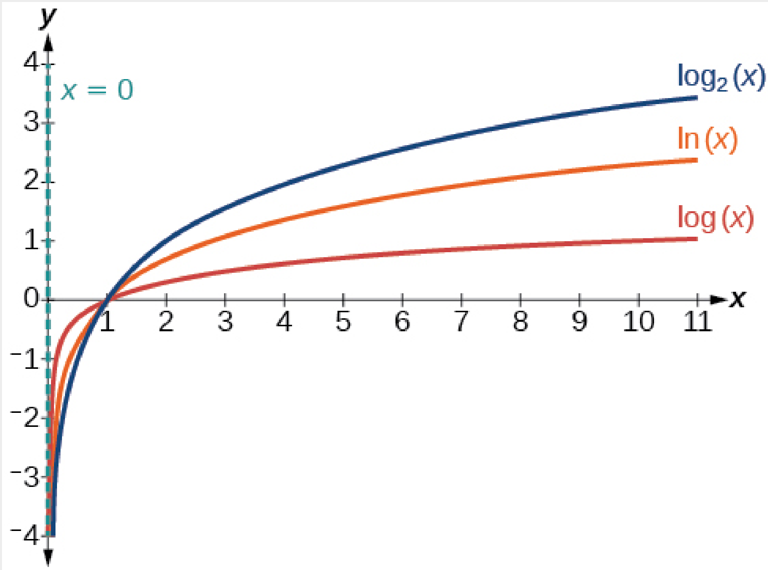
- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- *x*-intercept: $(1,0)$ and key point $(b,1)$
- *y*-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See [link].



[link] shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718.$)

The graphs of three logarithmic functions with different bases, all greater than 1.



Given a logarithmic function with the form $f(x)= \log_b(x)$, graph the function.

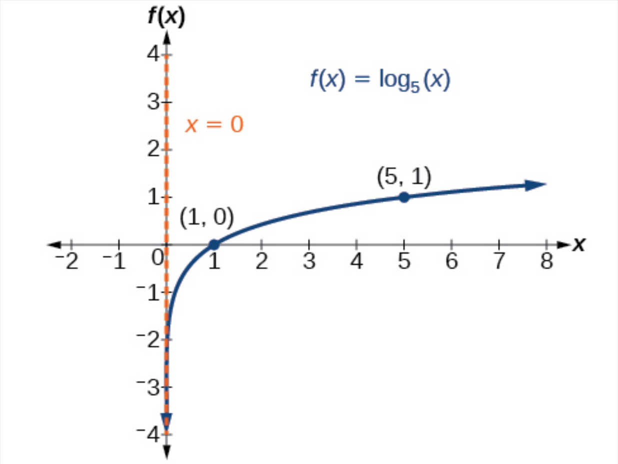
1. Draw and label the vertical asymptote, $x=0$.
2. Plot the *x*-intercept, $(1,0)$.
3. Plot the key point $(b,1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0,\infty)$, the range, $(-\infty,\infty)$, and the vertical asymptote, $x=0$.

Graphing a Logarithmic Function with the Form $f(x) = \log_b(x)$.

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Before graphing, identify the behavior and key points for the graph.

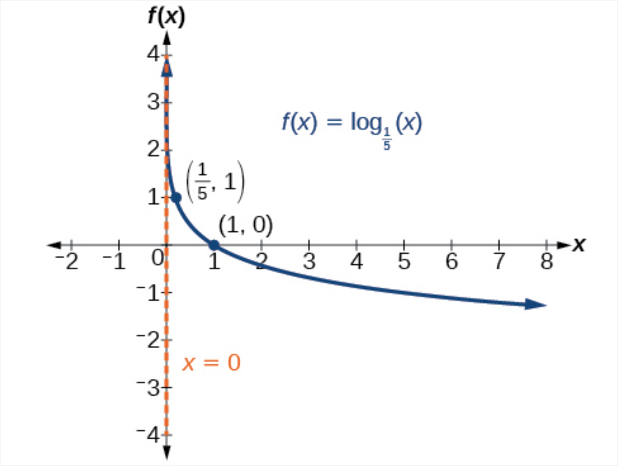
- Since $b=5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x=0$, and the right tail will increase slowly without bound.
- The *x*-intercept is $(1,0)$.
- The key point $(5,1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see [link]).



The domain is $(0,\infty)$, the range is $(-\infty,\infty$

), and the vertical asymptote is $x=0$.

Graph $f(x)= \log 1 5 (x)$. State the domain, range, and asymptote.



The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

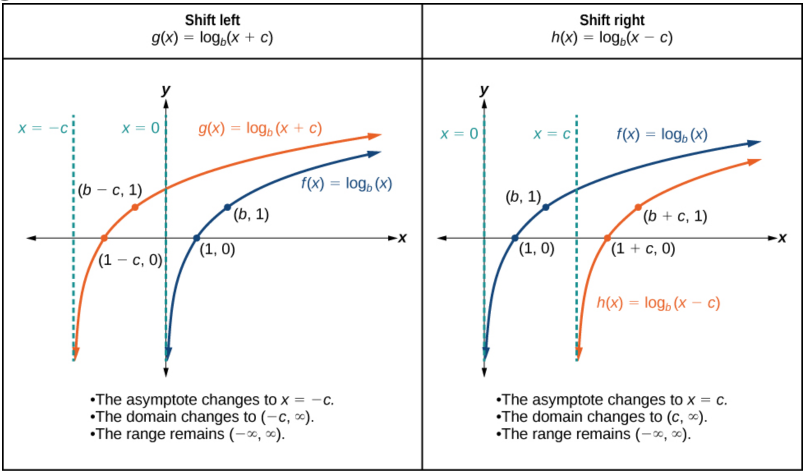
Graphing Transformations of Logarithmic

Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of *f*(*x*) = log*b*(*x*)

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See [link].



Horizontal Shifts of the Parent Function $y = \log_b(x)$

For any constant c, the function $f(x) = \log_b(x+c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_b(x+c)$, graph the translation.

1. Identify the horizontal shift:
    1. If $c > 0$, shift the graph of $f(x) = \log_b(x)$ left c units.
    2. If $c < 0$, shift the graph of $f(x) = \log_b(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.

5. The Domain is $(-c, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log 3\,(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Since the function is $f(x) = \log 3\,(x-2)$, we notice $x + (-2) = x–2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log 3\,(x)$ right 2 units.

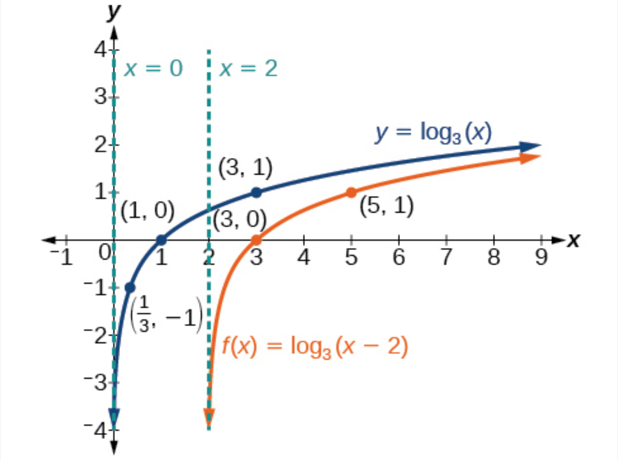
The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $(1\ 3, -1)$, $(1,0)$, and $(3,1)$.

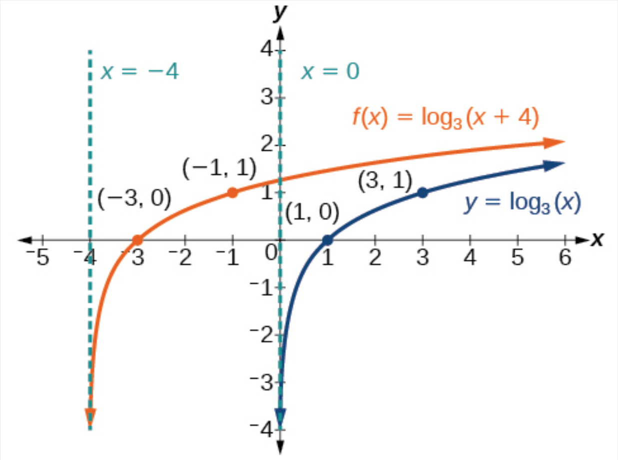
The new coordinates are found by adding 2 to the x coordinates.

Label the points $(7\ 3, -1)$, $(3,0)$, and $(5,1)$.

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.



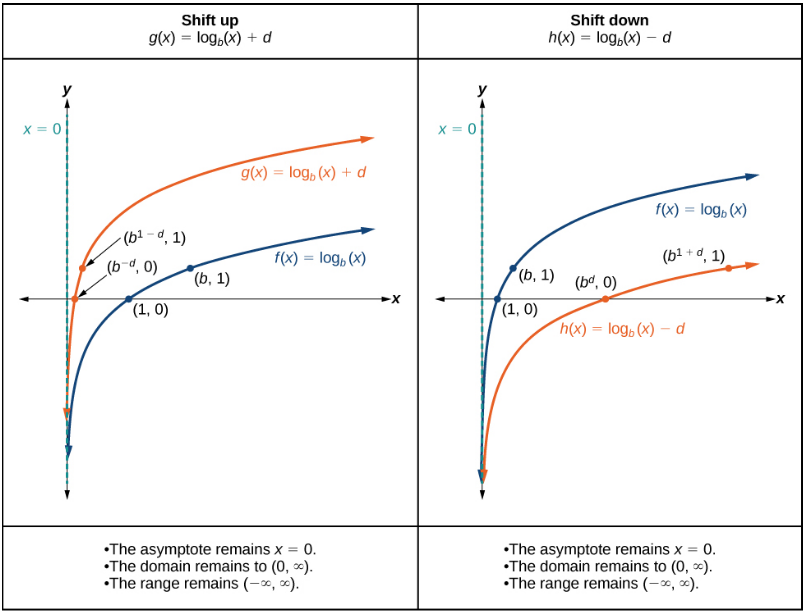
Sketch a graph of $f(x) = \log 3\ (x + 4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.



The domain is $(-4, \infty)$, the range $(-\infty, \infty)$, and the asymptote $x = -4$.

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See [link].



Vertical Shifts of the Parent Function $y = \log_b(x)$

For any constant d, the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down d units.

4. Draw the vertical asymptote $x = 0$.
5. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.
6. Label the three points.
7. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Since the function is $f(x) = \log_3(x) - 2$, we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

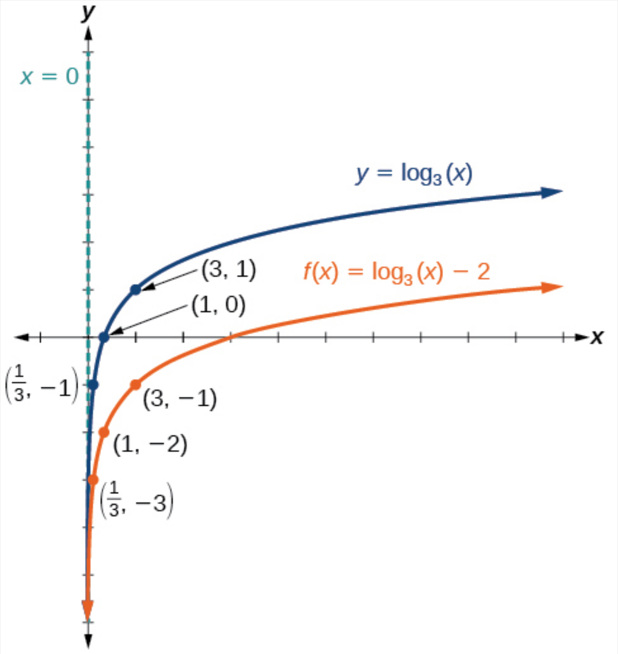
The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $(\frac{1}{3}, -1)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

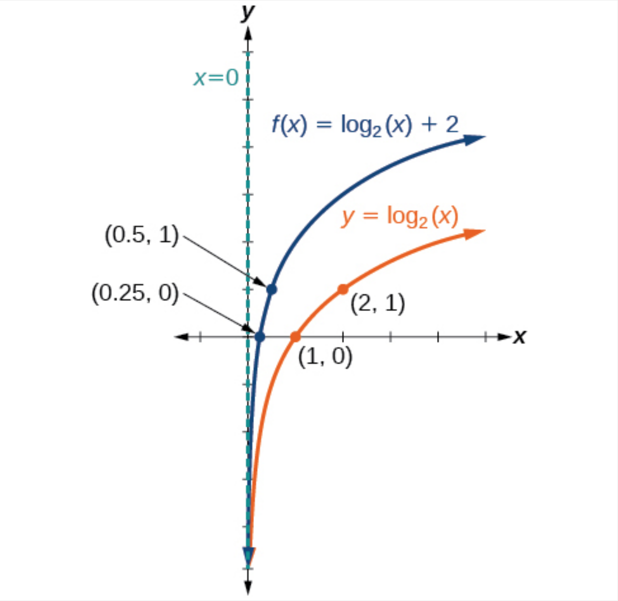
Label the points $(\frac{1}{3}, -3)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.



The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

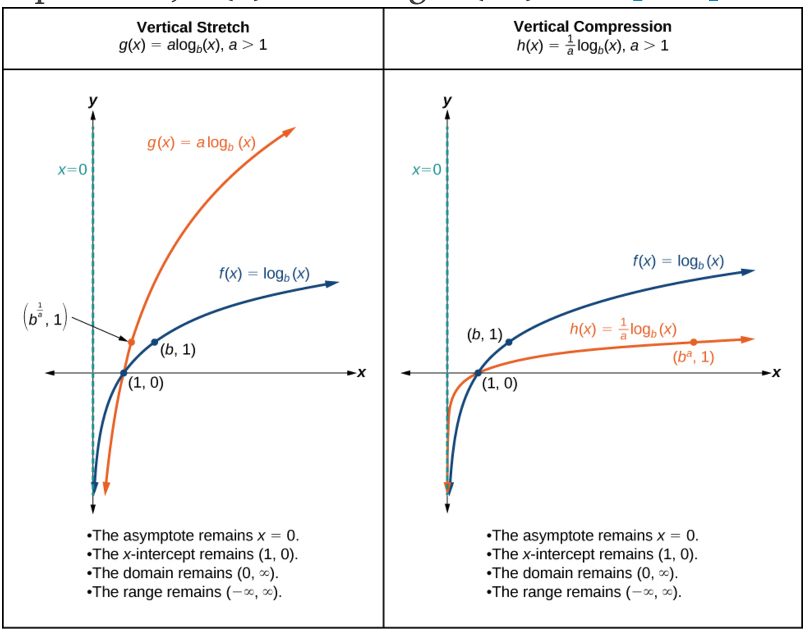
Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.



The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x)=\log_b(x)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f(x)=\log_b(x)$ alongside the vertical stretch, $g(x)=a\log_b(x)$ and the vertical compression, $h(x)=\frac{1}{a}\log_b(x)$. See [link].



Vertical Stretches and Compressions of the Parent Function $y = \log_b(x)$

For any constant $a>1$, the function $f(x)=a\log_b($

x)

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the *x*-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = a \log_b(x)$, $a > 0$, graph the translation.

1. Identify the vertical stretch or compressions:

    - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
    - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

4. Draw the vertical asymptote $x = 0$.
5. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
6. Label the three points.
7. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing a Stretch or Compression of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x)=2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Since the function is $f(x)=2\log_4(x)$, we will notice $a=2$.

This means we will stretch the function $f(x)=\log_4(x)$ by a factor of 2.

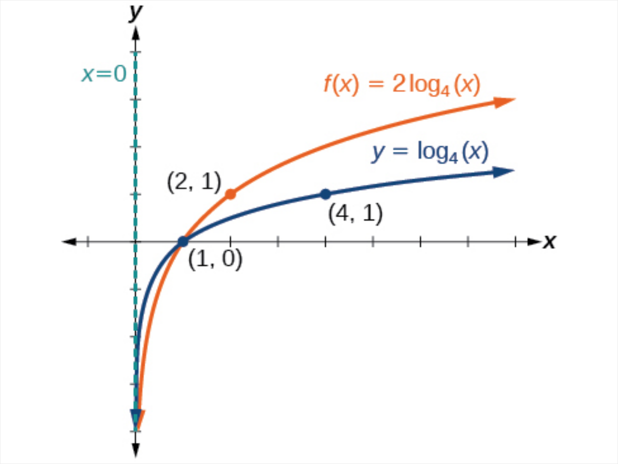
The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{4},-1\right)$, $(1,0)$, and $(4,1)$.

The new coordinates are found by multiplying the y coordinates by 2.

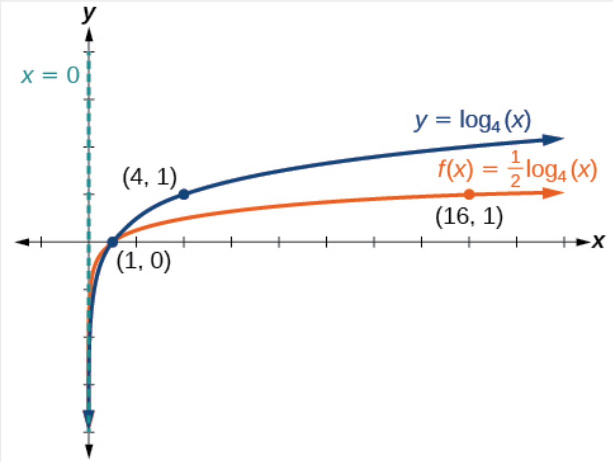
Label the points $\left(\frac{1}{4},-2\right)$, $(1,0)$, and $(4,2)$.

The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$. See [link].



The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Sketch a graph of $f(x)=\frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.



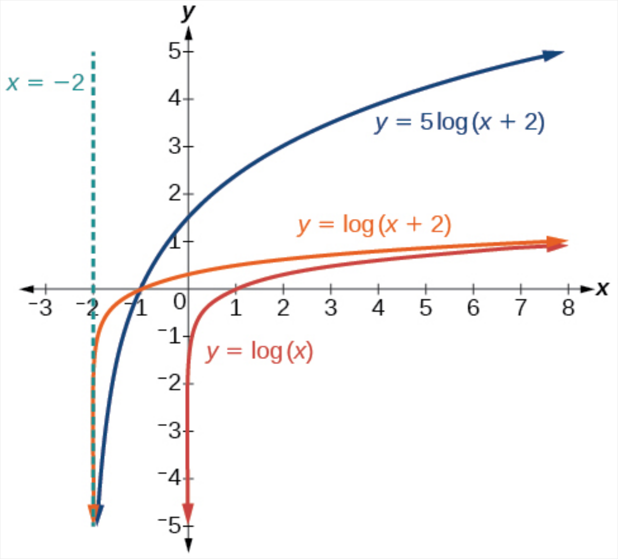
The domain is $(0,\infty)$, the range is $(-\infty,\infty)$, and the vertical asymptote is $x=0$.

Combining a Shift and a Stretch

Sketch a graph of $f(x)=5\log(x+2)$. State the domain, range, and asymptote.

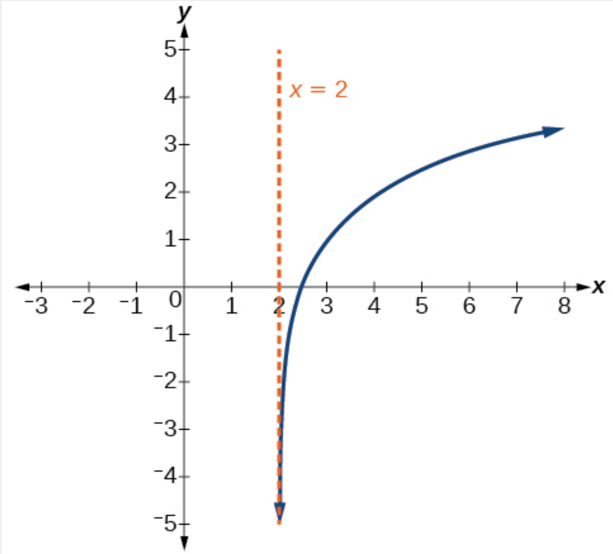
Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in [link]. The vertical asymptote will be shifted to $x=-2$. The *x*-intercept will be $(-1,0)$. The domain will be $(-2,\infty)$. Two points will help give the shape of the graph: $(-1,0)$ and $(8,5)$. We chose $x=8$ as

the *x*-coordinate of one point to graph because when $x=8$, $x+2=10$, the base of the common logarithm.



The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

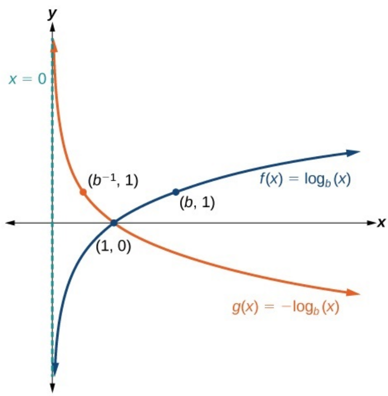
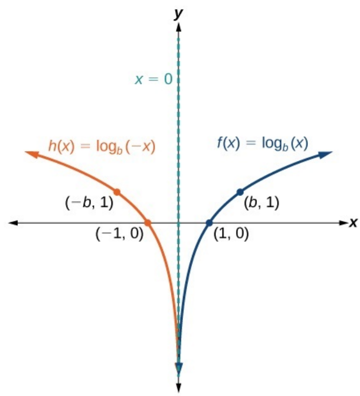
Sketch a graph of the function $f(x)=3\log(x-2)+1$. State the domain, range, and asymptote.



The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the *input* is multiplied by -1, the result is a reflection about the *y*-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the *x*-axis, $g(x) = -\log_b(x)$ and the reflection about the *y*-axis, $h(x) = \log_b(-x)$.

Reflection about the x-axis $g(x)=-\log_b x,\ b>1$	**Reflection about the y-axis** $h(x)=\log_b(-x),\ b>1$
 •The reflected function is decreasing as x moves from zero to infinity. •The asymptote remains $x = 0$. •The x-intercept remains (1, 0). •The key point changes to $(b^{-1}, 1)$ •The domain remains $(0, \infty)$. •The range remains $(-\infty, \infty)$.	 •The reflected function is decreasing as x moves from negative infinity to zero. •The asymptote remains $x = 0$. •The x-intercept changes to $(-1, 0)$. •The key point changes to $(-b, 1)$ •The domain changes to $(-\infty, 0)$. •The range remains $(-\infty, \infty)$.

Reflections of the Parent Function $y = \log_b(x)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the y-axis.

- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x=0$, which are unchanged from the parent function.

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$		If $f(x) = \log_b(-x)$

1. Draw the vertical asymptote, $x=0$.
1. Plot the x intercept, $(1,0)$.
1. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the x axis.
1. Draw a smooth curve through the points.
1. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x=0$.

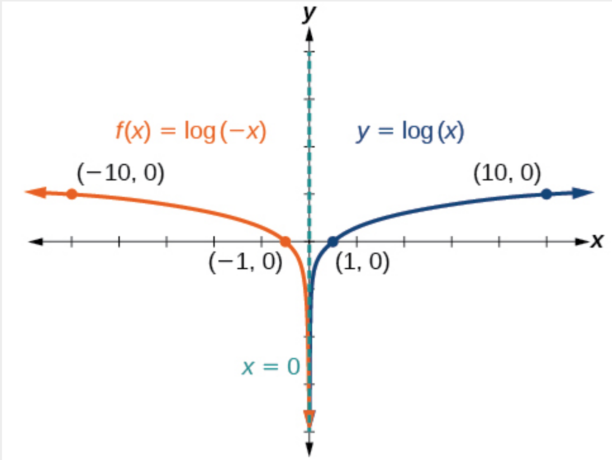
Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its

parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

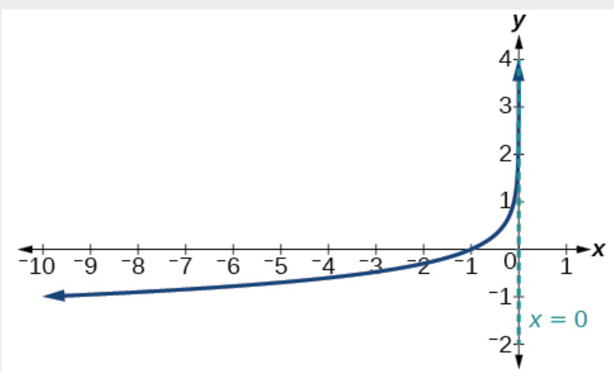
Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the *y*-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The *x*-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.



The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x=0$.

Graph $f(x)=-\log(-x)$. State the domain, range, and asymptote.



The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x=0$.

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y1=** and, if needed, **Y2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x, for the point(s) of intersection.

Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x)+1=-2\ln(x-1)$ graphically. Round to the nearest thousandth.

Press **[Y=]** and enter $4\ln(x)+1$ next to **Y1=**. Then enter $-2\ln(x-1)$ next to **Y2=**. For a window, use the values 0 to 5 for x and –10 to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x=1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x\approx 1.339$.

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

$x\approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in [link] to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y=\log_b(x)$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$y=\log_b(x+c)+d$
Stretch and Compress • Stretch if $\lvert a\rvert>1$ • Compression if $\lvert a\rvert<1$	$y=a\log_b(x)$
Reflect about the *x*-axis	$y=-\log_b(x)$
Reflect about the *y*-axis	$y=\log_b(-x)$
General equation for all translations	$y=a\log_b(x+c)+d$

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y=\log_b(x)$, have the form

$f(x)=a\log_b(x+c)+d$

where the parent function, $y=\log_b(x), b>1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a|>0$.
- compressed vertically by a factor of $|a|$ if $0<|a|<1$.
- reflected about the *x*-axis when $a<0$.

For $f(x)=\log(-x)$, the graph of the parent function is reflected about the *y*-axis.

Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x)=-2\log_3(x+4)+5$?

The vertical asymptote is at $x=-4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical

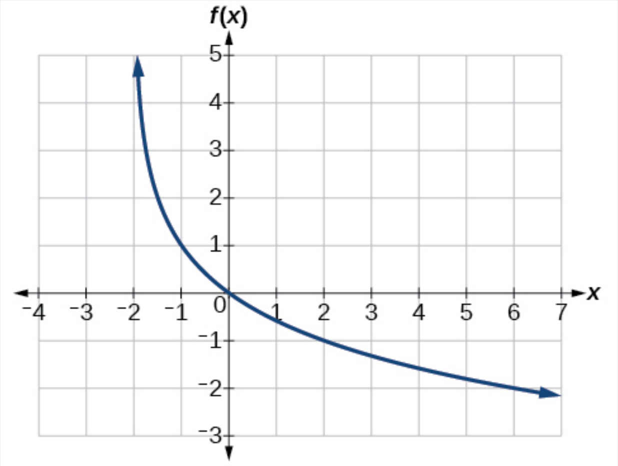
asymptote to $x = -4$.

What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

$x = 1$

Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in [link].



This graph has a vertical asymptote at $x=-2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$f(x)=-a\log(x+2)+k$

It appears the graph passes through the points $(-1,1)$ and $(2,-1)$. Substituting $(-1,1)$,

$1=-a\log(-1+2)+k$ Substitute $(-1,1)$.
$1=-a\log(1)+k$ Arithmetic. $1=k$ $\log(1)=0$.

Next, substituting in $(2,-1)$,

$-1=-a\log(2+2)+1$ Plug in $(2,-1)$. $-2=-a\log(4)$ Arithmetic. $a=2\log(4)$
Solve for a.

This gives us the equation $f(x)=-2\log(4)\log(x+2)+1$.

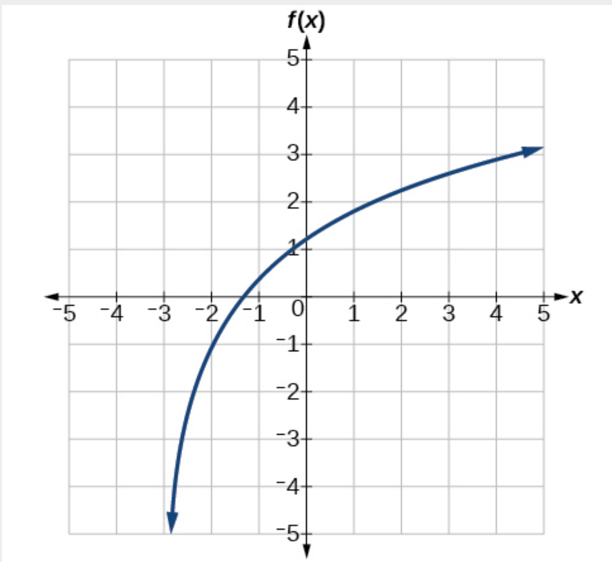
Analysis

We can verify this answer by comparing the function values in [link] with the points on the graph in [link].

x	−1	0	1	2	3

f(x)	1	0	−0.58496	1	−1.3219
x	4	5	6	7	8
f(x)	−1.5850	−1.8074	−2	−2.1699	−2.3219

Give the equation of the natural logarithm graphed in [link].



$f(x) = 2\ln(x+3) - 1$

Is it possible to tell the domain and range and describe the end behavior of a function just by

looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in [link]. The graph approaches $x=-3$ *(or thereabouts) more and more closely, so* $x=-3$ *is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right,* $\{x\,|\,x>-3\}$. *The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as* $x\to-3^+$, $f(x)\to-\infty$ *and as* $x\to\infty, f(x)\to\infty$.

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function
- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Equations

General Form for the Translation of the Parent Logarithmic Function $f(x)=\log_b(x)$	$f(x)=a\log_b(x+c)+d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See [link] and [link]
- The graph of the parent function $f(x)=\log_b(x)$ has an *x*-intercept at $(1,0)$, domain $(0,\infty)$, range $(-\infty,\infty)$, vertical asymptote $x=0$, and
  - if $b>1$, the function is increasing.
  - if $0<b<1$, the function is decreasing.

  See [link].
- The equation $f(x)=\log_b(x+c)$ shifts the parent function $y=\log_b(x)$ horizontally
  - left c units if $c>0$.
  - right c units if $c<0$.

  See [link].
- The equation $f(x)=\log_b(x)+d$ shifts the parent function $y=\log_b(x)$ vertically

- up d units if $d>0$.
- down d units if $d<0$.

See [link].
- For any constant $a>0$, the equation $f(x)=a\log_b(x)$
  - stretches the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|>1$.
  - compresses the parent function $y=\log_b(x)$ vertically by a factor of a if $|a|<1$.

  See [link] and [link].
- When the parent function $y=\log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
  - The equation $f(x)=-\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
  - The equation $f(x)=\log_b(-x)$ represents a reflection of the parent function about the *y*-axis.

  See [link].
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See [link].

- All translations of the logarithmic function can be summarized by the general equation $f(x)=a\log_b(x+c)+d$. See [link].
- Given an equation with the general form $f(x)=a\log_b(x+c)+d$, we can identify the vertical asymptote $x=-c$ for the transformation. See [link].
- Using the general equation $f(x)=a\log_b(x+c)+d$, we can write the equation of a logarithmic function given its graph. See [link].

Section Exercises

Verbal

The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Since the functions are inverses, their graphs are mirror images about the line $y=x$. So for every point (a,b) on the graph of a logarithmic function, there is a corresponding point (b,a) on the graph of its inverse exponential function.

What type(s) of translation(s), if any, affect the range of a logarithmic function?

What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?

Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

$f(x)=\log_3(x+4)$

$h(x)=\ln\left(\frac{1}{2}-x\right)$

Domain: $\left(-\infty,\frac{1}{2}\right)$; Range: $(-\infty,\infty)$

$g(x)=\log_5(2x+9)-2$

$h(x)=\ln(4x+17)-5$

Domain: $\left(-\frac{17}{4},\infty\right)$; Range: $(-\infty,\infty)$

$f(x)=\log_2(12-3x)-3$

For the following exercises, state the domain and the vertical asymptote of the function.

$f(x)=\log_b(x-5)$

Domain: $(5,\infty)$; Vertical asymptote: $x=5$

$g(x)=\ln(3-x)$

$f(x)=\log(3x+1)$

Domain: $\left(-\frac{1}{3},\infty\right)$; Vertical asymptote: $x=-\frac{1}{3}$

$f(x)=3\log(-x)+2$

$g(x)=-\ln(3x+9)-7$

Domain: $(-3,\infty)$; Vertical asymptote: $x=-3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

$f(x)=\ln(2-x)$

$f(x)=\log\left(x-\frac{3}{7}\right)$

Domain: $\left(\frac{3}{7},\infty\right)$;
Vertical asymptote: $x=\frac{3}{7}$; End behavior: as $x\to\left(\frac{3}{7}\right)^{+}, f(x)\to-\infty$ and as $x\to\infty, f(x)\to\infty$

$h(x)=-\log(3x-4)+3$

$g(x)=\ln(2x+6)-5$

Domain: $(-3,\infty)$; Vertical asymptote: $x=-3$
;
End behavior: as $x\to-3^{+}$, $f(x)\to-\infty$ and as $x\to\infty$, $f(x)\to\infty$

$f(x)=\log_3(15-5x)+6$

For the following exercises, state the domain, range, and *x*- and *y*-intercepts, if they exist. If they do not exist, write DNE.

$h(x)=\log_4(x-1)+1$

Domain: $(1,\infty)$; Range: $(-\infty,\infty)$; Vertical asymptote: $x=1$; *x*-intercept: $\left(\frac{5}{4},0\right)$; *y*-intercept: DNE

$f(x)=\log(5x+10)+3$

$g(x)=\ln(-x)-2$

Domain: $(-\infty,0)$; Range: $(-\infty,\infty)$; Vertical asymptote: $x=0$; *x*-intercept: $(-e^2,0)$; *y*-intercept: DNE

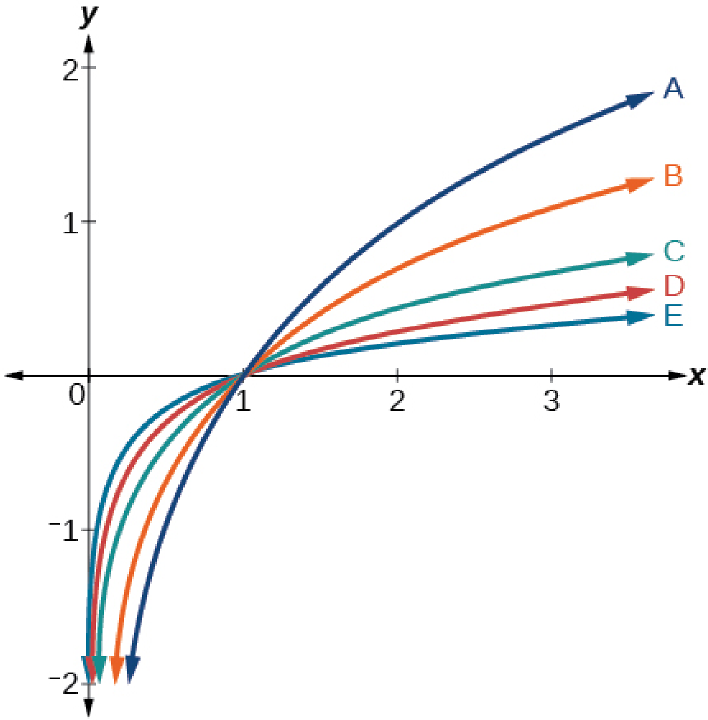
$f(x) = \log_2(x+2) - 5$

$h(x) = 3\ln(x) - 9$

Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; Vertical asymptote: $x = 0$; *x*-intercept: $(e^3, 0)$; *y*-intercept: DNE

Graphical

For the following exercises, match each function in [link] with the letter corresponding to its graph.



$d(x) = \log(x)$

$f(x) = \ln(x)$

B

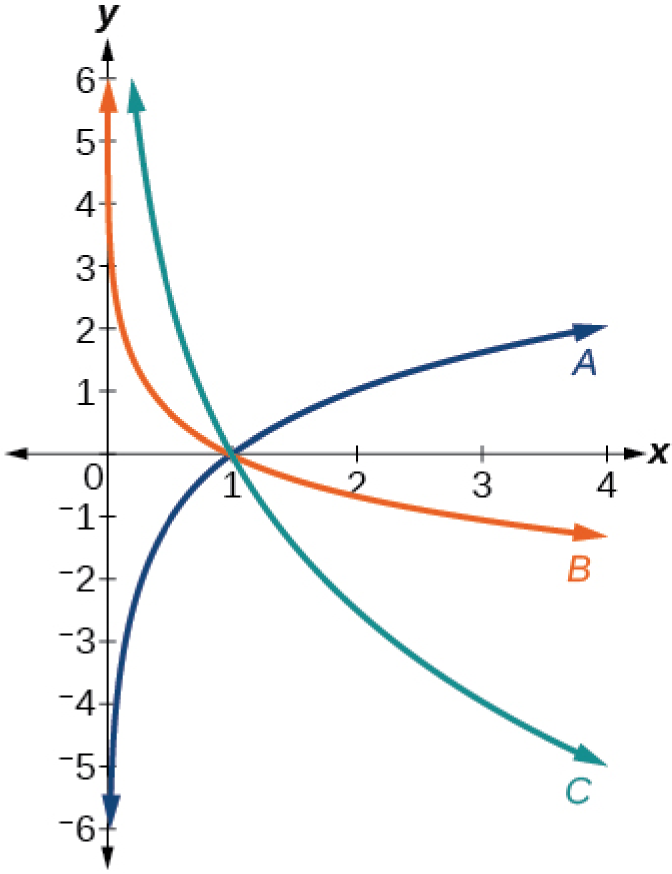
$g(x) = \log_2(x)$

$h(x) = \log_5(x)$

C

$j(x) = \log_{25}(x)$

For the following exercises, match each function in [link] with the letter corresponding to its graph.



$f(x) = \log_{\frac{1}{3}}(x)$

B

$g(x) = \log_2(x)$

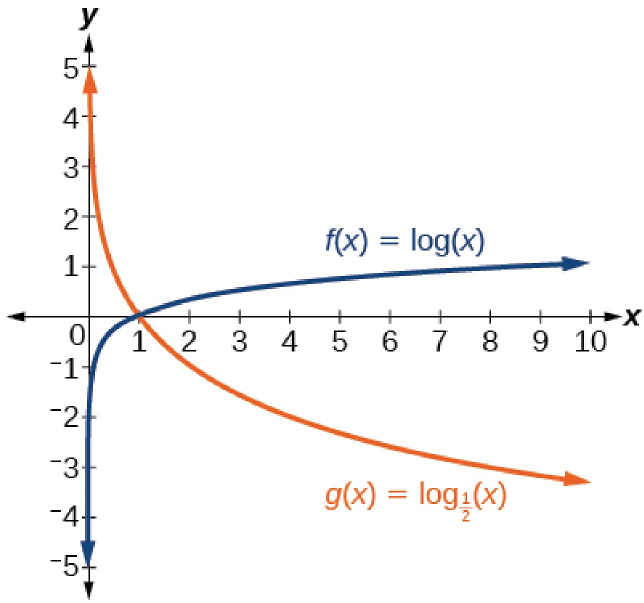
$h(x) = \log_{\frac{3}{4}}(x)$

C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

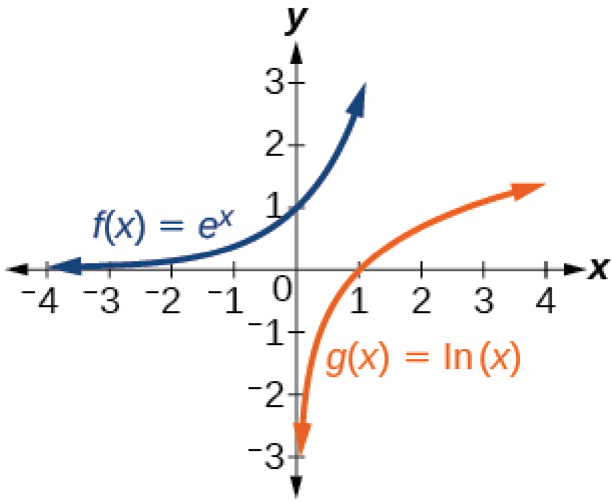
$f(x)=\log(x)$ and $g(x) = 10^x$

$f(x)=\log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

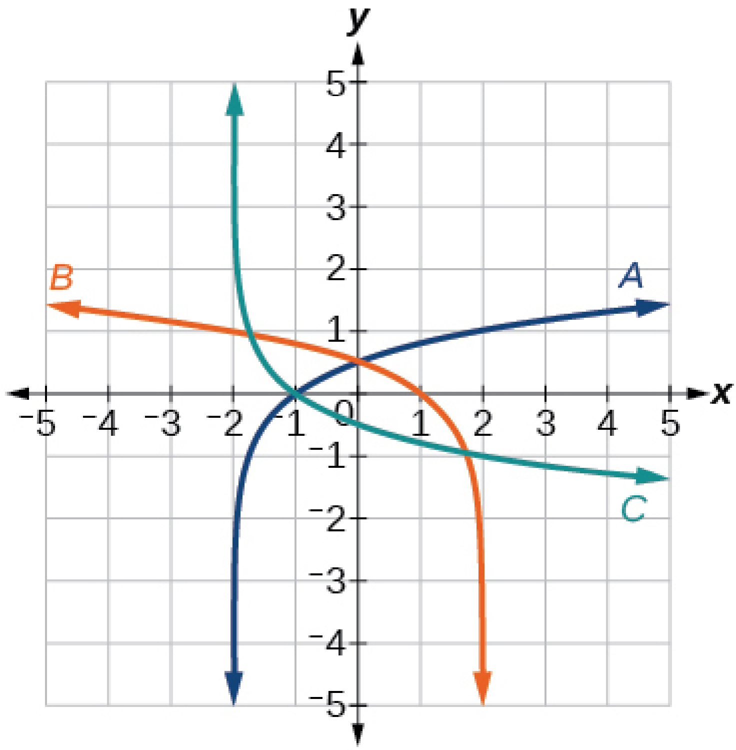


$f(x) = \log 4 (x)$ and $g(x) = \ln(x)$

$f(x) = e x$ and $g(x) = \ln(x)$



For the following exercises, match each function in [link] with the letter corresponding to its graph.



$f(x) = \log_4(-x+2)$

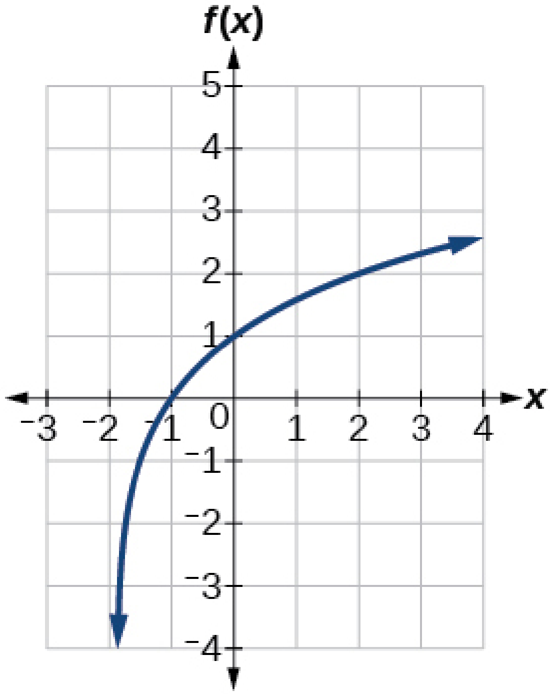
$g(x) = -\log_4(x+2)$

C

$h(x) = \log_4(x+2)$

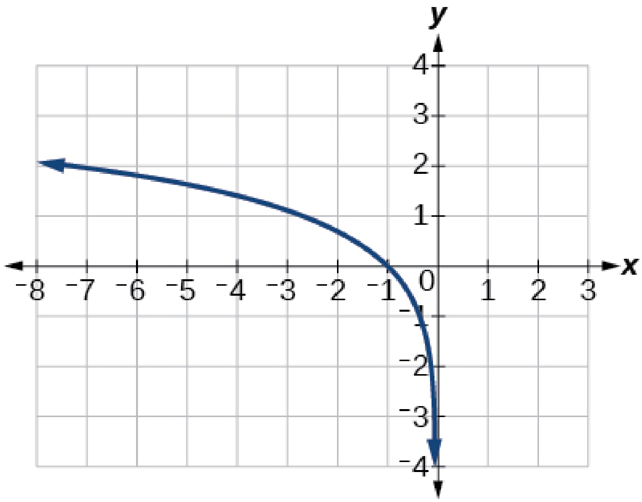
For the following exercises, sketch the graph of the indicated function.

$f(x) = \log 2\ (x+2)$



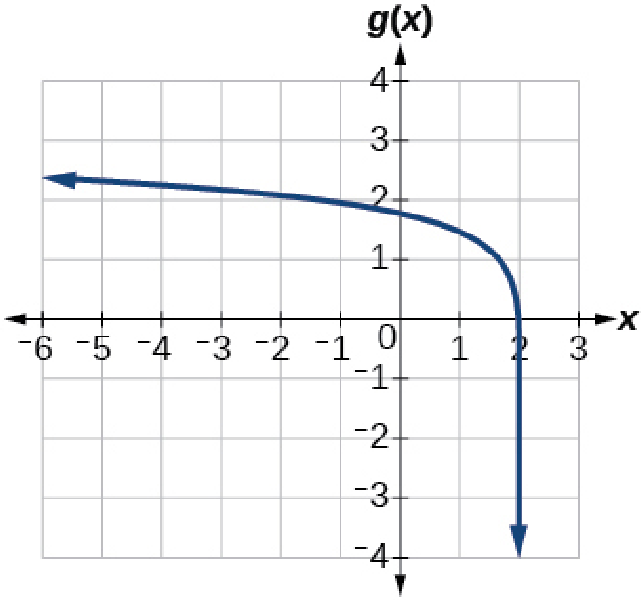
$f(x) = 2\log(x)$

$f(x) = \ln(-x)$



$g(x) = \log(\ 4x + 16\ ) + 4$

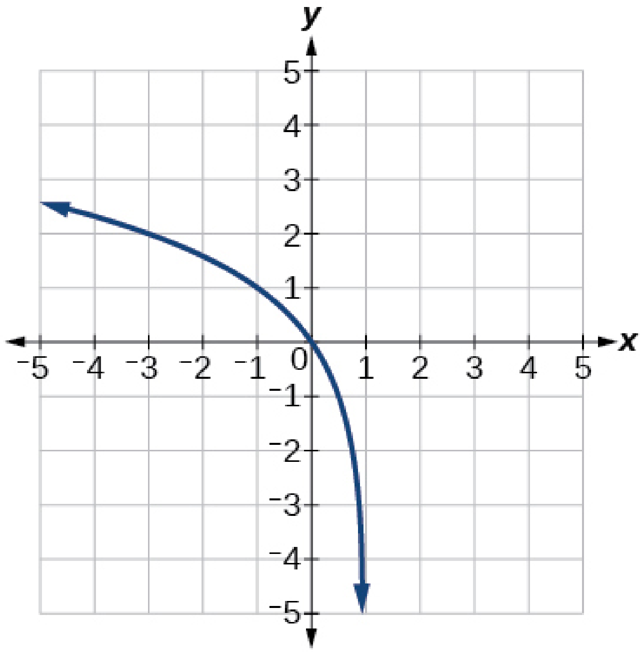
$g(x) = \log(\ 6 - 3x\ ) + 1$



$h(x) = -\frac{1}{2}\ln(x+1) - 3$

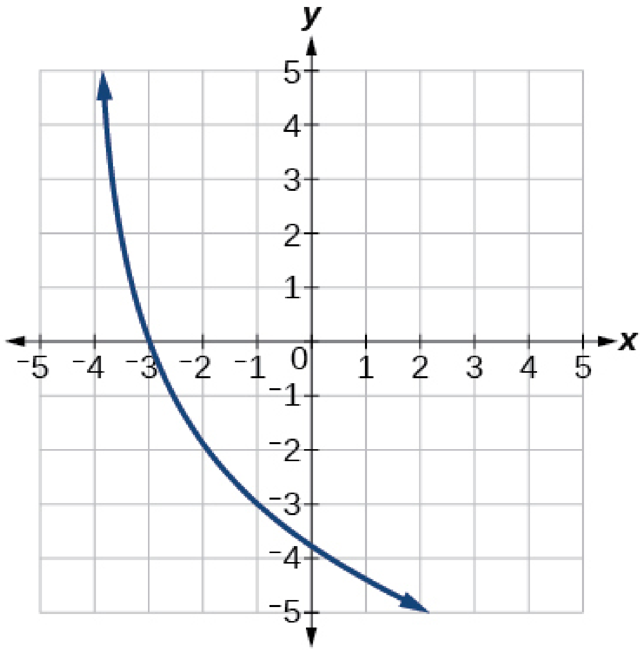
For the following exercises, write a logarithmic equation corresponding to the graph shown.

Use $y = \log_2(x)$ as the parent function.

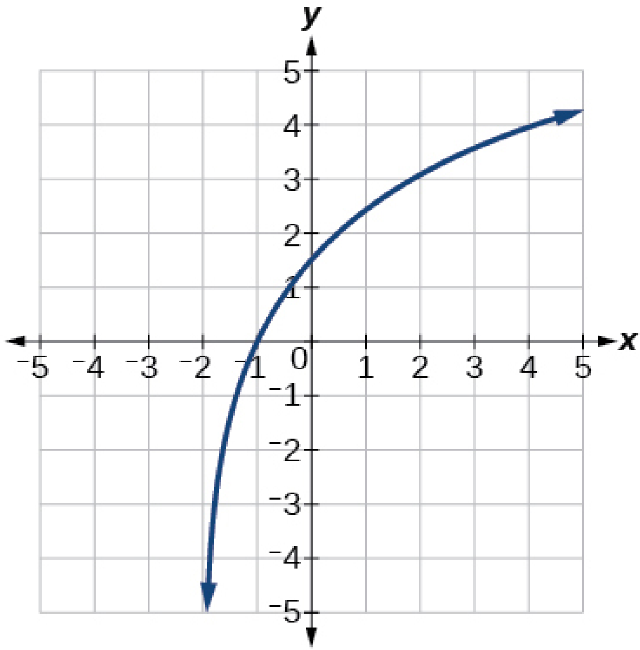


$f(x) = \log_2(-(x-1))$

Use $f(x) = \log_3(x)$ as the parent function.

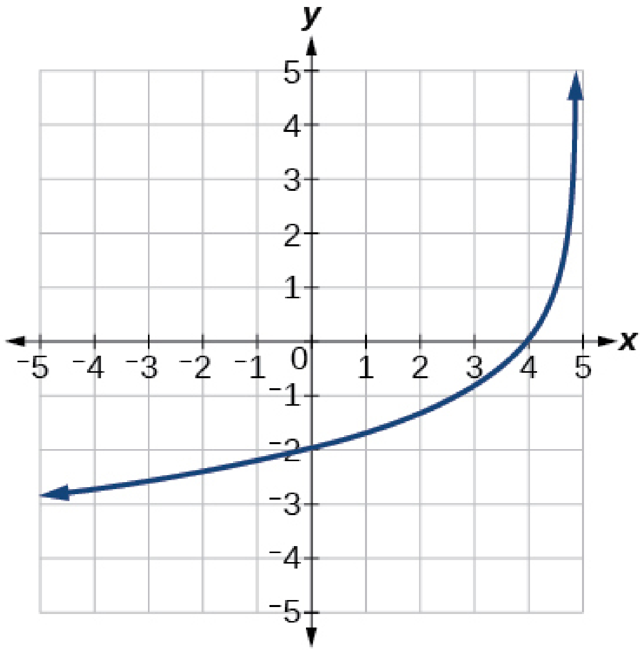


Use $f(x) = \log_4(x)$ as the parent function.



$f(x) = 3\log_4(x+2)$

Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

$$\log(x-1)+2=\ln(x-1)+2$$

$x=2$

$$\log(2x-3)+2=-\log(2x-3)+5$$

$\ln(x-2) = -\ln(x+1)$

$x \approx 2.303$

$2\ln(5x+1) = \frac{1}{2}\ln(-5x) + 1$

$\frac{1}{3}\log(1-x) = \log(x+1) + \frac{1}{3}$

$x \approx -0.472$

Extensions

Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

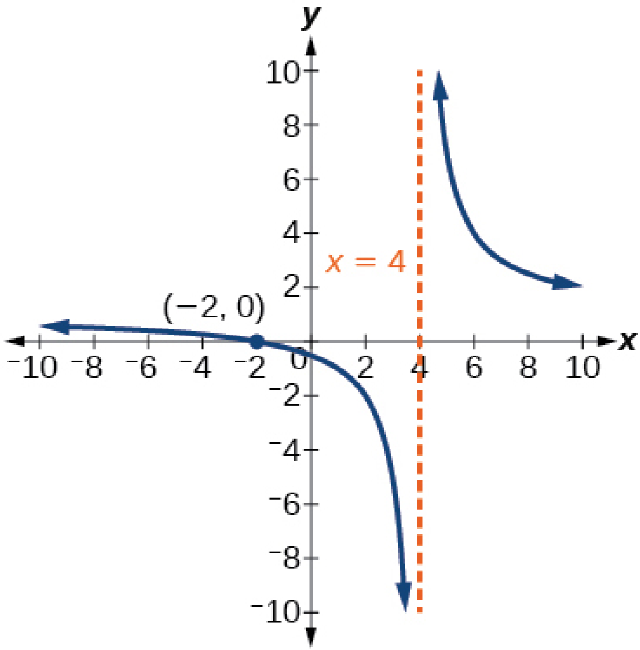
Explore and discuss the graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.

The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same; Conjecture: for any positive base $b \neq 1$, $\log_b(x) = -\log_{\frac{1}{b}}(x)$.

Prove the conjecture made in the previous exercise.

What is the domain of the function $f(x)=\ln(x +2 x-4)$? Discuss the result.

Recall that the argument of a logarithmic function must be positive, so we determine where $x+2 x-4 >0$. From the graph of the function $f(x)= x+2 x-4$, note that the graph lies above the *x*-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$. Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$.



Use properties of exponents to find the *x*-intercepts of the function $f(x)=\log(x^2+4x+4)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

Logarithmic Properties

In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity

or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution

$$\text{pH} = -\log([H^+]) \qquad = \log\left(\frac{1}{[H^+]}\right)$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0 \quad \log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x \qquad b^{\log_b x} = x, x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x:

$$3x = 2x + 5 \quad \text{Set the arguments equal.} \quad x = 5$$

Subtract 2x.

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$\log_b(MN) = \log_b(M) + \log_b(N)$.

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$\log_b(MN) = \log_b(b^m b^n)$
Substitute for M and N. $= \log_b(b^{m+n})$
Apply the product rule for exponents. $= m+n$
Apply the inverse property of logs. $= \log_b(M) + \log_b(N)$ Substitute for m and n.

Note that repeated applications of the product rule

for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

We begin by factoring the argument completely, expressing 30 as a product of primes.

$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$

Next we write the equivalent equation by summing the logarithms of each factor.

$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$

Expand $\log_b(8k)$.

$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms.

Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b}=x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b\neq 1$, we will show
$\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$.

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m=M$ and $b^n=N$. It follows that
$\log_b\left(\frac{M}{N}\right)=\log_b\left(\frac{b^m}{b^n}\right)$
Substitute for M and N. $=\log_b\left(b^{m-n}\right)$
Apply the quotient rule for exponents. $=m-n$
Apply the inverse property of logs. $=\log_b(M)-\log_b(N)$ Substitute for m and n.

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,
$\log\left(\frac{2x^2+6x}{3x+9}\right)=\log\left(\frac{2x(x+3)}{3(x+3)}\right)$
Factor the numerator and denominator.
$=\log\left(\frac{2x}{3}\right)$
Cancel the common factors.

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of

the numerator. Then we apply the product rule.

$$\begin{aligned}\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3)\end{aligned}$$

The Quotient Rule for Logarithms

The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x(x-1)\right) - \log_2\left((3x+4)(2-x)\right)$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\log_b(x^2) = \log_b(x \cdot x) = \log_b x + \log_b x = 2\log_b x$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base of the exponential expression. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base of the exponential expression.

log b (M n) = n log b M

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base of the exponential expression.

Expanding a Logarithm with Powers

Expand log 2 x 5 .

The argument is already written as a power, so we identify the exponent, 5, and the base of the exponential expression, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base of the exponential expression.

$\log_2(x^5)=5\log_2 x$

Expand $\ln x^2$.

$2\ln x$

Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Expressing the argument as a power, we get $\log_3(25)=\log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base of the exponential expression.

$\log_3(5^2)=2\log_3(5)$

Expand ln(1 x 2).

$-2\ln(x)$

Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Because the logarithm of a power is the product of the exponent times the logarithm of the exponential base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the exponential base, and rewrite the product as a logarithm of a power: $4\ln(x) =$ ln(x 4).

Rewrite 2 log 3 4 using the power rule for

logs to a single logarithm with a leading coefficient of 1.

log 3 16

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:
$\log_b\left(\frac{6x}{y}\right)=\log_b(6x)-\log_b y=\log_b 6+\log_b x-\log_b y$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:
$\log_b\left(\frac{A}{C}\right)=\log_b\left(AC^{-1}\right)=\log_b(A)+\log_b\left(C^{-1}\right)=\log_b A+(-1)\log_b C=\log_b A-\log_b C$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a

product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln(x \ 4 \ y \ 7)$ as a sum or difference of logs.

First, because we have a quotient of two expressions, we can use the quotient rule:
$\ln(x \ 4 \ y \ 7) = \ln(x \ 4 \ y) - \ln(7)$

Then seeing the product in the first term, we use the product rule:
$\ln(x \ 4 \ y) - \ln(7) = \ln(x \ 4) + \ln(y) - \ln(7)$

Finally, we use the power rule on the first term:
$\ln(x \ 4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$

Expand $\log(x\ 2\ y\ 3\ z\ 4)$.
$2\log x + 3\log y - 4\log z$

Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression Expand $\log(x)$.
$\log(x) = \log x (1\ 2) = 1\ 2 \log x$

Expand $\ln(x\ 2\ 3)$.
$2\ 3 \ln x$

Properties of Exponential and Logarithmic Functions

Let b, M and N be positive real numbers and m, n be

any real numbers.If a is a positive real number, then the following properties of exponential and logarithmic functions hold.

	Properties of Exponential Functions	Properties of Logarithmic Functions
Sum Rule	$b^{m+n}=b^m b^n$	$\log_b(MN)=\log_b(M)+\log_b(N)$
Difference Rule	$b^{m-n}=\frac{b^m}{b^n}$	$\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$
Product Rule	$b^{mn}=\left(b^m\right)^n=\left(b^n\right)^m$	$\log_b\left(M^n\right)=n\log_b(M)$

Notice that for the Sum, Difference, and Product rules, each property involves an operation inside the function corresponding to an operation outside the function. For example $b^{m+n}=b^m b^n$ says that addition inside an exponential function is equivalent to multiplication outside the function. Whereas $\log_b(MN)=\log_b(M)+\log_b(N)$ says that multiplication inside a logarithmic function is equivalent to addition outside the function. Due to the inverse relationship between exponential and logarithmic functions, which operation is on the

inside and which is on the outside FLIPS when you change from exponential to logarithmic or vice versa.

Can we expand $\ln(x^2 + y^2)$?
No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Expanding Complex Logarithmic Expressions

Expand $\log_6 (64x^3(4x+1)(2x-1))$.

We can expand by applying the Product and Quotient Rules.
$\log_6 (64x^3(4x+1)(2x-1)) = \log_6 64 + \log_6 x^3 + \log_6 (4x+1) - \log_6 (2x-1)$
Apply the Quotient Rule. $= \log_6 2^6 + \log_6 x^3 + \log_6 (4x+1) - \log_6 (2x-1)$
Simplify by writing 64 as 2^6. $= 6\log_6 2 + 3\log_6 x + \log_6 (4x+1) - \log_6 (2x-1)$
Apply the Power Rule.

Expand $\ln\left(\frac{(x-1)(2x+1)^2}{(x^2-9)}\right)$.
$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite

sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Using the product rule
$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$

This reduces our original expression to
$\log_3(40) - \log_3(2)$

Then, using the quotient rule
$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$

Condense $\log 3 - \log 4 + \log 5 - \log 6.$

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$.

We apply the power rule first:
$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(x-1)-\log_2((x+3)^6)$

Next we apply the product rule to the sum:
$\log_2(x^2)+\log_2(x-1)-\log_2((x+3)^6)=\log_2(x^2\,x-1)-\log_2((x+3)^6)$

Finally, we apply the quotient rule to the difference:
$\log_2(x^2\,x-1)-\log_2((x+3)^6)=\log_2\frac{x^2\,x-1}{(x+3)^6}$

Rewrite $\log(5)+0.5\log(x)-\log(7x-1$
$)+3\log(x-1)$ as a single logarithm.

$\log(5\ (x-1)\ 3\ x\ (7x-1)\ )$

Rewriting as a Single Logarithm

Rewrite $2\log x-4\log(x+5)+\ 1\ x\ \log(3x+5)$ as a single logarithm.

We apply the power rule first:
$\log(x+5)\ +\ 1x\ \log(3x+5)\ =\log(\ x\ 2\ )-\log\ ($
$x+5\ )\ 4\ +\log(\ (3x+5)\ x\ -1\ )$

Next we rearrange and apply the product rule to the sum:
$\log(\ x\ 2\ )-\log\ (\ x+5)\ 4\ +\log(\ (3x+5)\ x\ -1\ )$
$=\ \log(\ x\ 2\ )\ +\log(\ (3x+5)\ x\ -1\ )-\log\ (\ x$
$+5)\ 4$
$=\log(x2\ (3x+5)\ x\ -1\ )\ -\log(x+5)4$

Finally, we apply the quotient rule to the difference:
$=\ \log\ (\ x\ 2\ (\ 3\ x\ +\ 5\ )\ x\ -1\ )\ -\ \log\ (\ x\ +\ 5\ )$
$4\ =\ \log\ x\ 2\ (\ 3\ x\ +\ 5\ )\ x\ -1\ (\ x\ +\ 5\ )\ 4$

Condense $4(3\log(x)+\log(x+5)-\log(2x+3))$.

$\log x\,12\,(x+5)\,4\,(2x+3)\,4$; this answer could also be written $\log(x\,3\,(x+5)\,(2x+3))\,4$.

Applying of the Laws of Logs

Recall that, in chemistry, $pH=-\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P=–\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH=-\log(2C)$$

Using the product rule of logs

$$pH=-\log(2C)=-(\log(2)+\log(C))=-\log(2)-\log(C)$$

Since $P=–\log(C)$, the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, we show
$\log_b M = \frac{\log_n M}{\log_n b}$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that
$\log_n(b^y) = \log_n M$ Apply the one-to-one property. $y \log_n b = \log_n M$
Apply the power rule for logarithms. $y = \frac{\log_n M}{\log_n b}$ Isolate y. $\log_b M = \frac{\log_n M}{\log_n b}$
Substitute for y.

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.
$\log_5 36 = \frac{\log(36)}{\log(5)}$
Apply the change of base formula using base 10.
≈ 2.2266
Use a calculator to evaluate to 4 decimal places.

The Change-of-Base Formula
The **change-of-base formula** can be used to evaluate a logarithm with any base.

For any positive real numbers $M, b,$ and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula

    - The numerator of the quotient will be a logarithm with base n and argument M.
    - The denominator of the quotient will be a logarithm with base n and argument b.

Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n=e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$\log_b M = \frac{\ln M}{\ln b} \quad \log_5 3 = \frac{\ln 3}{\ln 5}$

Change $\log_{0.5} 8$ to a quotient of natural logarithms.

$\frac{\ln 8}{\ln 0.5}$

Can we change common logarithms to natural

logarithms?

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$\log_2 10 = \frac{\ln 10}{\ln 2}$

Apply the change of base formula using base e.

≈ 3.3219

Use a calculator to evaluate to 4 decimal places.

Evaluate $\log_5(100)$ using the change-of-base formula.

ln100 ln5 ≈ 4.6051 1.6094 = 2.861

Properties of Exponential and Logarithmic Functions continued

Let a,b and M be positive real numbers. Then the following properties hold.

	Properties of Exponential Functions	Properties of Logarithmic Functions
Change of Base	a m = b lo g b (m)	log b (M) = log a (M) log a (b)

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n \log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b}$ $n>0, n \neq 1, b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See [link].
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See [link].
- We can use the power rule for logarithms to rewrite the log of a power as the product of the

exponent and the log of its base. See [link], [link], and [link].

- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See [link], [link], and [link].
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See [link], [link], [link], and [link].
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See [link].
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See [link].

Section Exercises

Verbal

How does the power rule for logarithms help when solving logarithms with the form $\log_b(x^n)$?

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

$\log_b(7x \cdot 2y)$

$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

$\ln(3ab \cdot 5c)$

$\log_b\left(\frac{13}{17}\right)$

$\log_b(13) - \log_b(17)$

$\log_4 (\quad x z \quad w)$

$\ln(1 \; 4 \; k)$

$-k\ln(4)$

$\log_2 (y \; x)$

For the following exercises, condense to a single logarithm if possible.

$\ln(7) + \ln(x) + \ln(y)$

$\ln(7xy)$

$\log_3 (2) + \log_3 (a) + \log_3 (11) + \log_3 (b)$

$\log_b (28) - \log_b (7)$

$\log_b (4)$

$\ln(a) - \ln(d) - \ln(c)$

$-\log_b\left(\frac{1}{7}\right)$

$\log_b(7)$

$\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

$\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

$15\log(x)+13\log(y)-19\log(z)$

$\ln\left(\frac{a^{-2}}{b^{-4}c^{5}}\right)$

$\log\left(\sqrt{x^{3}y^{-4}}\right)$

$\frac{3}{2}\log(x)-2\log(y)$

$\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

$\log\left(x^{2}y^{3}\sqrt[3]{x^{2}y^{5}}\right)$

$8\ 3\ \log(x)+\ 14\ 3\ \log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

$\log(\ 2\ x\ 4\ )+\log(\ 3\ x\ 5\ )$

$\ln(6\ x\ 9\ )-\ln(3\ x\ 2\ )$

$\ln(2\ x\ 7\ )$

$2\log(x)+3\log(x+1)$

$\log(x)-\ 1\ 2\ \log(y)+3\log(z)$

$\log(\ x\ z\ 3\ y\ )$

$4\ \log\ 7\ (\ c\ )+\ \log\ 7\ (\ a\ )\ 3\ +\ \log\ 7\ (\ b\ )\ 3$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

$\log_7(15)$ to base e

$\log_7(15) = \ln(15)\ \ln(7)$

$\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

$\log_{11}(5)$

$\log_{11}(5) = \log_5(5)\ \log_5(11) = 1\ b$

$\log_6(55)$

$\log_{11}(6\ 11)$

$\log_{11}(6\ 11) = \log_5(6\ 11)\ \log_5(11) =$
$\log_5(6) - \log_5(11)\ \log_5(11) = a - b\ b$
$= a\ b - 1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

$\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

$6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

3

$2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

$\log_3(22)$

2.81359

$\log_8(65)$

$\log_6(5.38)$

0.93913

$\log_4\left(\frac{15}{2}\right)$

$\log_{\frac{1}{2}}(4.7)$

-2.23266

Extensions

Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.

Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

$x=4$; By the quotient rule: $\log_6(x+2)-\log_6(x-3)=\log_6\left(\frac{x+2}{x-3}\right)=1.$

Rewriting as an exponential equation and solving for x:

$6^1=\frac{x+2}{x-3}\quad 0=\frac{x+2}{x-3}-6\quad 0=\frac{x+2}{}$

$x-3 - 6(x-3)(x-3)\ 0 = x+2-6x+18\ x$
$-3\ 0 = x-4\ x-3\ x = 4$

Checking, we find that $\log 6 (4+2)- \log 6 (4-3)= \log 6 (6)- \log 6 (1)$ is defined, so $x=4$.

Can the power property of logarithms be derived from the power property of exponents using the equation $b\ x = m$? If not, explain why. If so, show the derivation.

Prove that $\log b (n)= 1 \log n (b)$ for any positive integers $b>1$ and $n>1$.

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log b (n)= \log n (n) \log n (b) = 1 \log n (b)$.

Does $\log 81 (2401)= \log 3 (7)$? Verify the claim algebraically.

Glossary

change-of-base formula
: a formula for converting a logarithm with any base to a quotient of logarithms with any

other base.

power rule for logarithms
: a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms
: a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms
: a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Exponential and Logarithmic Equations
In this section, you will:

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the “rabbit plague.” (credit: Richard Taylor, Flickr)

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting.

Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b>0$, $b\neq 1$, $b S = b T$ if and only if $S=T$.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal

to one another, and solve for the unknown.

For example, consider the equation $3\ 4x-7 = 3\ 2x$ 3. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$3\ 4x-7 = 3\ 2x\ 3$ $\quad 3\ 4x-7 = 3\ 2x\ 3\ 1$
Rewrite 3 as $3\ 1$. $\quad 3\ 4x-7 = 3\ 2x-1$
Use the division property of exponents. $4x-7 = 2x-1$ Apply the one-to-one property of exponents.
$2x = 6$ Subtract $2x$ and add 7 to both sides. $\quad x = 3$
Divide by 3.

Using the One-to-One Property of Exponential Functions to Solve Exponential Equations

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$b\ S = b\ T$ if and only if $S = T$

Given an exponential equation with the form $b\ S = b\ T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if

necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S=T$, for the unknown.

Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

$2^{x-1} = 2^{2x-4}$ The common base is 2.
$x-1=2x-4$ By the one-to-one property the exponents must be equal.
$x=3$ Solve for x.

Solve $5^{2x} = 5^{3x+2}$.

$x=-2$

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:
$256 = 4^{x-5}$ $2^8 = (2^2)^{x-5}$
Rewrite each side as a power with base 2. $2^8 = 2^{2x-10}$ Use the one-to-one property of exponents.
$8 = 2x - 10$ Apply the one-to-one property of exponents. $18 = 2x$
Add 10 to both sides. $x = 9$ Divide by 2.

Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the

exponents equal.
4. Solve the resulting equation, $S=T$, for the unknown.

Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2}=16^{x+1}$.

$8^{x+2}=16^{x+1}$

$(2^3)^{x+2}=(2^4)^{x+1}$ Write 8 and 16 as powers of 2.

$2^{3x+6}=2^{4x+4}$ To take a power of a power, multiply exponents.

$3x+6=4x+4$ Use the one-to-one property to set the exponents equal.

$x=2$ Solve for x.

Solve $5^{2x}=25^{3x+2}$.

$x=-1$

Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve 2 5x = 2 .

2 5x = 2 1 2
Write the square root of 2 as a power of 2.
5x= 1 2 Use the one-to-one property. x= 1
10 Solve for x.

Solve 5 x = 5 .

x= 1 2

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?
No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

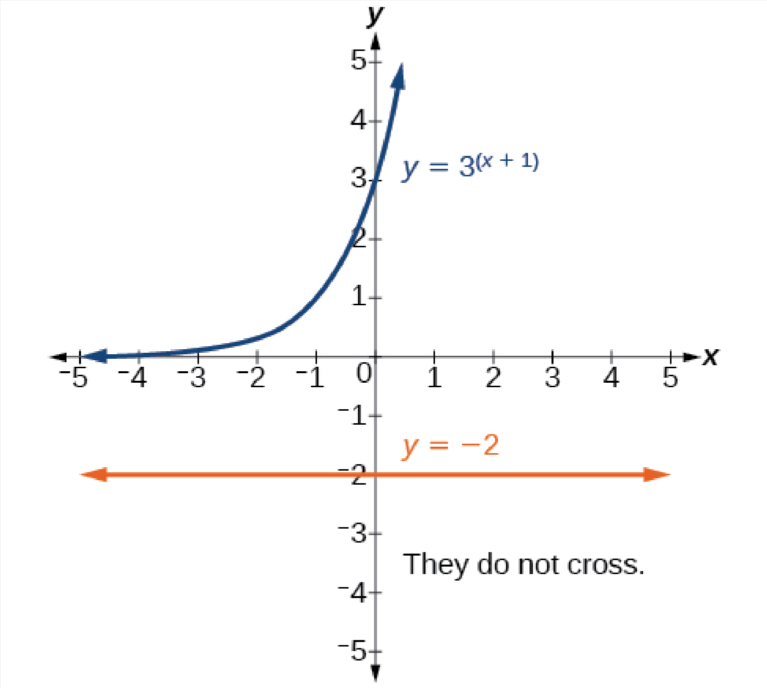
Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

[link] shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.



Solve $2 x = -100$.

The equation has no solution.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

Given an exponential equation in which a common base cannot be found, solve for the unknown.

1. Apply the logarithm of both sides of the equation.
    - If one of the terms in the equation has

base 10, use the common logarithm.
- If none of the terms in the equation has base 10, use the natural logarithm.

4. Use the rules of logarithms to solve for the unknown.

Solving an Equation Containing Powers of Different Bases

Solve 5 x+2 = 4 x .

5 x+2 = 4 x

There is no easy way to get the powers to have the same

ln 5 x+2 =ln 4 x Take ln of both sides.

$(x+2)\ln 5 = x\ln 4$ Use laws of logs.

$x\ln 5 + 2\ln 5 = x\ln 4$ Use the distributive law.

$x\ln 5 - x\ln 4 = -2\ln 5$

Get terms containing x on one side, terms without x on t

$x(\ln 5 - \ln 4) = -2\ln 5$

On the left hand side, factor out an x.

xln(5 4)=ln(1 25) Use the laws of logs.

x= ln(1 25) ln(5 4)

Divide by the coefficient of x.

Solve $2 x = 3 x+1$.

$x= \ln 3 \ln(2 3)$

Is there any way to solve $2 x = 3 x$?
Yes. The solution is 0.

Equations Containing *e*

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

Given an equation of the form $y=A e kt$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Solve algebraically the equation $7 x 2 = 5 1 - x$

Solution

$\ln(7 x 2) = \ln(5 1 - x)$ $x 2 \ln(7) = (1 - x)\ln(5)$
$x 2 \ln(7) = \ln(5) - x\ln(5)$

Solve an Equation of the Form $y = Aekt$

Solve $100 = 20 e 2t$.

$100 = 20 e 2t$ $5 = e 2t$
Divide by the coefficient of the power. $\ln5$
$= 2t$
Take ln of both sides. Use the fact that ln(x) and
$e x$ are inverse functions. $t = \ln5 2$
Divide by the coefficient of t.

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln 5$. If we want a decimal approximation of the answer, we use a calculator.

Solve $3 e 0.5t = 11$.

$t = 2\ln(11 3)$ or $\ln (11 3) 2$

Does every equation of the form $y=Ae^{kt}$ have a solution?

No. There is a solution when $k\neq 0$, and when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is $2=-3e^{t}$.

Solving an Equation That Can Be Simplified to the Form $y=Ae^{kt}$

Solve $4e^{2x}+5=12$.

$4e^{2x}+5=12$ $4e^{2x}=7$
Combine like terms. $e^{2x}=\frac{7}{4}$
Divide by the coefficient of the power.
$2x=\ln\left(\frac{7}{4}\right)$ Take ln of both sides. $x=\frac{1}{2}\ln\left(\frac{7}{4}\right)$ Solve for x.

Solve $3+e^{2t}=7e^{2t}$.

$t=\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln(2)$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^{x} = 56$.

$e^{2x} - e^{x} = 56$ $e^{2x} - e^{x} - 56 = 0$
Get one side of the equation equal to zero. $(e^{x} + 7)(e^{x} - 8) = 0$
Factor by the FOIL method. $e^{x} + 7 = 0$ or $e^{x} - 8 = 0$
If a product is zero, then one factor must be zero. $e^{x} = -7$ or $e^{x} = 8$
Isolate the exponentials. $e^{x} = 8$
Reject the equation in which the power equals a negativ

$x = \ln 8$

Solve the equation in which the power equals a positive

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

Solve $e^{2x} = e^x + 2$.

$x = \ln 2$

Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x-5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$\log_2(2) + \log_2(3x-5) = 3$

$\log_2(2(3x-5)) = 3$ Apply the product rule of logarithms.

$\log_2(6x-10) = 3$ Distribute.

$2^3 = 6x-10$ Apply the definition of a logarithm.

$8 = 6x-10$ Calculate 2^3.

$18 = 6x$ Add 10 to both sides.

$x = 3$ Divide by 6.

Using the Definition of a Logarithm to Solve Logarithmic Equations

For any algebraic expression S and real numbers

b and c, where $b>0$, $b\neq 1$,
$\log_b(S)=c$ if and only if $b^c=S$

Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x+3=7$.

$2\ln x+3=7$ $\quad 2\ln x=4$ Subtract 3.
$\ln x=2$ Divide by 2. $\quad x=e^2$
Rewrite in exponential form.

Solve $6+\ln x=10$.

$x=e^4$

Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x)=7$.

$2\ln(6x)=7 \quad \ln(6x)=\frac{7}{2}$ Divide by 2.
$6x=e^{\left(\frac{7}{2}\right)}$ Use the definition of ln.
$x=\frac{1}{6}e^{\left(\frac{7}{2}\right)}$ Divide by 6.

Solve $2\ln(x+1)=10.$

$x=e^{5}-1$

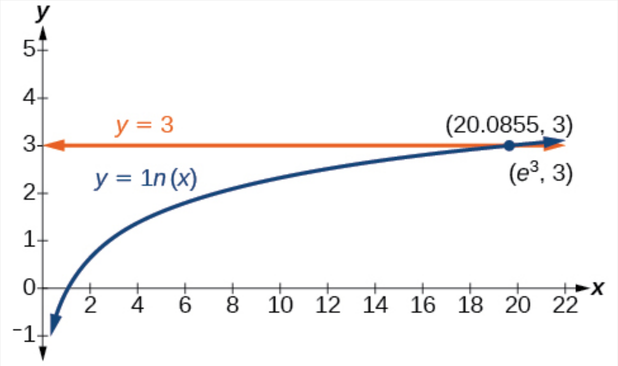
Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x=3.$

$\ln x=3 \quad x=e^{3}$
Use the definition of the natural logarithm.

[link] represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^{3}\approx 20.$ A calculator gives a better approximation: $e^{3}\approx 20.0855.$
The graphs of $y=\ln x$ and $y=3$ cross at the point $(e^{3},3)$, which is approximately

(20.0855, 3).



Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

$x \approx 9.97$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-

to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x>0$, $S>0$, $T>0$ and any positive real number b, where $b\neq 1$,
$\log_b S=\log_b T$ if and only if $S=T$.

For example,
If $\log_2(x-1)=\log_2(8)$, then $x-1=8$.

So, if $x-1=8$, then we can solve for x, and we get $x=9$. To check, we can substitute $x=9$ into the original equation: $\log_2(9-1)=\log_2(8)=3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2)-\log(2)=\log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:
$\log(3x-2)-\log(2)=\log(x+4)$ $\quad$ $\log\left(\frac{3x-2}{2}\right)=\log(x+4)$
Apply the quotient rule of logarithms.

$3x-2\ 2\ =x+4$
Apply the one to one property of a logarithm.
$$3x-2=2x+8$$
Multiply both sides of the equation by 2.
$$x=10$$ Subtract 2x and add 2.

To check the result, substitute $x=10$ into $\log(\ 3x\ -2\ )-\log(\ 2\ )=\log(\ x+4\ )$.
$\log(3(10)-2)-\log(2)=\log((10)+4)$
$\log(28)-\log(2)=\log(14)$ $\log(\ 28\ 2\ )=\log(14)$ The solution checks.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations
For any algebraic expressions S and T and any positive real number b, where $b\neq 1$,
$\log b\ S=\ \log b\ T$ if and only if $S=T$
Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Given an equation containing logarithms, solve it using the one-to-one property.

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log b\ S=\ \log b\ T$.

2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S=T$, for the unknown.

Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2)=\ln(2x+3)$.

$$\ln(x^2)=\ln(2x+3)$$

$x^2=2x+3$ Use the one-to-one property of the logarithm.

$x^2-2x-3=0$ Get zero on one side before factoring.

$(x-3)(x+1)=0$ Factor using FOIL.

$x-3=0$ or $x+1=0$

If a product is zero, one of the factors must be zero.

$x=3$ or $x=-1$ Solve for x.

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Solve $\ln(x^2) = \ln 1$.

$x = 1$ or $x = -1$

Solving logarithmic equation using logarithmic properties

Solve the equation $\log_7(1-2x) = 1 - \log_7(3-x)$. [footnote]

http://www.stitz-zeager.com/ Accessed 5/1/2018

Solution

$\log_7(1-2x) = 1 - \log_7(3-x)$

$\log_7(1-2x) + \log_7(3-x) = 1$ Moving the term $\log_7(3-x)$ to the right

$\log_7((1-2x)(3-x)) = 1$ Using the product property of logarithm

$\log_7(3-6x-x+2x^2) = 1$ Multyplying the factors

$\log_7(2x^2-7x+3) = 1$ Combining like terms and arranging

$7^1 = 2x^2-7x+3$ Using equivalence property of l

$2x^2-7x-4=0$ Get zero on one side

$(2x+1)(x-4)=0$ Factoring

$x=-\frac{1}{2}$ or $x=4$ Solving for x

Now we check the answers.

Substituting in $x=-\frac{1}{2}$ into the original equation

gives us
$\log_7\left(1-2\left(-\frac{1}{2}\right)\right)=1-\log_7\left(3-\left(-\frac{1}{2}\right)\right)$
$\log_7(2)=1-\log_7\left(\frac{7}{2}\right)$ Doing the arithmetic $\log_7(2)=1-(\log_7(7)-\log_7(2))$ Using quotient property of logarithms
$\log_7(2)=1-\log_7(7)+\log_7(2)$ Distributing the negative sign $\log_7(2)=1-1+\log_7(2)$ Using $\log_7(7)=1$ $\log_7(2)=\log_7(2)$
Hence $x=-\frac{1}{2}$ is a solution.

Substituting in $x=4$ into the original equation gives us
$\log_7(1-2(4))=1-\log_7(3-4)$ $\log_7(-7)=1-\log_7(-1)$
Since $\log_7(-7)$ is not defined, $x=4$ is not a solution.

Solving logarithmic equation using logarithmic properties

Solve the equation $1+2\log_4(x+1)=2\log_2 x$.
[footnote]
http://www.stitz-zeager.com/ Accessed 5/1/2018

Solution

$1+2\log_4(x+1)=2\log_2 x$

First, we will convert the expression with base 4 to base 2.

2 log 4 (x+1)=2· log 2 (x+1) log 2 4

Using change of base formula.

=2· log 2 (x+1) log 2 2 2

Writing 4 in exponential form with base 2.

=2· log 2 (x+1) 2 log 2 2

Using the power rule for logarithms.

= 2 · log 2 (x+1) 2 log 2 2

Canceling the common number.

= log 2 (x+1

) Using the property, log of any number to its own bas

Hence our original equation becomes

1+ log 2 (x+1)=2 log 2 (x) 1=2 log 2 (x)−

log 2 (x+1) Gathering the logs on one side.

1= log 2 (x 2)− log 2 (x+1

) Using the power property of logarithms. 1=

log 2 (x 2 x+1

) Using the quotient rule of logarithms

(x 2 x+1

)=2 Rewriting in exponential form

x 2 =2x

+2 Multyping both sides by x+1.

x 2 −2x−2=0 Making one side zero.

x= −(−2)± (−2) 2 −4(1)(−2) 2(1) =

2± 12 2 = 2±2 3 2 =1± 3

Now, we check our answers, first substituting in

x=1+ 3 .

1+2 log 4 (1+1+ 3) ? = 2 log 2 (1+ 3) 1+2(

2+ 3) ? = 2 log 2 (1+ 3)

$2.899968627=2.899968627$

Next, we check when $x=1-\ 3$.
$1+2\log 4(1+1-\ 3)\ ?\ =\ 2\log 2(1-\ 3)$
Since $1-\ 3<0$, then $\log 2(1-\ 3)$ is not defined so $1-\ 3$ is not a solution.

$\log(x)-\log(2)=\log(x+8)-\log(x+2)$.
[footnote]
http://www.stitz-zeager.com/ Accessed 5/1/2018

4

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve

logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. [link] lists the half-life for several of the more common radioactive substances.

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for

radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \quad A(t) = A_0 e^{\ln(0.5)\frac{t}{T}}$$
$$A(t) = A_0 \left(e^{\ln(0.5)}\right)^{\frac{t}{T}} \quad A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

$$y = 1000e^{\frac{\ln(0.5)}{703,800,000}t}$$
$$900 = 1000e^{\frac{\ln(0.5)}{703,800,000}t}$$
After 10% decays, 900 grams are left.
$$0.9 = e^{\frac{\ln(0.5)}{703,800,000}t}$$
Divide by 1000. $\ln(0.9) = \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right)$ Take ln of both sides.
$\ln(0.9) = \frac{\ln(0.5)}{703,800,000}t$ $\ln(e^M) = M$
$$t = 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} \text{ years}$$
Solve for t. $t \approx 106,979,777$ years

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

$t = 703{,}800{,}000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years $\approx$ 226,572,993 years.

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being

equal, set the exponents equal to one another and solve for the unknown. See [link].

- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See [link], [link], and [link].
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See [link].
- We can solve exponential equations with base e, by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See [link] and [link].
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See [link].
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See [link] and [link].
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See [link].

- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See [link].
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See [link].

Section Exercises

Verbal

How can an exponential equation be solved?

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

When does an extraneous solution occur? How can an extraneous solution be recognized?

When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

$4^{-3v-2} = 4^{-v}$

$64 \cdot 4^{3x} = 16$

$x = -\frac{1}{3}$

$3^{2x+1} \cdot 3^{x} = 243$

$2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

$n = -1$

$625 \cdot 5^{3x+3} = 125$

$\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

$b = \frac{6}{5}$

$\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

For the following exercises, use logarithms to solve.

$9^{x-10} = 1$

$x = 10$

$2e^{6x} = 13$

$e\ r+10\ -10=-42$

No solution

$2\cdot\ 10\ 9a\ =29$

$-8\cdot\ 10\ p+7\ -7=-24$

$p=\log(\ 17\ 8\ )-7$

$7\ e\ 3n-5\ +5=-89$

$e\ -3k\ +6=44$

$k=-\ \ln(\ 38\ )\ 3$

$-5\ e\ 9x-8\ -8=-62$

$-6\ e\ 9x+8\ +2=-74$

$x=\ \ln(\ 38\ 3\ )-8\ 9$

$2\ x+1\ =\ 5\ 2x-1$

$e\ 2x\ -\ e\ x\ -132=0$

$x=\ln 12$

$7\ e\ 8x+8\ -5=-95$

$10\ e\ 8x+3\ +2=8$

$x=\ \ln(\ 3\ 5\ )-3\ 8$

$4\ e\ 3x+3\ -7=53$

$8\ e\ -5x-2\ -4=-90$

---

no solution

$3\ 2x+1\ =\ 7\ x-2$

$e\ 2x\ -\ e\ x\ -6=0$

$x=\ln(3)$

$3e^{3-3x}+6=-31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

$\log\left(\frac{1}{100}\right)=-2$

$10^{-2}=\frac{1}{100}$

$\log_{324}(18)=\frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

$5\log_7 n=10$

$n=49$

$-8\log_9 x=16$

$4+\log_2(9k)=2$

$k=\frac{1}{36}$

$2\log(8n+4)+6=10$

$10-4\ln(9-8x)=6$

$x=\frac{9-e}{8}$

For the following exercises, use the one-to-one property of logarithms to solve.

$\ln(10-3x)=\ln(-4x)$

$\log_{13}(5n-2)=\log_{13}(8-5n)$

$n=1$

$\log(x+3)-\log(x)=\log(74)$

$\ln(-3x)=\ln(x^2-6x)$

No solution

$\log_4(6-m) = \log_4 3m$

$\ln(x-2) - \ln(x) = \ln(54)$

No solution

$\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

$\ln(x^2 - 10) + \ln(9) = \ln(10)$

$x = \pm \frac{10}{3}$

For the following exercises, solve each equation for x.

$\log(x+12) = \log(x) + \log(12)$

$\ln(x) + \ln(x-3) = \ln(7x)$

$x = 10$

$\log_2(7x+6) = 3$

$$\ln(7)+\ln(2-4x^2)=\ln(14)$$

$$x=0$$

$$\log_8(x+6)-\log_8(x)=\log_8(58)$$

$$\ln(3)-\ln(3-3x)=\ln(4)$$

$$x=\frac{3}{4}$$

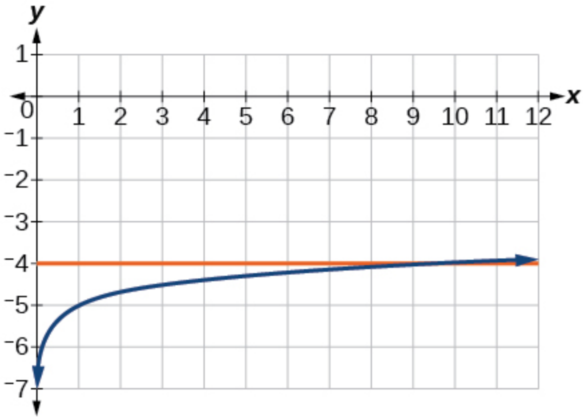
$$\log_3(3x)-\log_3(6)=\log_3(77)$$

Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

$$\log_9(x)-5=-4$$

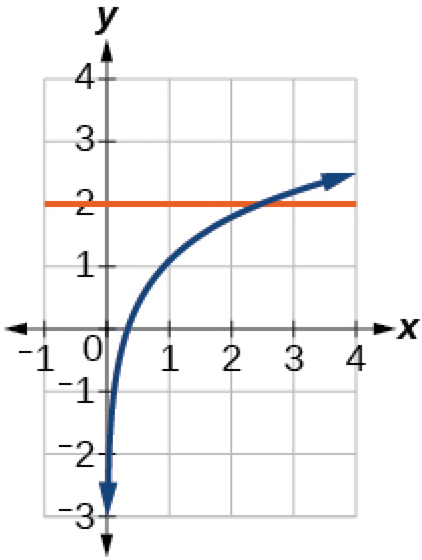
$$x=9$$



$$\log_3(x)+3=2$$

$$\ln(3x)=2$$

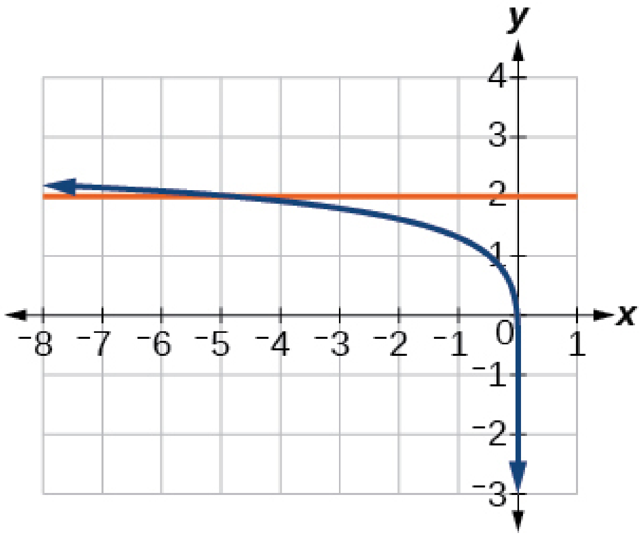
$$x=\frac{e^2}{3}\approx 2.5$$



$\ln(x-5)=1$

$\log(4)+\log(-5x)=2$

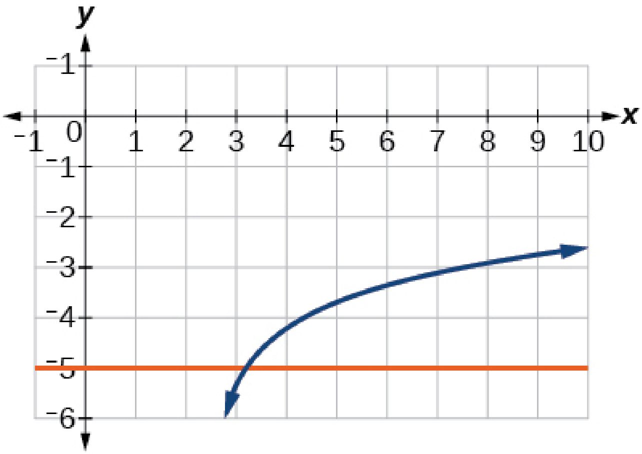
$x=-5$



$-7 + \log 3 (4 - x) = -6$

$\ln(4x - 10) - 6 = -5$

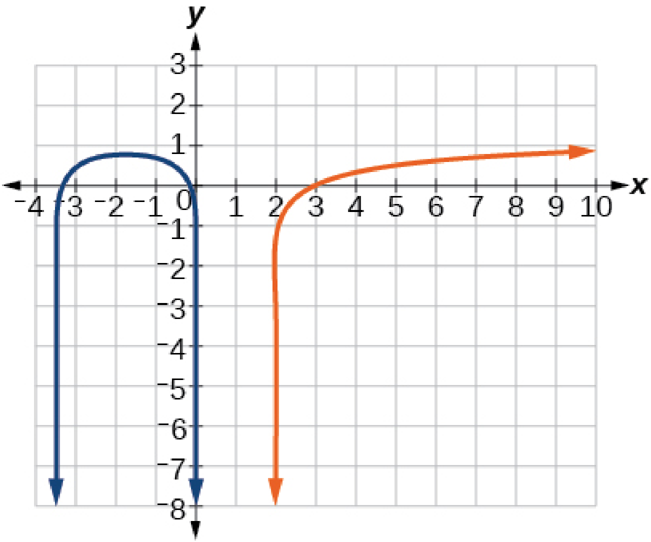
$x = e + 10\ 4 \approx 3.2$



$$\log(4-2x)=\log(-4x)$$

$$\log_{11}(-2x^2-7x)=\log_{11}(x-2)$$

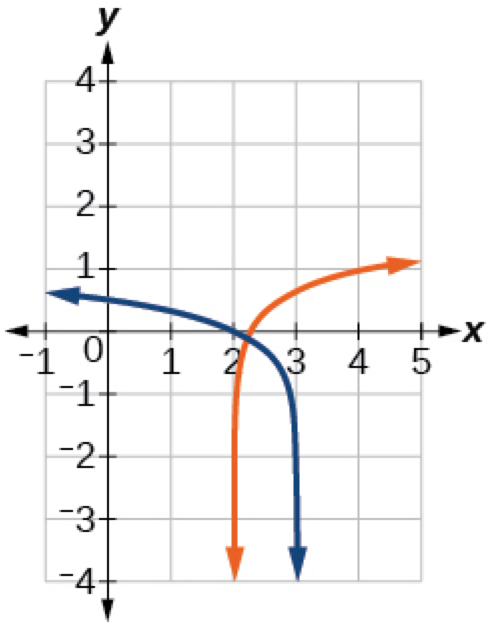
No solution



$$\ln(2x+9)=\ln(-5x)$$

$$\log_9(3-x)=\log_9(4x-8)$$

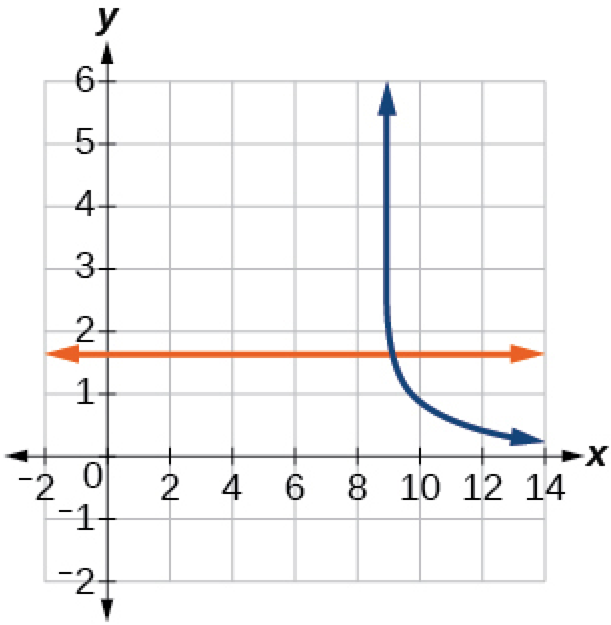
$$x=\frac{11}{5}\approx 2.2$$



$$\log(x^2+13)=\log(7x+3)$$

$$3\log_2(10)-\log(x-9)=\log(44)$$

$$x=\frac{101}{11}\approx 9.2$$



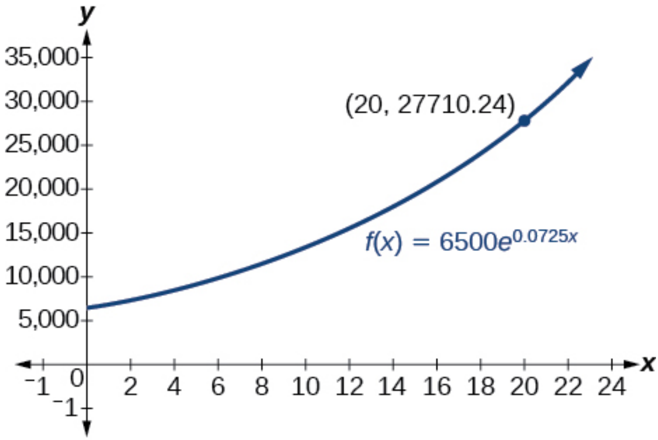
$$\ln(x) - \ln(x+3) = \ln(6)$$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

An account with an initial deposit of $6,500 earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

---

about $27,710.24

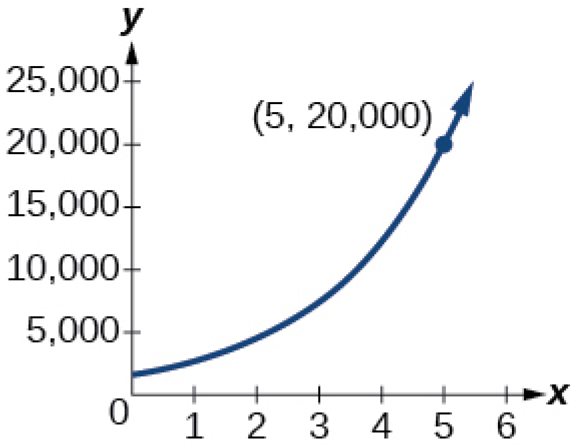


The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3 \cdot 10^{2}$ watts per square meter?

The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

---

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

$1000(1.03)^t = 5000$ using the common log.

$e^{5x} = 17$ using the natural log

$\frac{\ln(17)}{5} \approx 0.567$

$3(1.04)^{3t} = 8$ using the common log

$3^{4x-5}=38$ using the common log

$x=\dfrac{\log(38)+5\log(3)}{4\log(3)}\approx 2.078$

$50e^{-0.12t}=10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

$7e^{3x-5}+7.9=47$

$x\approx 2.2401$

$\ln(3)+\ln(4.4x+6.8)=2$

$\log(-0.7x-9)=1+5\log(5)$

$x\approx -44655.7143$

Atmospheric pressure P in pounds per square inch is represented by the formula $P=14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric

pressure of 8.369 pounds per square inch? (*Hint*: there are 5280 feet in a mile)

The magnitude *M* of an earthquake is represented by the equation $M=\frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0=10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4\cdot 10^{13}$ joules of energy?

---

about 5.83

Extensions

Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x}=x$.

Recall the formula for continually compounding interest, $y=Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

$t=\ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$

Recall the compound interest formula $A=a\left(1+\frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

Newton’s Law of Cooling states that the temperature T of an object at any time *t* can be described by the equation $T=T_s+(T_0-T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

$t=\ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$

Glossary

extraneous solution
: a solution introduced while solving an equation that does not satisfy the conditions of the original equation

Angles

In this section, you will:

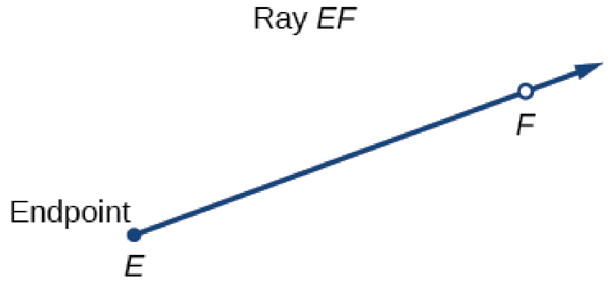
- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.
Angle theta, shown as ∠θQuadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

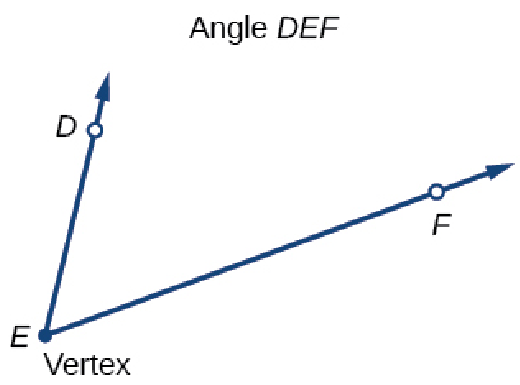
Drawing Angles in Standard Position

Properly defining an angle first requires that we

define a ray. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in [link] can be named as ray EF, or in symbol form $\overrightarrow{EF}$.



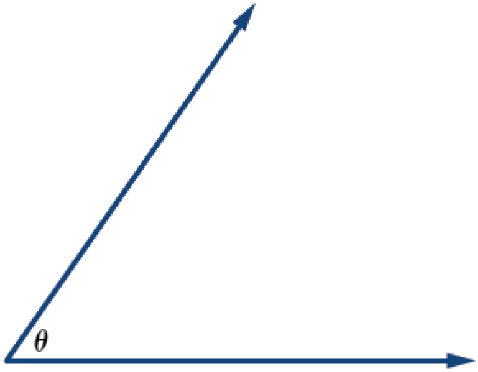
An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in [link] is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.



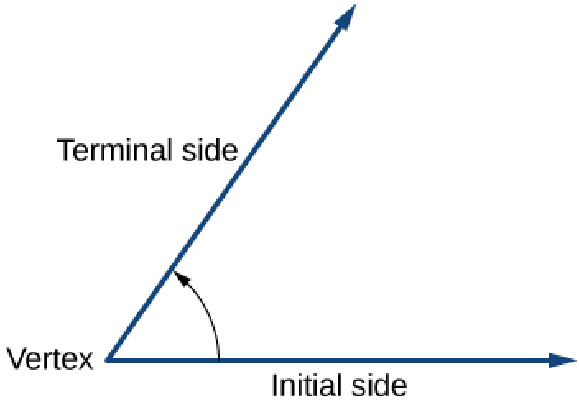
Greek letters are often used as variables for the

measure of an angle. [link] is a list of Greek letters commonly used to represent angles, and a sample angle is shown in [link].

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

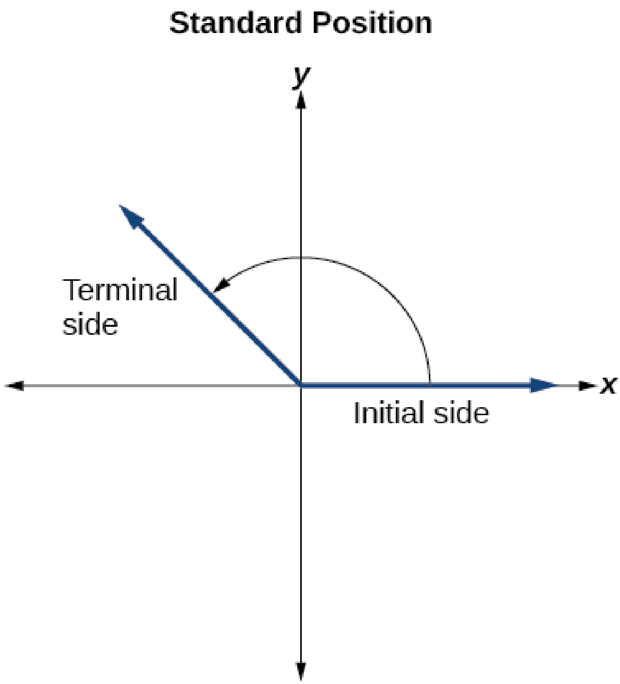


Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in [link].



As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is 1 360 of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol $^\circ$. For example, $90 \text{ degrees} = 90^\circ$.

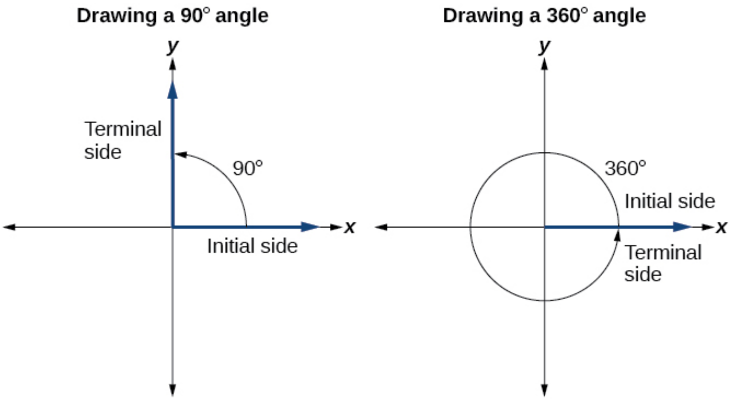
To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See [link].



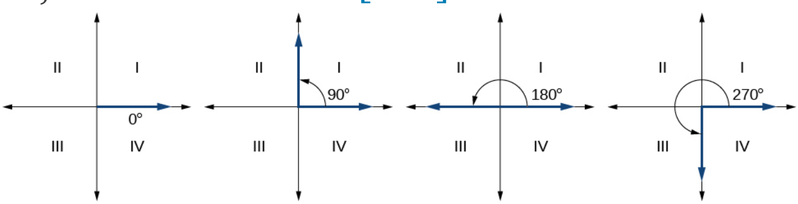
If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving

counterclockwise from the positive x-axis. To draw a 360° angle, we calculate that $360^\circ\ 360^\circ\ = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive x-axis. In this case, the initial side and the terminal side overlap. See [link].



Since we define an angle in standard position by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See [link].



Quadrantal Angles

Quadrantal angles are angels in standard position whose terminal side lies on an axis, including 0°,

90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position.

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Drawing an Angle in Standard Position Measured in Degrees

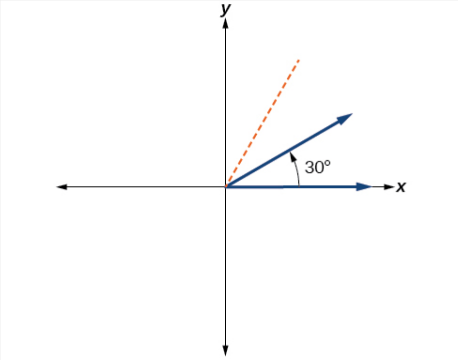
1. Sketch an angle of 30° in standard position.
2. Sketch an angle of $-135°$ in standard position.

1. Divide the angle measure by 360°.
   $30° \; 360° \; = 1 \; 12$

   To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in [link].



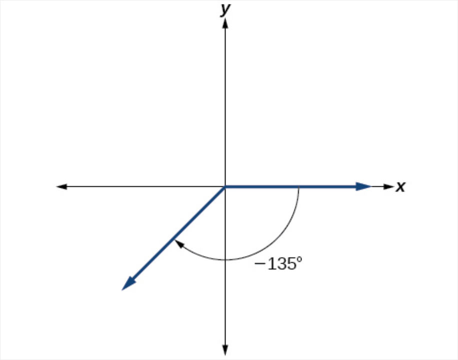
2. Divide the angle measure by 360°.

$$\frac{-135°}{360°} = -\frac{3}{8}$$

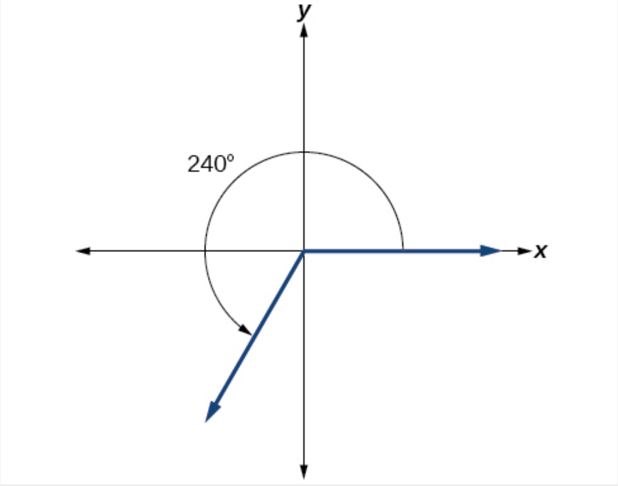
In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in [link].



Show an angle of 240° on a circle in standard position.



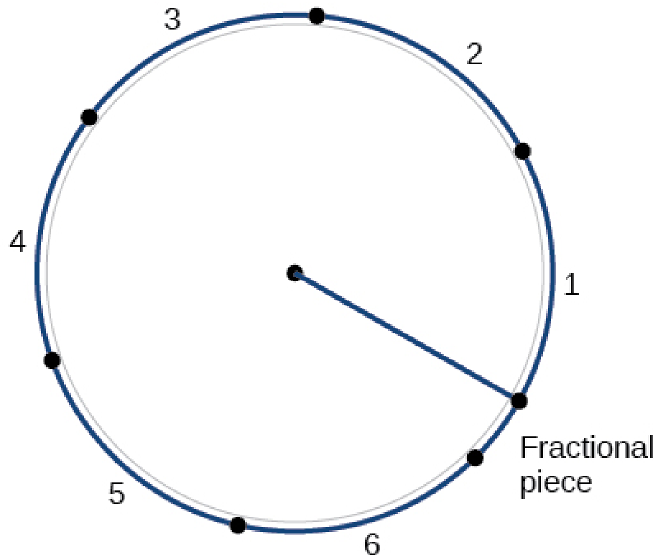
The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle. (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius. Commonly encountered angles measured in degrees Commonly encountered angles measured in radians

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the circumference of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a

string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in [link].

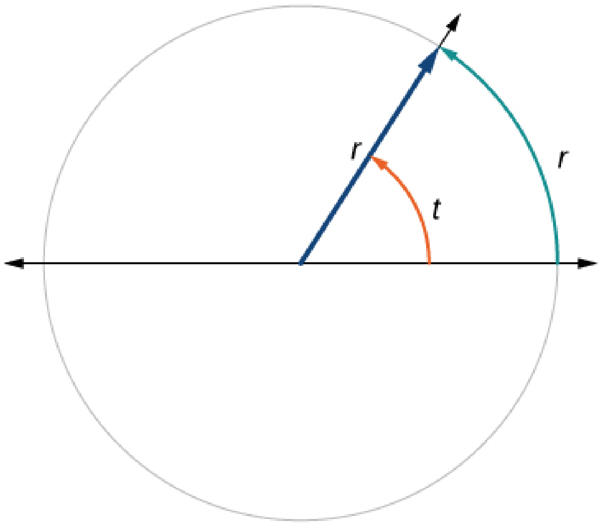


This brings us to our new angle measure. One radian is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians} = 360^\circ \quad \pi \text{ radians} = \frac{360^\circ}{2} = 180^\circ$$
$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$

See [link]. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3

radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.



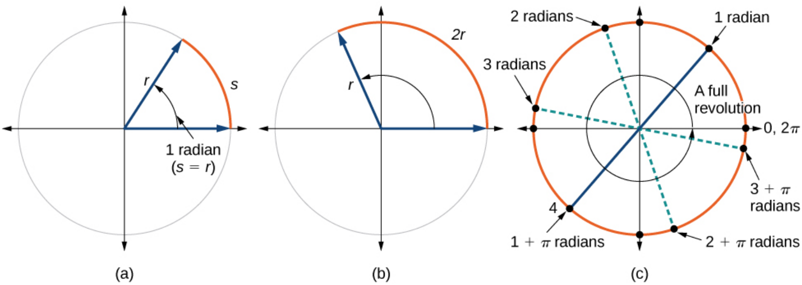
Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

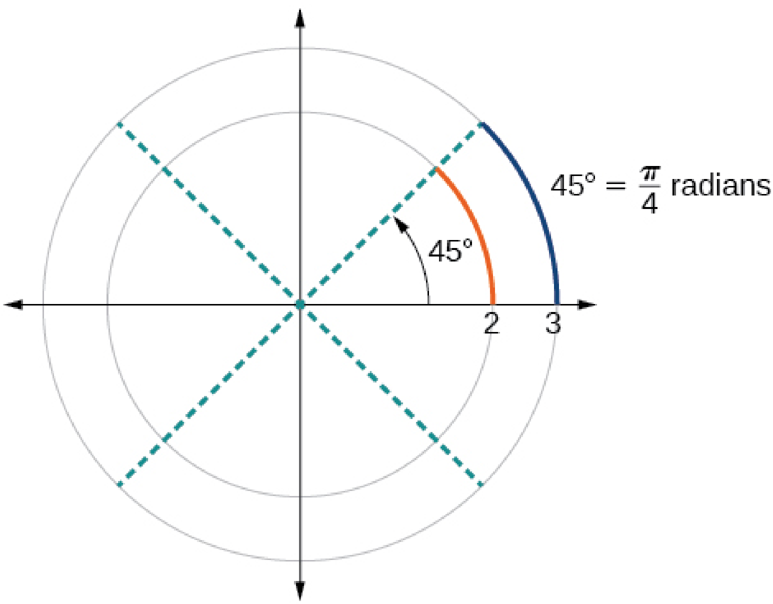
This ratio, called the radian measure, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r. See [link].

$s=r\theta$ $\theta=$ s r

If $s=r$, then $\theta=$ r r $=$ 1 radian.



To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C=2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2)=4\pi$ and the larger has circumference $2\pi(3)=6\pi$. Now we draw a $45°$ angle on the two circles, as in [link].



Notice what happens if we find the ratio of the arc length divided by the radius of the circle.
Smaller circle: 1 2 π 2 = 1 4 π Larger circle: 3 4 π 3 = 1 4 π

Since both ratios are 1 4 π, the angle measures of both circles are the same, even though the arc length and radius differ.

Radians

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures s r radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360°}{2\pi} \approx 57.3°$.

Using Radians

Because radian measure is the ratio of two lengths, it is a unitless measure. For example, in [link], suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

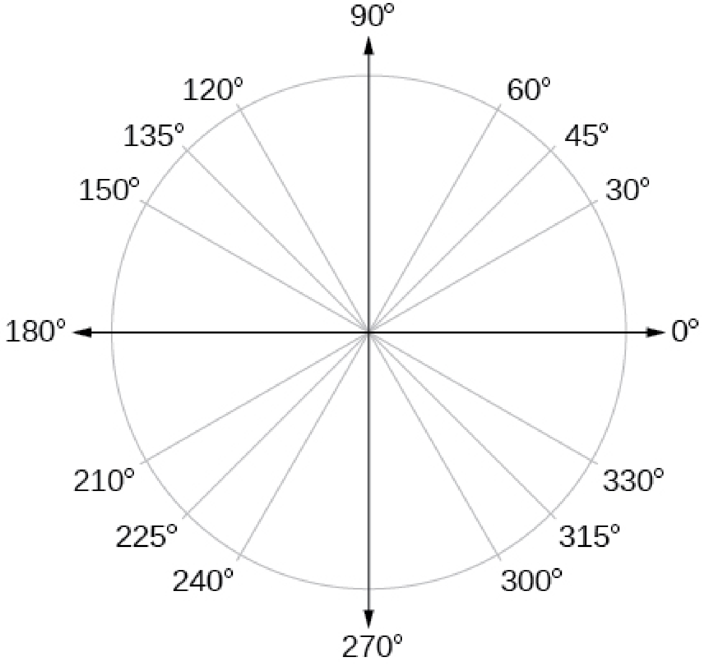
Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C=2\pi r$, and for the unit circle $C=2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$1 \text{ rotation} = 360° = 2\pi \text{ radians} \quad \frac{1}{2} \text{ rotation} = 180° = \pi \text{ radians} \quad \frac{1}{4} \text{ rotation} = 90° = \frac{\pi}{2} \text{ radians}$$

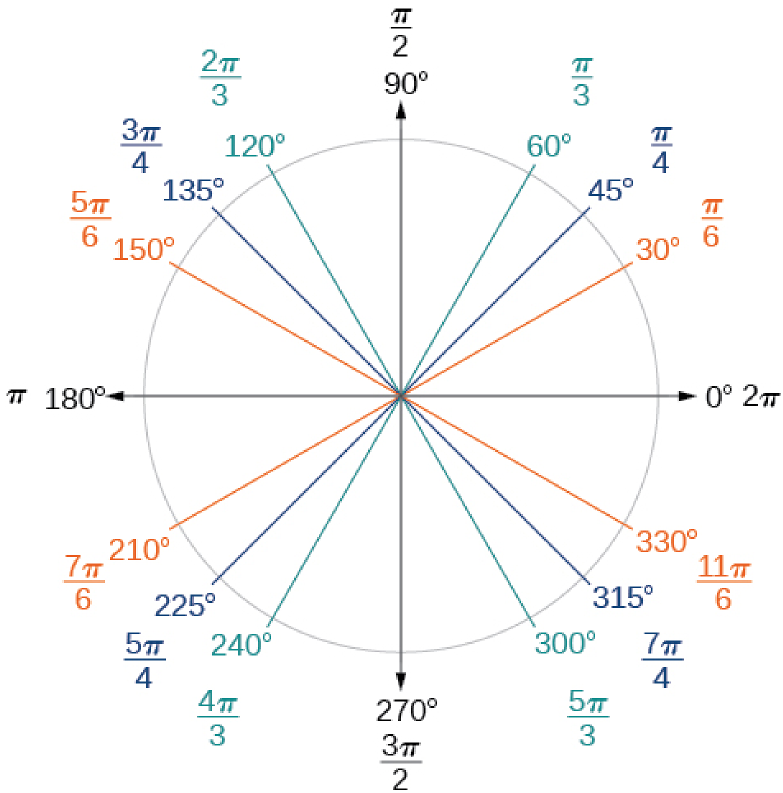
Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees

and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in [link]. Memorizing these angles will be very useful as we study the properties associated with angles.



Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in [link], which are shown in [link]. Be sure you can verify each of these measures.



Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$1 \text{ rotation} = 2\pi r$

So,

$s = \frac{1}{3}(2\pi r) = \frac{2\pi r}{3}$

The radian measure would be the arc length divided by the radius.

$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} = \frac{2\pi r}{3r} = \frac{2\pi}{3}$

Find the radian measure of three-fourths of a full rotation.

$\frac{3\pi}{2}$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$\frac{\theta}{180} = \frac{\theta^R}{\pi}$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

Degrees 180 = Radians π

Converting between Radians and Degrees
To convert between degrees and radians, use the proportion
θ 180 = θ R π

Converting Radians to Degrees

Convert each radian measure to degrees.

1. π 6
2. 3

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.
   θ 180 = θ R π θ 180 = π 6 π θ = 180 6 θ = 30 ∘
2. We use the proportion, substituting the given information.
   θ 180 = θ R π θ 180 = 3 π θ = 3(180) π θ ≈ 172 ∘

Convert $-\frac{3\pi}{4}$ radians to degrees.

-135°

Converting Degrees to Radians

Convert 15 degrees to radians.

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \frac{15}{180} = \frac{\theta^R}{\pi} \quad \frac{15\pi}{180} = \theta^R \quad \frac{\pi}{12} = \theta^R$$

Analysis

Another way to think about this problem is by remembering that $30^{\circ} = \frac{\pi}{6}$. Because $15^{\circ} = \frac{1}{2}(30^{\circ})$, we can find that $\frac{1}{2}\left(\frac{\pi}{6}\right)$ is $\frac{\pi}{12}$.

Convert 126° to radians.
7π 10

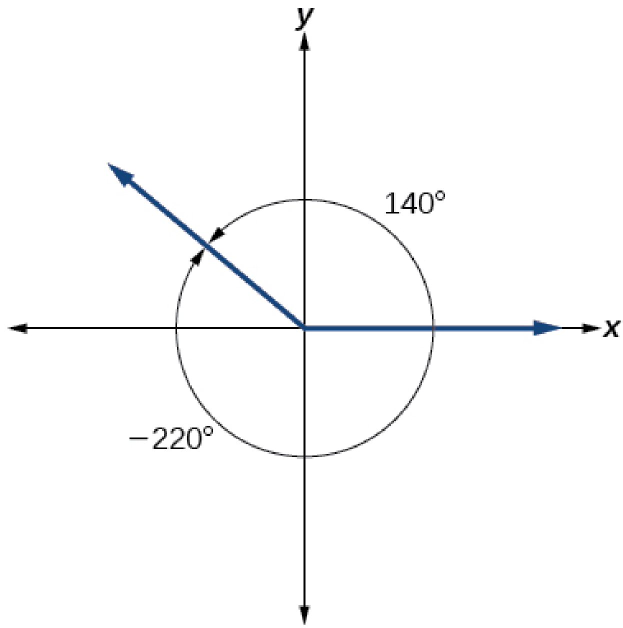
An angle of 140° and an angle of –220° are coterminal angles.

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at [link]. The angle of 140° is a positive angle, measured counterclockwise. The angle of –220° is a negative angle, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are coterminal angles.

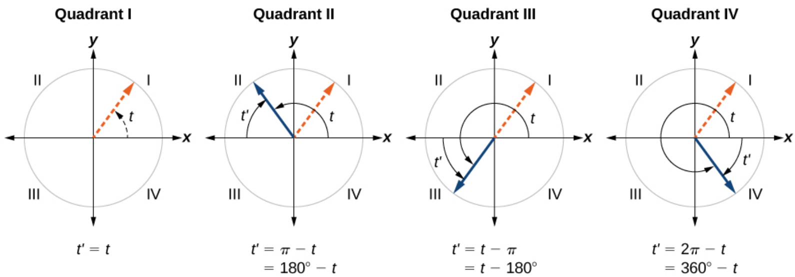
Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.



Any angle has infinitely many coterminal angles because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is -260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the

terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See [link] for examples of reference angles for angles in different quadrants.



Coterminal and Reference Angles

Coterminal angles are two angles in standard position that have the same terminal side.

An angle's **reference angle** is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°.

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.

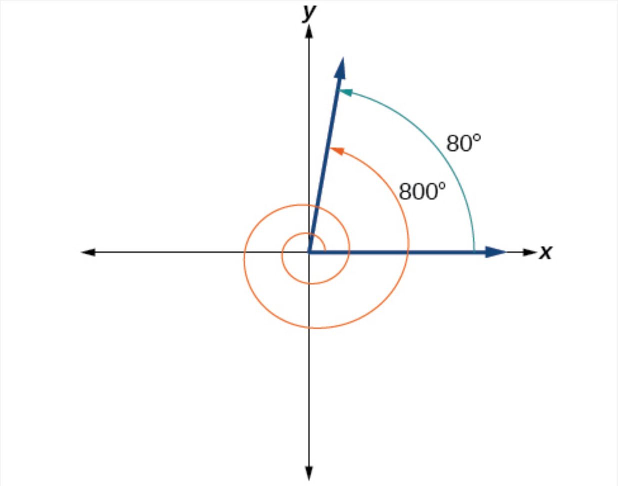
3. The resulting angle is coterminal with the original angle.

Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

An angle with measure 800° is coterminal with an angle with measure $800 - 360 = 440^\circ$, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: $440 - 360 = 80^\circ$.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in [link].



Find an angle α that is coterminal with an angle measuring $870°$, where $0° \le \alpha < 360°$.

$\alpha = 150°$

Given an angle with measure less than $0°$, find a coterminal angle having a measure between $0°$ and $360°$.

1. Add $360°$ to the given angle.
2. If the result is still less than $0°$, add $360°$ again until the result is between $0°$ and $360°$.

3. The resulting angle is coterminal with the original angle.

Finding an Angle Coterminal with an Angle Measuring Less Than 0°

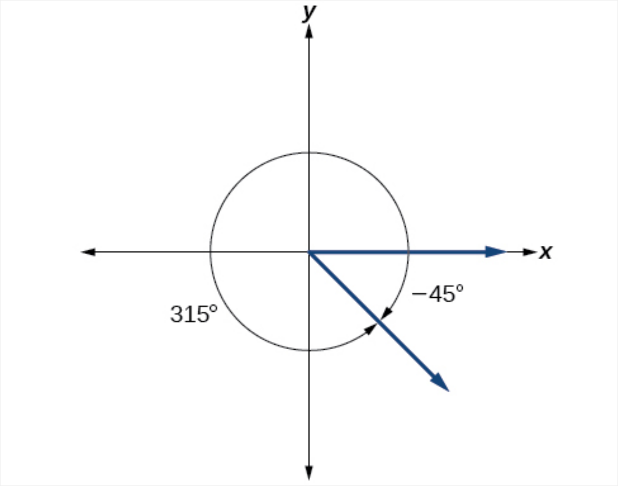
Show the angle with measure -45° on a circle and find a positive coterminal angle α such that $0^\circ \leq \alpha < 360^\circ$.

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ$$

We can then show the angle on a circle, as in [link].



Find an angle β that is coterminal with an angle measuring -300° such that $0^\circ \leq \beta < 360^\circ$.

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find coterminal angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

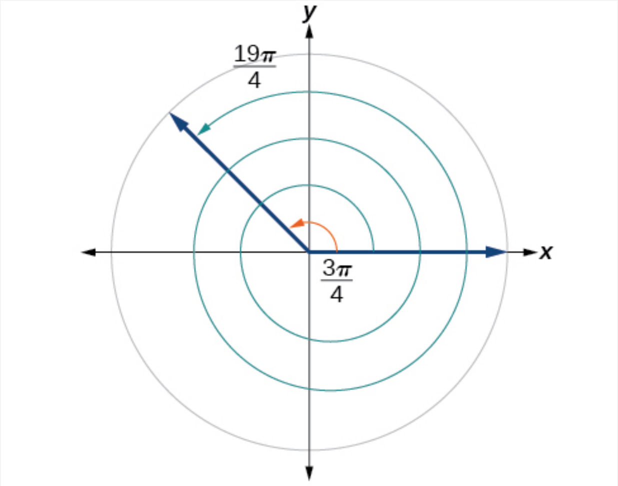
When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} = \frac{11\pi}{4}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in [link].



Find an angle of measure θ that is coterminal with an angle of measure − 17π 6 where 0 ≤ θ < 2π.

7π 6

Determining the Length of an Arc

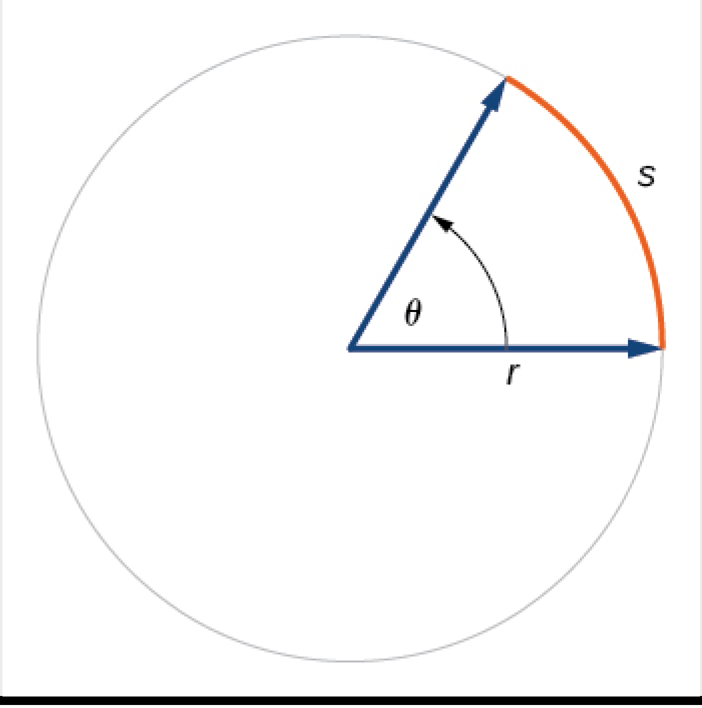
Recall that the radian measure θ of an angle was

defined as the ratio of the arc length s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

Arc Length on a Circle

In a circle of radius *r*, the length of an arc s subtended by an angle with measure θ in radians, shown in [link], is

$$s = r\theta$$



Given a circle of radius r**, calculate the length** s **of the arc subtended by a given angle of measure** θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ: $s = r\theta$.

Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

---

1. Let's begin by finding the circumference of Mercury's orbit.
   $C = 2\pi r \ = 2\pi(36 \text{ million miles})$
   $\approx 226 \text{ million miles}$

   Since Mercury completes 0.0114 of its total revolution in one Earth day, we can

now find the distance traveled:
(0.0114
)226 million miles = 2.58 million miles

2. Now, we convert to radians:
radian= arclength radius =
2.58 million miles 36 million miles
=0.0717

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

215π 18 =37.525 units

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a sector of a circle. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found

using the formula $A=\pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector}=\left(\frac{\theta}{2\pi}\right)\pi r^2=\frac{\theta\pi r^2}{2\pi}=\frac{1}{2}\theta r^2$$

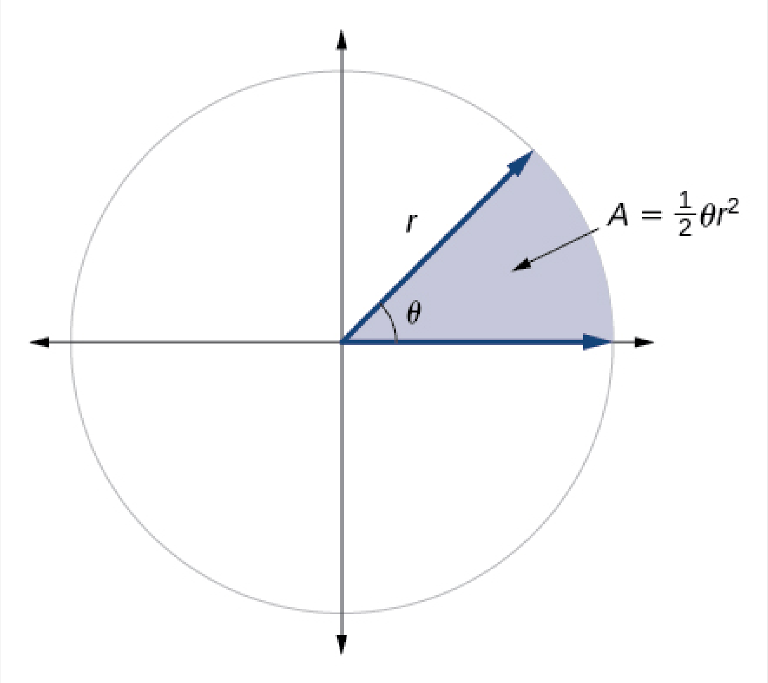
Area of a Sector

The **area of a sector** of a circle with radius r subtended by an angle θ, measured in radians, is

$$A=\frac{1}{2}\theta r^2$$

See [link].

The area of the sector equals half the square of the radius times the central angle measured in radians.



Given a circle of radius r, find the area of a sector defined by a given angle θ.

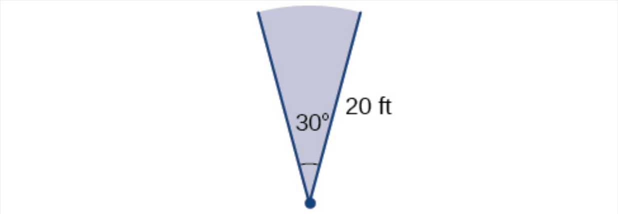
1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r: $A = \frac{1}{2}\theta r^2$.

Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance

of 20 feet while rotating 30 degrees, as shown in [link]. What is the area of the sector of grass the sprinkler waters?

The sprinkler sprays 20 ft within an arc of 30°.



First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30\text{ degrees} = 30 \cdot \frac{\pi}{180} = \frac{\pi}{6}\text{ radians}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \approx 104.72$$

So the area is about $104.72\ \text{ft}^2$.

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to

cover? Write the answer in radian measure to two decimal places.

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$v = s\ t$

Angular speed results from circular motion and can be determined by the angle through which a point

rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta = r\omega t$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} = \frac{r\omega t}{t} = r\omega$$

Angular and Linear Speed

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t}$$

The **linear speed**, v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$v = \frac{s}{t}$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$v = r\omega$

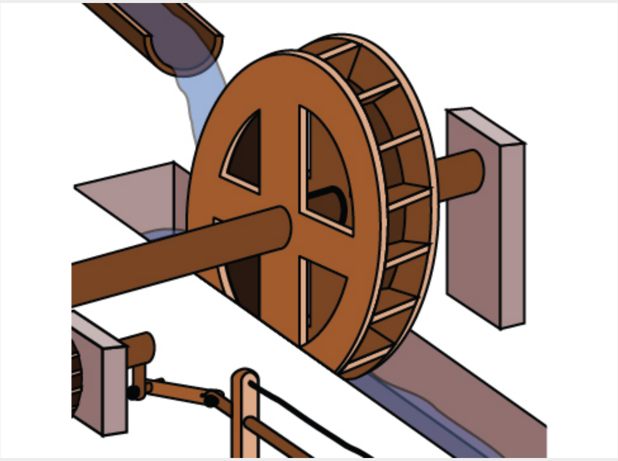
This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed.

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Finding Angular Speed

A water wheel, shown in [link], completes 1 rotation every 5 seconds. Find the angular speed in radians per second.



The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed.

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Here, we have an angular speed and need to

find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180 \text{ rotations minute} \cdot 2\pi\text{radians rotation} = 360\pi \text{ radians minute}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14\text{inches})(\ 360\pi \text{ radians minute}\ ) = 5040\pi \text{ inches minute}$$

Remember that radians are a unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \text{ inches minute} \cdot 1\ \text{ feet } 12\ \text{ inches} \cdot 1 \text{ mile } 5280\ \text{ feet} \cdot 60\ \text{ minutes } 1 \text{ hour} \approx 14.99\text{miles per hour (mph)}$$

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

1655 kilometers per hour

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See [link].
- In addition to degrees, the measure of an angle can be described in radians. See [link].
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See [link] and [link].
- Two angles that have the same terminal side

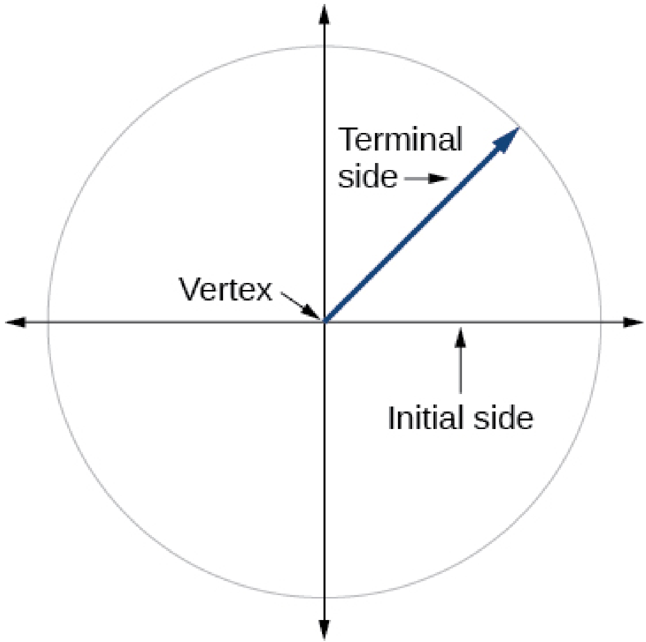
are called coterminal angles.

- We can find coterminal angles by adding or subtracting $360°$ or 2π. See [link] and [link].
- Coterminal angles can be found using radians just as they are for degrees. See [link].
- The length of a circular arc is a fraction of the circumference of the entire circle. See [link].
- The area of sector is a fraction of the area of the entire circle. See [link].
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See [link].
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See [link].

Section Exercises

Verbal

Draw an angle in standard position. Label the vertex, initial side, and terminal side.



Explain why there are an infinite number of angles that are coterminal to a certain angle.

State what a positive or negative angle signifies, and explain how to draw each.

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

Explain the differences between linear speed and angular speed when describing motion along a circular path.

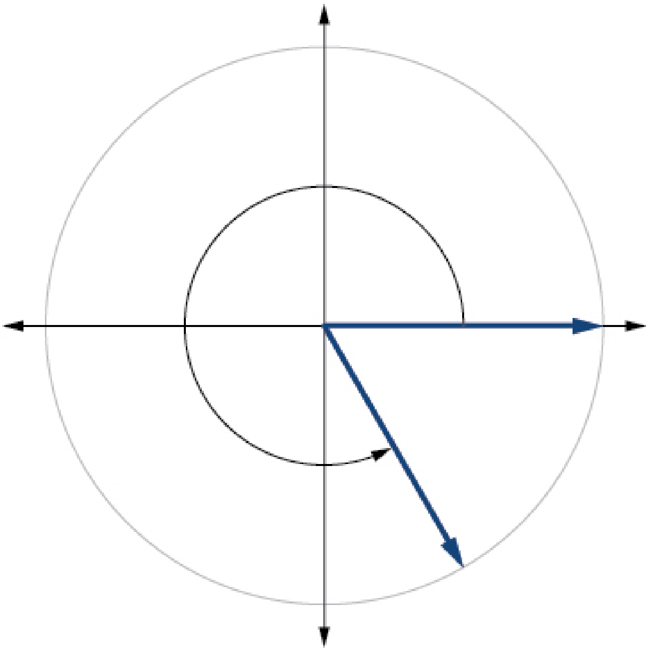
Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

Graphical

For the following exercises, draw an angle in standard position with the given measure.

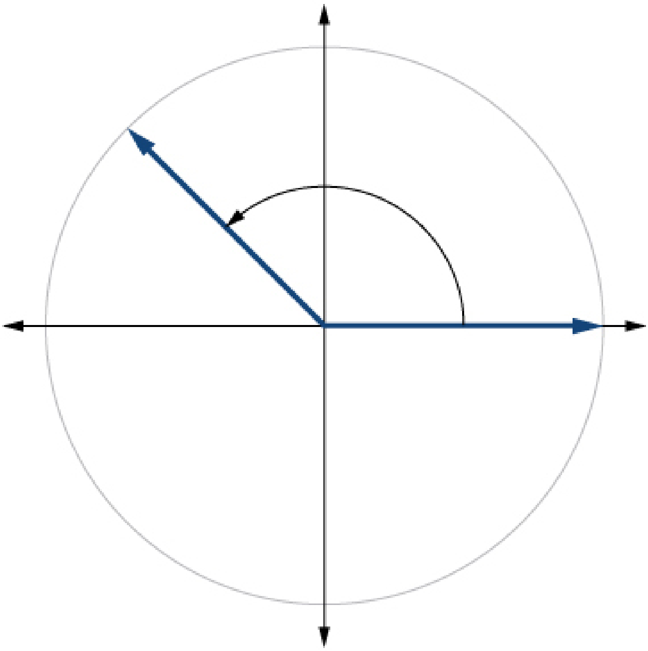
30°

300°



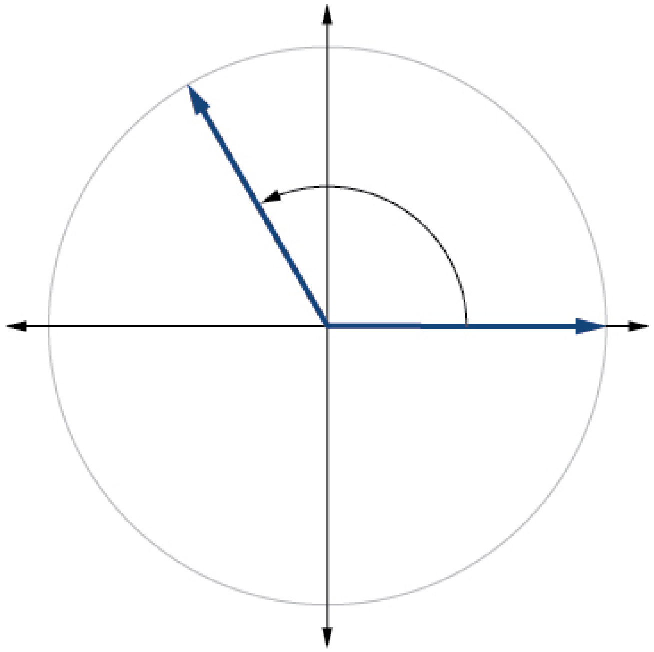
-80°

135°



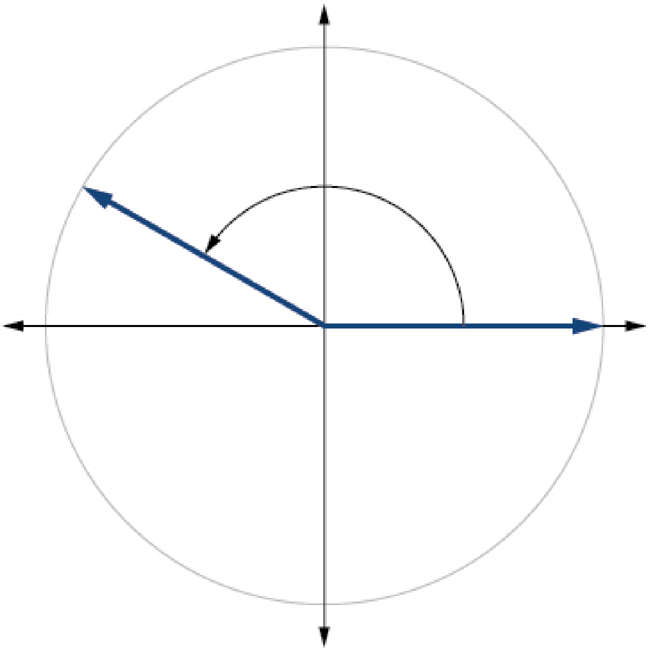
$-150°$

$2π 3$



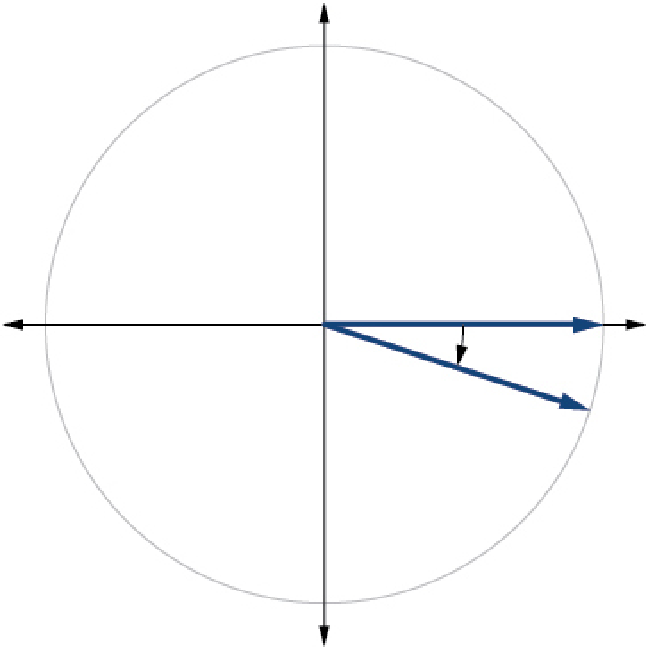
7π 4

5π 6



$\frac{\pi}{2}$

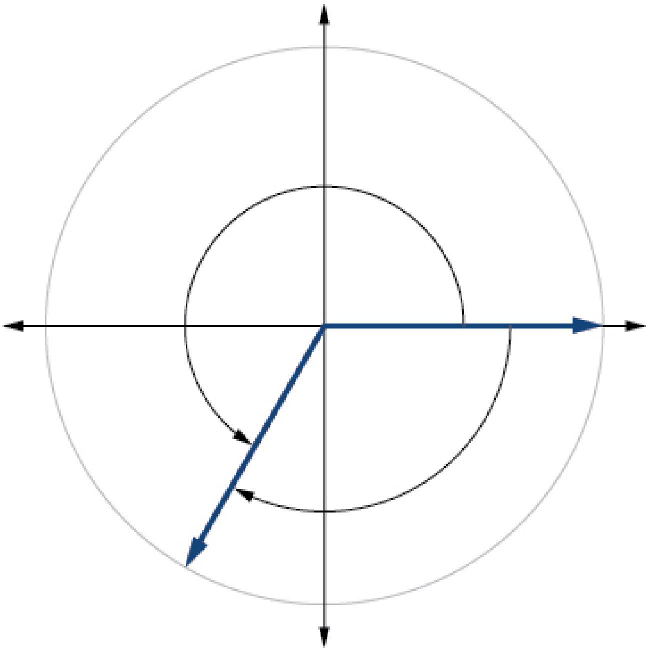
$-\frac{\pi}{10}$



$415°$

$-120°$

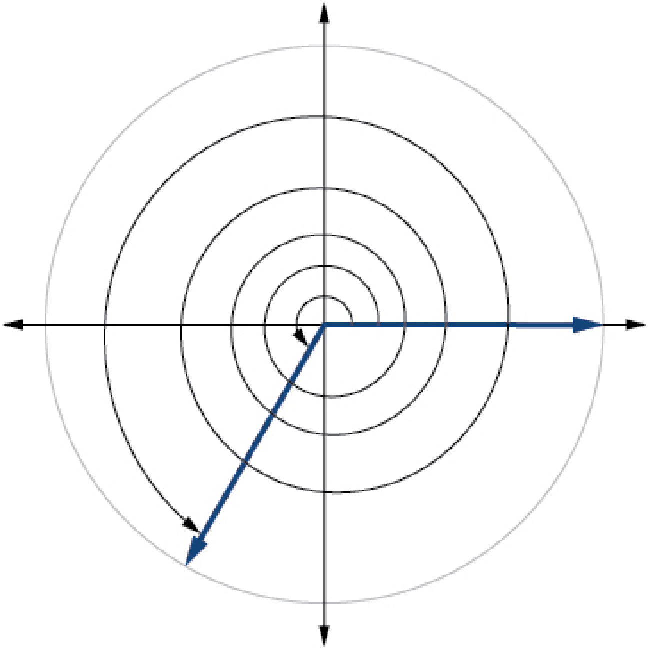
$240°$



-315°

$22\pi \ 3$

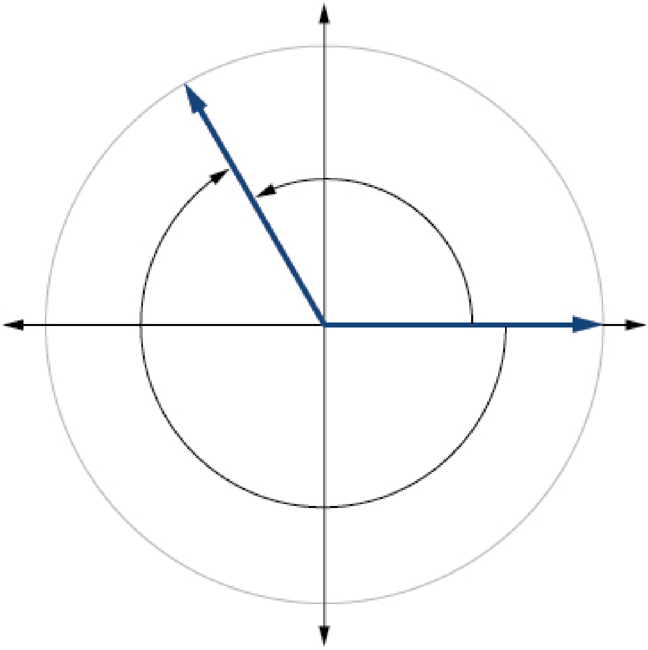
$4\pi \ 3$



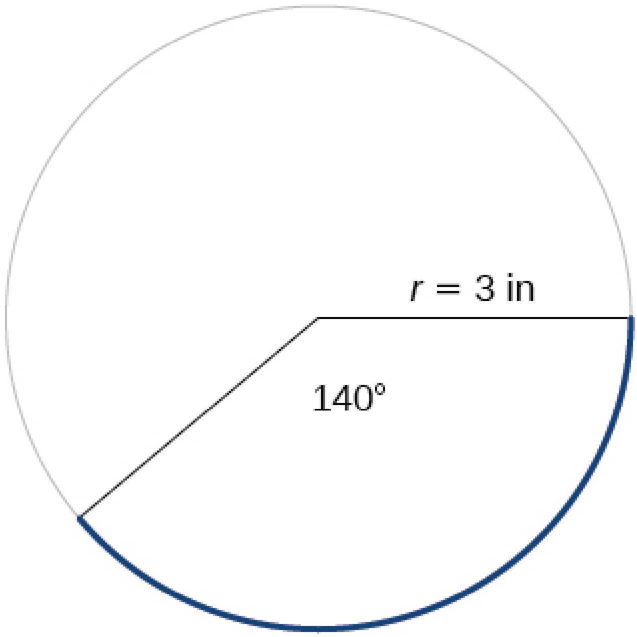
− π 6

− 4π 3

2π 3



For the following exercises, refer to [link]. Round to two decimal places.

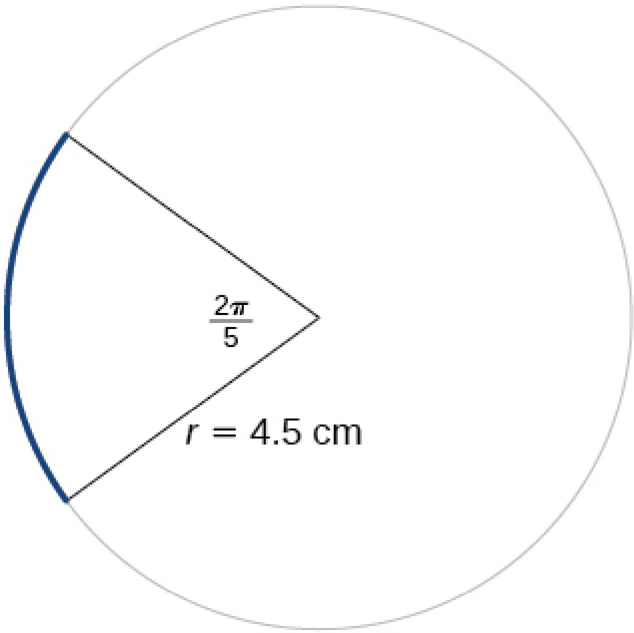


Find the arc length.

Find the area of the sector.

$\frac{7\pi}{2} \approx 11.00 \text{ in}^2$

For the following exercises, refer to [link]. Round to two decimal places.



Find the arc length.

Find the area of the sector.

$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For the following exercises, convert angles in radians to degrees.

3π 4 radians

π 9 radians

20°

$-$ 5π 4 radians

π 3 radians

60°

$-$ 7π 3 radians

$-$ 5π 12 radians

-75°

11π 6 radians

For the following exercises, convert angles in degrees to radians.

90°

π 2 radians

100°

-540°

-3π radians

-120°

180°

π radians

-315°

150°

5π 6 radians

For the following exercises, use to given information

to find the length of a circular arc. Round to two decimal places.

Find the length of the arc of a circle of radius 12 inches subtended by a central angle of π 4 radians.

Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of π 3 .

5.02π 3 ≈ 5.26 miles

Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of 5π 6 .

Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°.

25π 9 ≈ 8.73 centimeters

Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220°.

Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is 63°.

$21\pi\ 10\ \approx 6.60$ meters

For the following exercises, use the given information to find the area of the sector. Round to four decimal places.

A sector of a circle has a central angle of 45° and a radius 6 cm.

A sector of a circle has a central angle of 30° and a radius of 20 cm.

104.7198 cm2

A sector of a circle with diameter 10 feet and an angle of π 2 radians.

A sector of a circle with radius of 0.7 inches and an angle of π radians.

0.7697 in2

For the following exercises, find the angle between 0° and 360° that is coterminal to the given angle.

-40°

-110°

250°

700°

1400°

320°

For the following exercises, find the angle between 0 and 2π in radians that is coterminal to the given angle.

$-\frac{\pi}{9}$

$\frac{10\pi}{3}$

$\frac{4\pi}{3}$

13π 6

44π 9

8π 9

Real-World Applications

A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

1320 rad 210.085 RPM

A wheel of radius 8 inches is rotating 15°/s. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

A wheel of radius 14 inches is rotating 0.5 rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

---

7 in./s, 4.77 RPM, 28.65 deg/s

A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

When being burned in a writable CD-R drive, the angular speed of a CD varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4,800 RPM (revolutions per minute. Find the linear speed if the CD has diameter of 120millimeters.

1,809,557.37 mm/min = 30.16 m/s

A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes (1 minute $= \frac{1}{60}$ degree) . The radius of Earth is 3960 miles.

5.76 miles

Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes (1 minute $= \frac{1}{60}$ degree). The radius of Earth is 3960 miles.

Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

120°

Extensions

Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

---

794 miles per hour

A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

2,234 miles per hour

A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is travelling down the road.

A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

11.5 inches

A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

Glossary

angle
: the union of two rays having a common endpoint

angular speed
: the angle through which a rotating object travels in a unit of time

arc length
: the length of the curve formed by an arc

area of a sector
: area of a portion of a circle bordered by two radii and the intercepted arc; the fraction θ 2π multiplied by the area of the entire circle

coterminal angles

description of positive and negative angles in standard position sharing the same terminal side

degree
: a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
: the side of an angle from which rotation begins

linear speed
: the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
: the amount of rotation from the initial side to the terminal side

negative angle
: description of an angle measured clockwise from the positive x-axis

positive angle
: description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
: an angle whose terminal side lies on an axis

radian measure
: the ratio of the arc length formed by an angle divided by the radius of the circle

radian
: the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
: one point on a line and all points extending in one direction from that point; one side of an angle

reference angle
: the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position
: the position of an angle having the vertex at the origin and the initial side along the positive x-axis

terminal side
: the side of an angle at which rotation ends

vertex
: the common endpoint of two rays that form an angle

Solve Proportion and Similar Figure Applications

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications

Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Solve n3 = 30.
   If you missed this problem, review [link].
2. The perimeter of a triangular window is 23 feet. The lengths of two sides are ten feet and six feet. How long is the third side?
   If you missed this problem, review [link].

Solve Proportions

When two rational expressions are equal, the equation relating them is called a *proportion*.

Proportion

A **proportion** is an equation of the form $ab=cd$, where $b \neq 0, d \neq 0$.

The proportion is read “a is to b, as c is to d.”

The equation $12=48$ is a proportion because the two fractions are equal. The proportion $12=48$ is read “1 is to 2 as 4 is to 8.”

Proportions are used in many applications to ‘scale up’ quantities. We’ll start with a very simple example so you can see how proportions work. Even if you can figure out the answer to the example right away, make sure you also learn to solve it using proportions.

Suppose a school principal wants to have 1 teacher for 20 students. She could use proportions to find the number of teachers for 60 students. We let x be the number of teachers for 60 students and then set up the proportion:

$$1\text{teacher}20\text{students}=x\text{teachers}60\text{students}$$

We are careful to match the units of the numerators and the units of the denominators—teachers in the numerators, students in the denominators.

Since a proportion is an equation with rational expressions, we will solve proportions the same way

we solved equations in Solve Rational Equations. We’ll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

So let’s finish solving the principal’s problem now. We will omit writing the units until the last step.

	[redacted]
Multiply both sides by the LCD, 60.	[redacted]
Simplify.	[redacted]
	The principal needs 3 teachers for 60 students.

Now we’ll do a few examples of solving numerical proportions without any units. Then we will solve applications using proportions.

Solve the proportion: x63 = 47.

Solution

[redacted]

To isolate x, multiply both sid [redacted] LCD, 63.

Simplify.

[redacted]

Divide the common fa [redacted]

Check. To check our answer, we substitute into the original proportion.

[redacted]

$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$

Show common factors.

$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$

Simplify.

$\frac{4}{7} = \frac{4}{7} ✓$

Solve the proportion: n84 = 1112.
77

Solve the proportion: y96 = 1312.
104

When we work with proportions, we exclude values that would make either denominator zero, just like we do for all rational expressions. What value(s) should be excluded for the proportion in the next example?

Solve the proportion: 144a=94.
Solution

$$\frac{144}{a}=\frac{9}{4}$$

Multiply both sides by the LC

$$\frac{144}{a}\cdot 4a=\frac{9}{4}\cdot 4a$$

Remove common fac... on each side.

$$4\cdot 144=a\cdot 9$$

Simplify.

$$576=9a$$

Divide both sides by 9.

[redacted]

Simplify.

[redacted]

Check.

[redacted]

Show common factors.

[redacted]

Simplify.

[redacted]

Solve the proportion: $91b = 75$.

65

Solve the proportion: 39c = 138.
24

Solve the proportion: nn + 14 = 57.

Solution

[redacted]

Multiply both sides by the LC [redacted]

Remove common fa [redacted] side.

Simplify.

[redacted]

Solve for n.

[redacted]

[redacted]

Check.

[redacted]

[redacted] [redacted].

Simplify.

[redacted]

Show common factors.

[redacted]

Simplify.

[redacted]

Solve the proportion: $yy+55=38$.

33

Solve the proportion: zz − 84 = − 15.
14

Solve: p + 129 = p − 126.
Solution

$$\frac{p+12}{9} = \frac{p-12}{6}$$

Multiply both sides by the LC

$$18\left(\frac{p+12}{9}\right) = 18\left(\frac{p-12}{6}\right)$$

Simplify.

[redacted]

Distribute.

[redacted]

Solve for p.

[redacted]

Check.

[redacted]

Simplify.

[redacted]

Divide.

[redacted]

Solve: $v+308=v+6612$.

42

Solve: $2x + 159 = 7x + 315$.

6

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match and the units in the denominators must match.

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let a = ml of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion–be careful of the units. ml [redacted]	
Multiply both sides by the LC[redacted]	
Remove common fac[redacted] side.	
Simplify, but don't multiply on [redacted] the left.	
Notice what	

the next step will be.

Solve for a.

$$\frac{16 \cdot 5}{5} = \frac{5a}{5}$$

Check.

$$16 = a$$

Is the answer reasonable?

Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense.

Substitute $a = 16$ in the original proportion.

$$\frac{5}{25} = \frac{a}{80}$$

$$\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$$

$$\frac{1}{5} = \frac{1}{5} \checkmark$$

Write a complete sentence.

The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

12ml

For every 1 kilogram (kg) of a child’s weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

180ml

A 16-ounce iced caramel macchiato has 230

calories. How many calories are there in a 24-ounce iced caramel macchiato?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many calories are in a 24 ounce iced caramel macchiato?
	Let c = calories in 24 ounces.
Write a sentence that gives the information to find it.	If there are 230 calories in 16 ounces, then how many calories are in 24 ounces?
Translate into a proportion–be careful of the units. cal [redacted] unce	
Multiply both sides by the	

LC[redacted]

Remove common fac[redacted] side.

Simplify.

[redacted]

Solve for c.

[redacted]

[redacted]

Check.

Is the answer reasonable?

Yes, 345 calories for 24 ounces is more than 290 calories for 16 ounces, but not too much more.

[redacted]

Write a complete sentence.

There are 345 calories in a 24-ounce iced caramel macchiato.

At a fast-food restaurant, a 22-ounce chocolate shake has 850 calories. How many calories are in their 12-ounce chocolate shake? Round your answer to nearest whole number.

464calories

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many

pieces can she have in her snack?

5pieces

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

What are you asked to find?	How many Mexican pesos did Josiah get?
Assign a variable.	Let p = the number of Mexican pesos.
Write a sentence that gives the information to	If $1 US is equal to 12.54 Mexican pesos, then $325 is

find it.	how many pesos?
Translate into a proportion–be careful of the units.	
$pesos = $pesos	

[redacted]

Multiply both sides by the LC[redacted]	
Remove common fac[redacted] side.	
Simplify.	

[redacted]

Check.	
Is the answer reasonable?	
Yes, $100 would be 1,254 pesos. $325 is a little more than 3 times this amount, so our answer of 4075.5 pesos makes sense.	

[redacted]

Write a complete sentence.

Josiah got 4075.5 pesos for his spring break trip.

Yurianna is going to Europe and wants to change $800 dollars into Euros. At the current exchange rate, $1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

590.4Euros

Corey and Nicole are traveling to Japan and need to exchange $600 into Japanese yen. If each dollar is 94.1 yen, how many yen will

they get?
56,460yen

In the example above, we related the number of pesos to the number of dollars by using a proportion. We could say the number of pesos *is proportional to* the number of dollars. If two quantities are related by a proportion, we say that they are proportional.

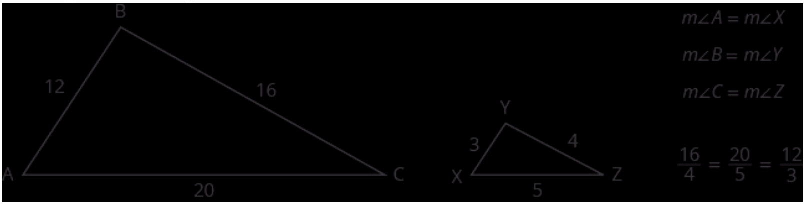
Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be *similar.* One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides are in the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

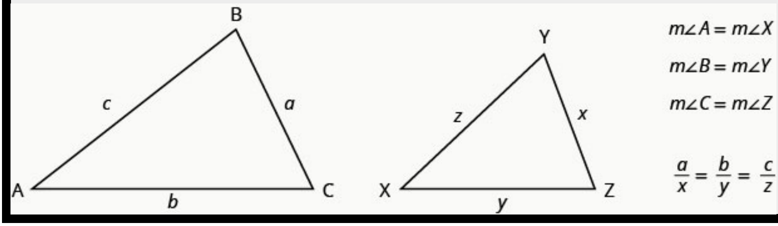
For example, the two triangles in [link] are similar. Each side of ΔABC is 4 times the length of the corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides are in the same ratio.

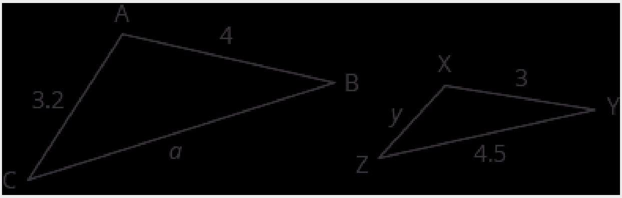


To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

Solve geometry applications.

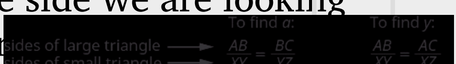
Read the problem and make all the words and ideas are understood. Draw the figure and label it with the given information. **Identify** what we are looking for. **Name** what we are looking for by choosing a variable to represent it. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information. **Solve the equation** using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given. Find the lengths of the third sides.



Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	Figure is given.
Step 2. Identify what we are looking for.	the length of the sides of similar triangles
Step 3. Name the variables.	Let a = length of the third side of ΔABC. y = length of the third side of ΔXYZ
Step 4. Translate.	Since the triangles are similar, the corresponding sides are proportional.
We need to write an equation that compares the side we are looking for	ABXY = BCYZ = ACXZ.



Since the side $AB = 4$

corresponds to the side $XY = 3$ we know ABXY=43. So we write equations with ABXY to find the sides we are looking for. Be careful to match up corresponding sides correctly.

Substitute.

$$\frac{4}{3} = \frac{a}{4.5}$$

$$\frac{4}{3} = \frac{3.2}{y}$$

Step 5. Solve the equation.

$$3a = 4(4.5)$$

$$4y = 3(3.2)$$

$$a = 6$$

$$y = 2.4$$

Step 6. Check.

43=?64.54(4.5)=?

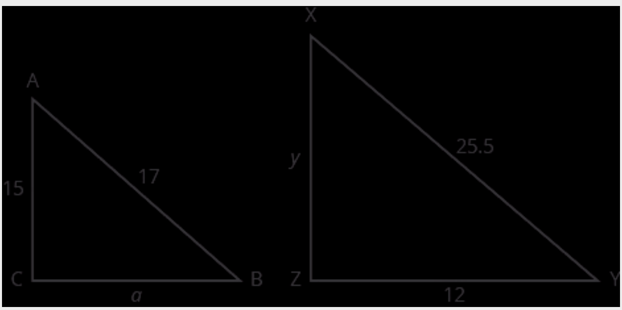
6(3)18=18✓43=?

3.22.44(2.4)=?

$3.2(3)9.6 = 9.6$✓

Step 7. Answer the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of side a.

8

ΔABC is similar to ΔXYZ. The lengths of two

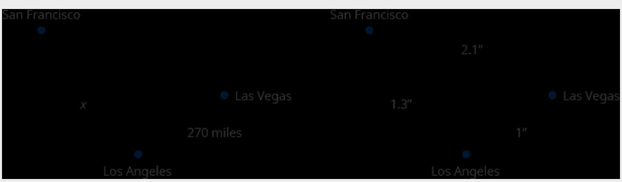
sides of each triangle are given in the figure.

Find the length of side y.

22.5

The next example shows how similar triangles are used with maps.

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. If the actual distance from Los Angeles to Las Vegas is 270 miles find the distance from Los Angeles to San Francisco.



Solution

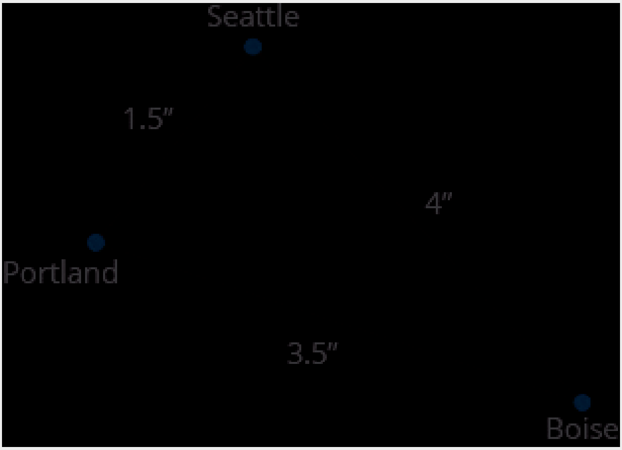
Read the problem. Draw the figures and label with the given information.	The figures are shown above.
Identify what we are looking for.	The actual distance from Los Angeles to San Francisco.
Name the variables.	Let $x =$ distance from Los Angeles to San Francisco.
Translate into an equation. Since the tri[redacted] are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches."	
Solve the equation.	
[redacted]	
[redacted]	
Check. On the map, the distance from Los Angeles to San Francisco is more than the distance from	

Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.

[redacted]

Answer the question. | The distance from Los Angeles to San Francisco is 351 miles.

On the map, Seattle, Portland, and Boise form a triangle whose sides are shown in the figure below. If the actual distance from Seattle to Boise is 400 miles, find the distance from Seattle to Portland.



150 miles

Using the map above, find the distance from Portland to Boise.

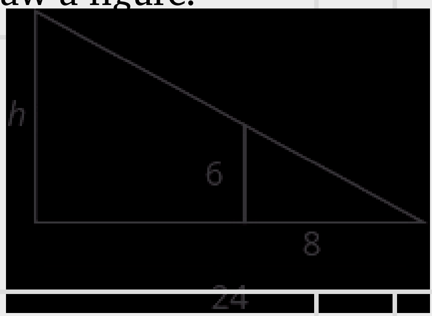
350 miles

We can use similar figures to find heights that we cannot directly measure.

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure.



We are looking for h, the height of the tree.

We will use similar triangles to write an equation.

The small triangle is similar to the large tri

$$\frac{h}{24} = \frac{6}{8}$$

Solve the proportion.

[redacted]

Simplify.

[redacted]

Check.
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow.

[redacted]

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

40 feet

A pine tree casts a shadow of 80 feet next to a 30-foot tall building which casts a 40 feet shadow. How tall is the pine tree?
60 feet

Key Concepts

- **Property of Similar Triangles**
  - If ΔABC is similar to ΔXYZ, then their corresponding angle measures are equal and their corresponding sides are in the same ratio.
- **Problem Solving Strategy for Geometry Applications**

  Read the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information. **Identify** what we are looking for. **Name** what we are looking for by choosing a variable to represent it. **Translate** into an equation by writing the

appropriate formula or model for the situation. Substitute in the given information. **Solve the equation** using good algebra techniques. **Check** the answer in the problem and make sure it makes sense. **Answer** the question with a complete sentence.

Practice Makes Perfect

Solve Proportions

In the following exercises, solve.

$\frac{x}{56}=\frac{7}{8}$

49

$\frac{n}{91}=\frac{8}{13}$

$\frac{49}{63}=\frac{z}{9}$

7

$\frac{56}{72}=\frac{y}{9}$

$5a = 65117$

9

$4b = 64144$

$98154 = -7p$

-11

$72156 = -6q$

$a - 8 = -4248$

7

$b - 7 = -3042$

$2.7j = 0.90.2$

0.6

$2.8k = 2.11.5$

$aa + 12 = 47$

16

$bb - 16 = 119$

$cc - 104 = -58$

-58

$dd - 48 = -133$

$m + 9025 = m + 3015$

60

$n + 104 = 40 - n6$

$2p + 48 = p + 186$

30

$q - 22 = 2q - 718$

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

9ml

Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child’s weight. How many milligrams will the doctor prescribe?

A veterinarian prescribed Sunny, a 65 pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

325mg

Belle, a 13 pound cat, is suffering from joint pain. How much medicine should the veterinarian prescribe if the dosage is 1.8 mg per pound?

A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

159calories

One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

A new 7 ounce lemon ice drink is advertised for having only 140 calories. How many ounces could Sally drink if she wanted to drink just 100 calories?

5oz

Reese loves to drink healthy green smoothies. A 16 ounce serving of smoothie has 170 calories. Reese drinks 24 ounces of these smoothies in

one day. How many calories of smoothie is he consuming in one day?

Janice is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.01 Canadian. How many Canadian dollars will she get for her trip?

252.5Canadian dollars

Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

Steve changed $600 into 480 Euros. How many Euros did he receive for each US dollar?

0.80Euros

Martha changed $350 US into 385 Australian dollars. How many Australian dollars did she receive for each US dollar?

When traveling to Great Britain, Bethany

exchanged her $900 into 570 British pounds. How many pounds did she receive for each American dollar?

0.63British pounds

A missionary commissioned to South Africa had to exchange his $500 for the South African Rand which is worth 12.63 for every dollar. How many Rand did he have after the exchange?

Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

3cups

Sarah drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

Elizabeth is returning to the United States from Canada. She changes the remaining 300 Canadian dollars she has to $230.05 in

American dollars. What was $1 worth in Canadian dollars?

1.30Canadian dollars

Ben needs to convert $1000 to the Japanese Yen. One American dollar is worth 123.3 Yen. How much Yen will he have?

A golden retriever weighing 85 pounds has diarrhea. His medicine is prescribed as 1 teaspoon per 5 pounds. How much medicine should he be given?

17tsp

Five-year-old Lacy was stung by a bee. The dosage for the anti-itch liquid is 150 mg for her weight of 40 pounds. What is the dosage per pound?

Karen eats 12 cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

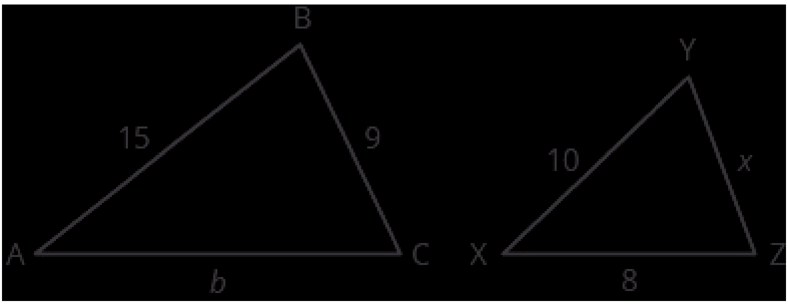
---

34 cup

An oatmeal cookie recipe calls for 12 cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



side *b*

12

side *x*

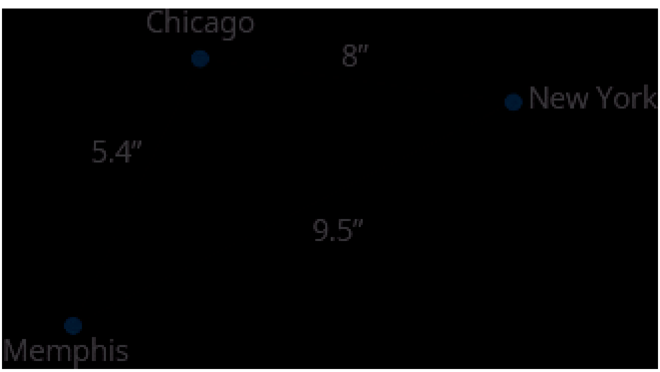
In the following exercises, ΔDEF is similar to ΔNPQ.

Find the length of side *d*.

7718

Find the length of side *q*.

In the following two exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle whose sides are shown in the figure below. The actual distance from New York to Chicago is 800 miles.

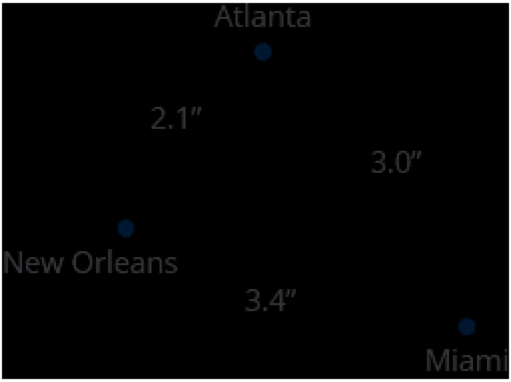


Find the actual distance from New York to Memphis.

950 miles

Find the actual distance from Chicago to Memphis.

In the following two exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle whose sides are shown in the figure below. The actual distance from Atlanta to New Orleans is 420 miles.



Find the actual distance from New Orleans to Miami.

680 miles

Find the actual distance from Atlanta to Miami.

A 2 foot tall dog casts a 3 foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

---

23 foot (8in)

Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry’s shadow was 8 feet and Tom’s was 7.75 feet long. What is Tom’s height?

The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven foot shadow. How long is the windmill’s shadow?

247.3feet

The height of the Statue of Liberty is 305 feet. Nicole, who is standing next to the statue, casts a 6 foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Everyday Math

Heart Rate At the gym, Carol takes her pulse for 10 seconds and counts 19 beats.

ⓐ How many beats per minute is this?
ⓑ Has Carol met her target heart rate of 140 beats per minute?

---

114 beats per minute
no

Heart Rate Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?
ⓑ Has Kevin met his target heart rate?

Cost of a Road Trip Jesse’s car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

---

19 gallons
$58.71

Cost of a Road Trip Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny's home and his car gets 18.5 miles per gallon.

ⓐ How many gallons of gas will Danny need to get to and from Phoenix?
ⓑ If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

Lawn Fertilizer Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

---

4 bags

House Paint April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

Cooking Natalia's pasta recipe calls for 2 pounds of pasta for 1 quart of sauce. How many pounds of pasta should Natalia cook if she has 2.5 quarts of sauce?

5

Heating Oil A 275 gallon oil tank costs $400 to fill. How much would it cost to fill a 180 gallon oil tank?

Writing Exercises

Marisol solves the proportion $144a=94$ by 'cross multiplying', so her first step looks like $4\cdot 144=9\cdot a$. Explain how this differs from the method of solution shown in [link].

Answers will vary.

Find a printed map and then write and solve an application problem similar to [link].

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

proportion
: A proportion is an equation of the form $ab=cd$, where $b\neq 0, d\neq 0$. The proportion is read "a is to b, as c is to d."

similar figures
: Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

Right Triangle Trigonometry

In this section you will:

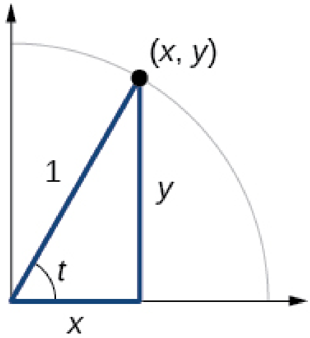
- Use right triangles to evaluate trigonometric functions.
- Find function values for 30°(π 6),45°(π 4), and 60°(π 3).
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

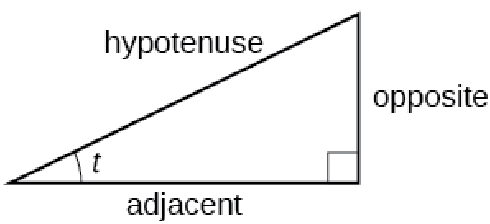
The sides of a right triangle in relation to angle t The side adjacent to one angle is opposite the other angle.Side lengths of special triangles

Using Right Triangles to Evaluate Trigonometric Functions

[link] shows a right triangle with a vertical side of length y and a horizontal side has length x. Notice that the triangle is inscribed in a circle of radius 1. Such a circle, with a center at the origin and a radius of 1, is known as a **unit circle**.



We can define the trigonometric functions in terms an angle *t* and the lengths of the sides of the triangle. The **adjacent side** is the side closest to the angle, *x*. (Adjacent means "next to.") The **opposite side** is the side across from the angle, *y*. The **hypotenuse** is the side of the triangle opposite the right angle, 1. These sides are labeled in [link].



Given a right triangle with an acute angle of t, the

first three trigonometric functions are listed.

$$\text{Sine} \quad \sin t = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\text{Cosine} \quad \cos t = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\text{Tangent} \quad \tan t = \frac{\text{opposite}}{\text{adjacent}}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "**S**ine is **o**pposite over **h**ypotenuse, **C**osine is **a**djacent over **h**ypotenuse, **T**angent is **o**pposite over **a**djacent."

For the triangle shown in [link], we have the following.

$$\sin t = \frac{y}{1} \quad \cos t = \frac{x}{1} \quad \tan t = \frac{y}{x}$$

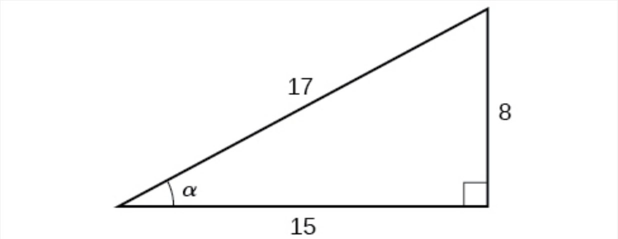
Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

Evaluating a Trigonometric Function of a

Right Triangle

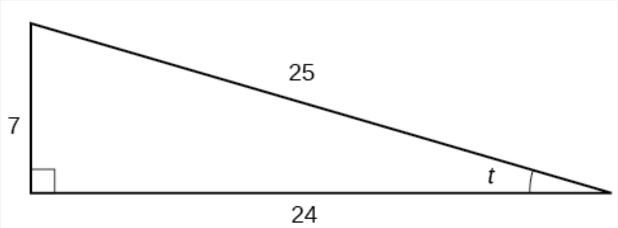
Given the triangle shown in [link], find the value of $\cos\alpha$.



The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17.

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$$

Given the triangle shown in [link], find the value of $\sin t$.



$\frac{7}{25}$

Reciprocal Functions

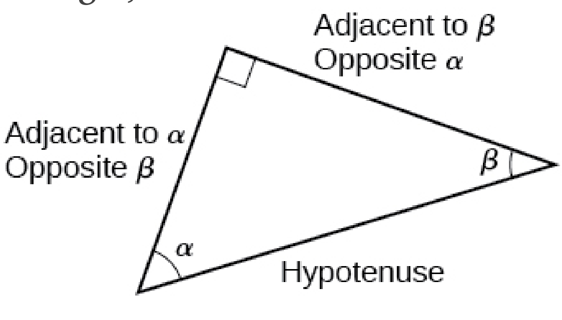
In addition to sine, cosine, and tangent, there are three more functions. These too are defined in terms of the sides of the triangle.

Secant $\sec t = \frac{\text{hypotenuse}}{\text{adjacent}}$
Cosecant $\csc t = \frac{\text{hypotenuse}}{\text{opposite}}$
Cotangent $\cot t = \frac{\text{adjacent}}{\text{opposite}}$

Take another look at these definitions. These functions are the reciprocals of the first three functions.

$\sin t = \frac{1}{\csc t}$ $\csc t = \frac{1}{\sin t}$ $\cos t = \frac{1}{\sec t}$ $\sec t = \frac{1}{\cos t}$ $\tan t = \frac{1}{\cot t}$ $\cot t = \frac{1}{\tan t}$

When working with right triangles, keep in mind that the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in [link]. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.



Many problems ask for all six trigonometric functions for a given angle in a triangle. A possible

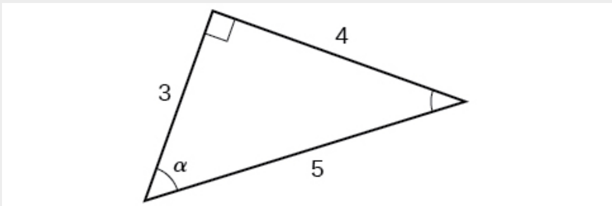
strategy to use is to find the sine, cosine, and tangent of the angles first. Then, find the other trigonometric functions easily using the reciprocals.

Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
    - sine as the ratio of the opposite side to the hypotenuse
    - cosine as the ratio of the adjacent side to the hypotenuse
    - tangent as the ratio of the opposite side to the adjacent side
    - secant as the ratio of the hypotenuse to the adjacent side
    - cosecant as the ratio of the hypotenuse to the opposite side
    - cotangent as the ratio of the adjacent side to the opposite side

Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in [link], evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha, \text{and} \cot\alpha$.



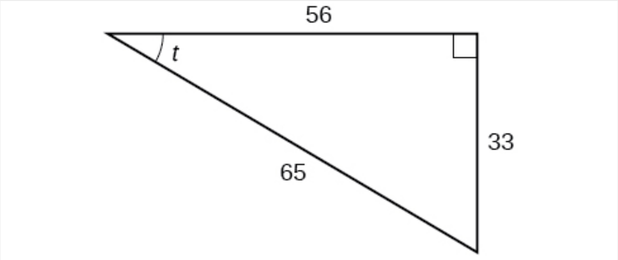
$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \quad \cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \quad \tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3} \quad \sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3} \quad \csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4} \quad \cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Analysis

Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$
$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$
$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Using the triangle shown in [link],evaluate sint,cost,tant,sect,csct,andcott.



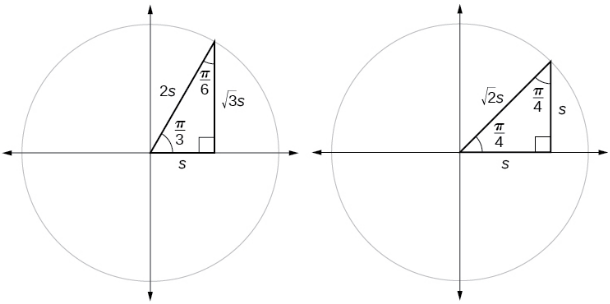
sin t = 33 65 , cos t = 56 65 , tan t = 33 56 , sec t = 65 56 , csc t = 65 33 , cot t = 56 33

Finding Trigonometric Functions of Special Angles Using Side Lengths

It is helpful to evaluate the trigonometric functions as they relate to the special angles—multiples of 30°,60°, and 45°. Remember, however, that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a 30°,60°,90° triangle, which can also be described as a π 6 , π 3 , π 2 triangle. The sides have lengths in the relation s, 3 s ,2s. The sides of a 45°,45°,90° triangle, which can also be described as a π 4 , π 4 , π 2 triangle, have lengths in the relation s,s, 2 s. These relations are shown in

[link].



We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in [link] for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of π 3 , using side lengths.

$\sin(\pi\ 3) = \text{opp hyp} = 3s\ 2s = 3\ 2\ \cos(\pi\ 3)$

$= \frac{adj}{hyp} = \frac{s}{2s} = \frac{1}{2}$ $\tan(\frac{\pi}{3}) = \frac{opp}{adj}$ $= \frac{\sqrt{3}s}{s} = \sqrt{3}$ $\sec(\frac{\pi}{3}) = \frac{hyp}{adj} = \frac{2s}{s} = 2$
$\csc(\frac{\pi}{3}) = \frac{hyp}{opp} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
$\cot(\frac{\pi}{3}) = \frac{adj}{opp} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, $\tan(\frac{\pi}{4}) = 1$, $\sec(\frac{\pi}{4}) = \sqrt{2}$, $\csc(\frac{\pi}{4}) = \sqrt{2}$, $\cot(\frac{\pi}{4}) = 1$

The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa. Cofunction identity of sine and cosine of complementary angles

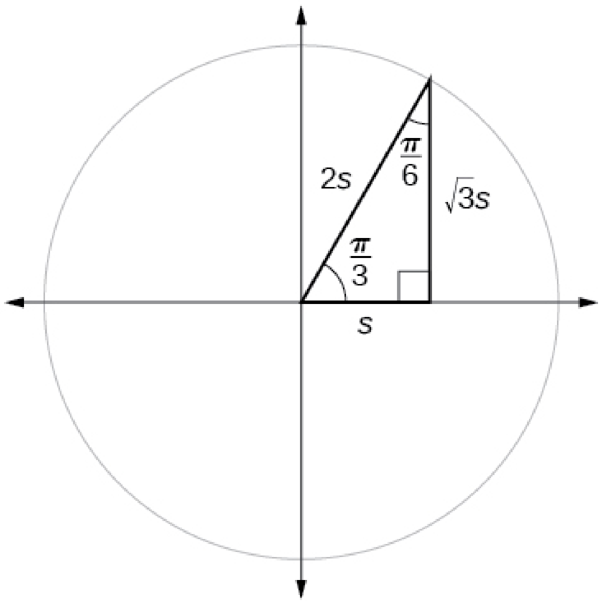
Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles, we notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$

, is also the cosine of π 3 .

sin π 3 =cos π 6 = 3 s 2s = 3 2 sin π 6 =cos π 3 = s 2s = 1 2

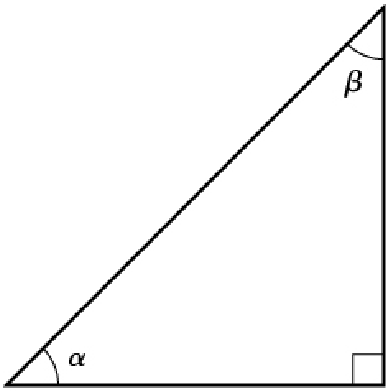
See [link].



This result should not be surprising because, as we see from [link], the side opposite the angle of π 3 is also the side adjacent to π 6 , so sin(π 3) and cos(π 6) are exactly the same ratio of the same two sides, 3 s and 2s. Similarly, cos(π 3) and sin(π 6) are also the same ratio using the same two sides, s and 2s.

The interrelationship between the sines and cosines of π 6 and π 3 also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one

angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in [link].



$$\sin\alpha = \cos\beta$$
$$\sin\beta = \cos\alpha$$

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a given angle $t, \cos t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

Cofunction Identities

The **cofunction identities** in radians are listed in [link].

$\cos t=\sin\left(\frac{\pi}{2}-t\right)$	$\sin t=\cos\left(\frac{\pi}{2}-t\right)$
$\tan t=\cot\left(\frac{\pi}{2}-t\right)$	$\cot t=\tan\left(\frac{\pi}{2}-t\right)$
$\sec t=\csc\left(\frac{\pi}{2}-t\right)$	$\csc t=\sec\left(\frac{\pi}{2}-t\right)$

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Using Cofunction Identities

If $\sin t=\frac{5}{12}$, find $\cos\left(\frac{\pi}{2}-t\right)$.

According to the cofunction identities for sine

and cosine, we have the following.
$\sin t = \cos\left(\frac{\pi}{2} - t\right)$

So
$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}$

If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

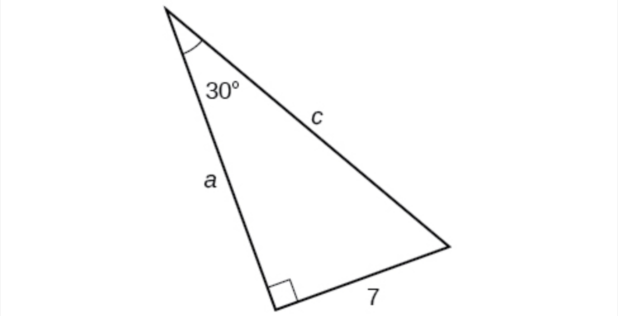
Given a right triangle, the length of one side,

and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in [link].



We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$\tan(30^\circ) = 7\ a$

We rearrange to solve for a.

$a = 7\ \tan(30^\circ) \approx 12.1$

We can use the sine to find the hypotenuse.

$\sin(30^\circ) = 7\ c$

Again, we rearrange to solve for c.

$c = 7\ \sin(30^\circ) = 14$

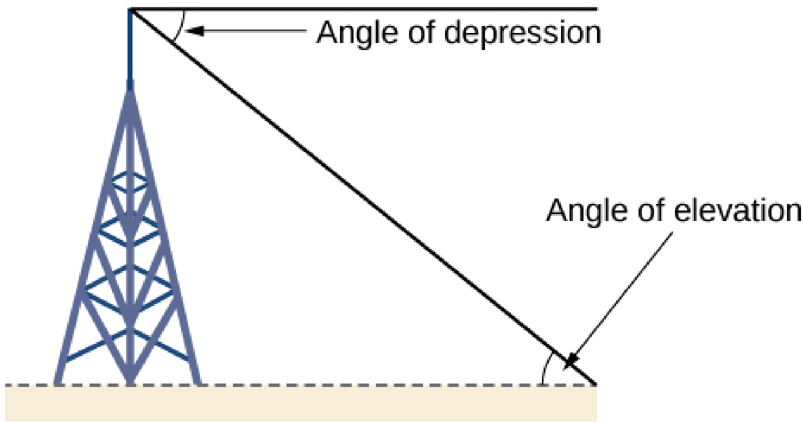
A right triangle has one angle of $\pi\ 3$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

$\text{adjacent} = 10; \text{opposite} = 10\ 3$; missing angle is $\pi\ 6$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height.

Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See [link].

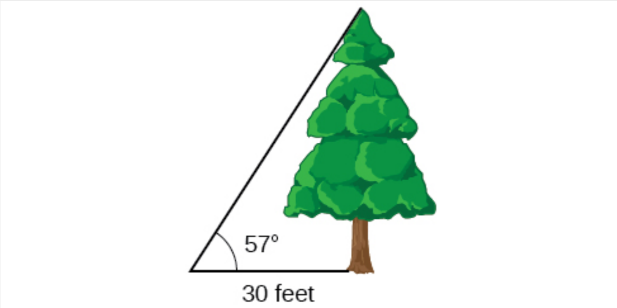


Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in [link]. Find the height of the tree.



We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ $\tan(57^\circ) = \frac{h}{30}$ Solve for h. $h = 30\tan(57^\circ)$ Multiply. $h \approx 46.2$ Use a calculator.

The tree is approximately 46 feet tall.

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of 5π 12 with the ground? Round to the nearest foot.

About 52 ft

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Key Equations

Trigonometric Functions	Sine $\sin t = \frac{\text{opposite}}{\text{hypotenuse}}$ Cosine $\cos t = \frac{\text{adjacent}}{\text{hypotenuse}}$ Tangent $\tan t = \frac{\text{opposite}}{\text{adjacent}}$ Secant $\sec t = \frac{\text{hypotenuse}}{\text{adjacent}}$ Cosecant $\csc t = \frac{\text{hypotenuse}}{\text{opposite}}$ Cotangent $\cot t = \frac{\text{adjacent}}{\text{opposite}}$
Reciprocal Trigonometric Functions	$\sin t = \frac{1}{\csc t}$ $\csc t = \frac{1}{\sin t}$ $\cos t = \frac{1}{\sec t}$ $\sec t = \frac{1}{\cos t}$ $\tan t = \frac{1}{\cot t}$ $\cot t = \frac{1}{\tan t}$
Cofunction Identities	$\cos t = \sin\left(\frac{\pi}{2} - t\right)$ $\sin t = \cos\left(\frac{\pi}{2} - t\right)$ $\tan t = \cot\left(\frac{\pi}{2} - t\right)$ $\cot t = \tan\left(\frac{\pi}{2} - t\right)$ $\sec t = \csc\left(\frac{\pi}{2} - t\right)$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See [link].
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See [link].
- We can evaluate the trigonometric functions of

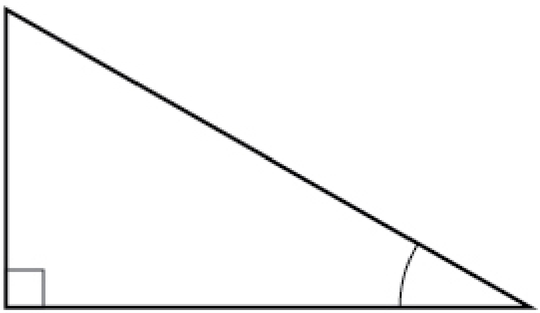
special angles, knowing the side lengths of the triangles in which they occur. See [link].
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See [link].
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See [link].
- Right-triangle trigonometry facilitates the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See [link].

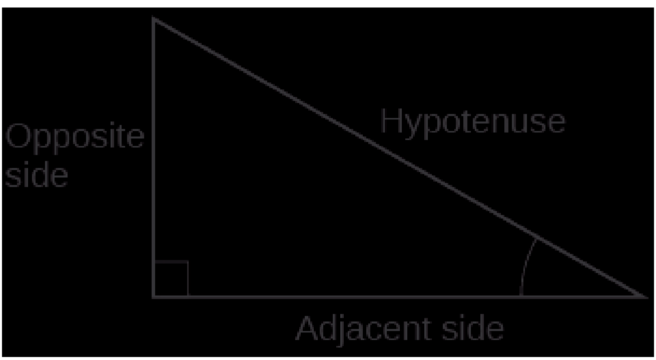
Section Exercises

Verbal

For the given right triangle, label the adjacent

side, opposite side, and hypotenuse for the indicated angle.





When a right triangle with a hypotenuse of 1 is placed in a circle of radius 1, which sides of the triangle correspond to the *x*- and *y*-coordinates?

The tangent of an angle compares which sides of the right triangle?

The tangent of an angle is the ratio of the opposite side to the adjacent side.

What is the relationship between the two acute angles in a right triangle?

Explain the cofunction identity.

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the following exercises, use cofunctions of complementary angles.

$\cos(34^\circ) = \sin(\_\_^\circ)$

$\cos(\pi\ 3) = \sin(\_\_)$

$\pi\ 6$

$\csc(21^\circ) = \sec(\_\_^\circ)$

$\tan(\pi 4) = \cot(\_\_)$

$\pi 4$

For the following exercises, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

$\cos B = 4 5, a = 10$

$\sin B = 1 2, a = 20$

$b = 20 3 3, c = 40 3 3$

$\tan A = 5 12, b = 6$

$\tan A = 100, b = 100$

$a = 10{,}000, c = 10{,}00.5$

$\sin B = 1 3, a = 2$

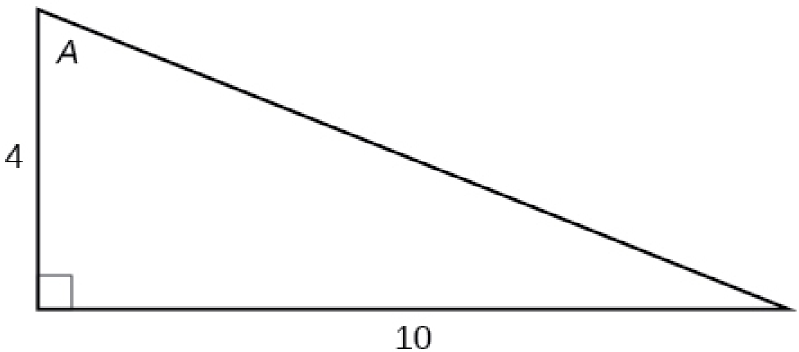
$a = 5, \measuredangle A = 60^\circ$

$b = \ 5\ 3\ 3\ , c = \ 10\ 3\ 3$

$c = 12, \measuredangle A = 45^\circ$

Graphical

For the following exercises, use [link] to evaluate each trigonometric function of angle A.



$\sin A$

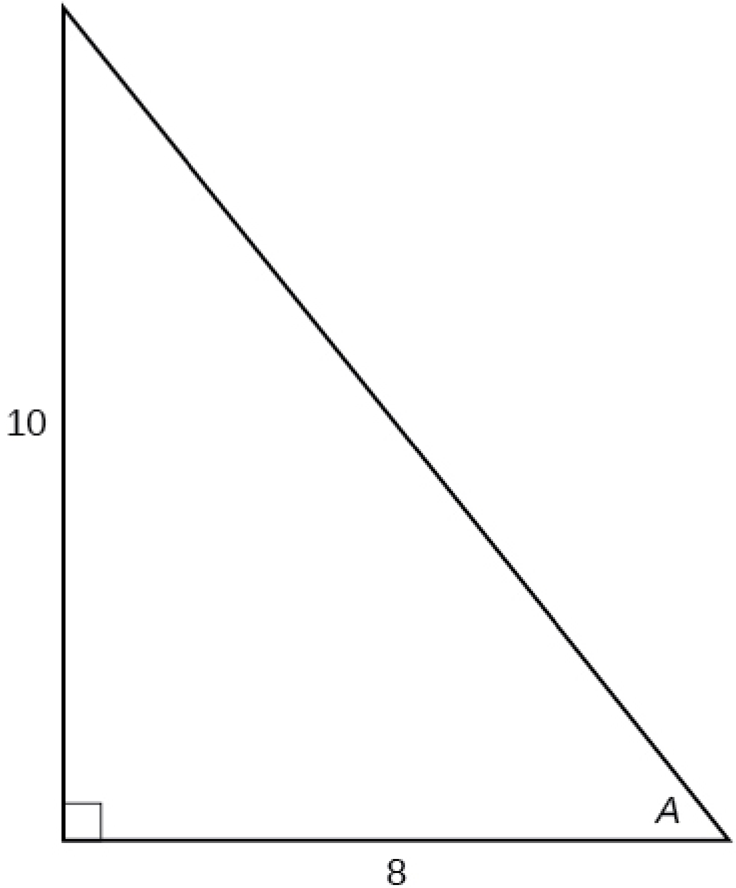
$5\ 29\ 29$

$\cos A$

$\tan A$

5 2
cscA
secA
29 2
cotA

For the following exercises, use [link] to evaluate each trigonometric function of angle A.



sinA

---

5 41 41

cosA

tanA

5 4

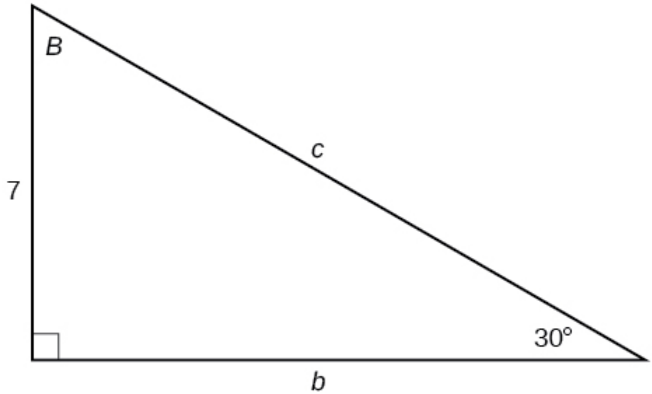
cscA

secA

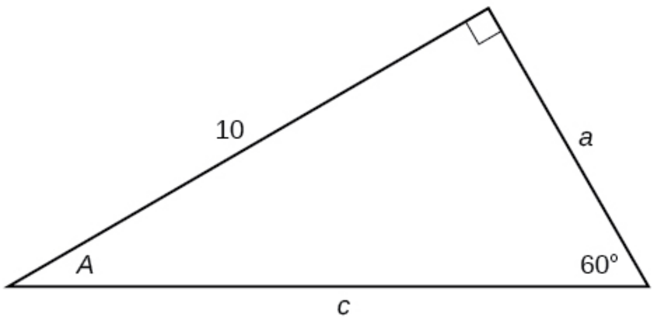
41 4

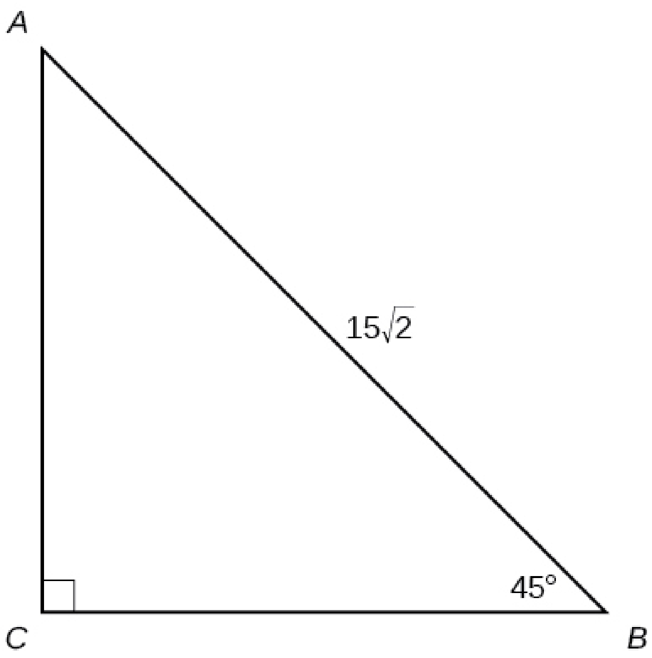
cotA

For the following exercises, solve for the unknown sides of the given triangle.



c=14,b=7 3

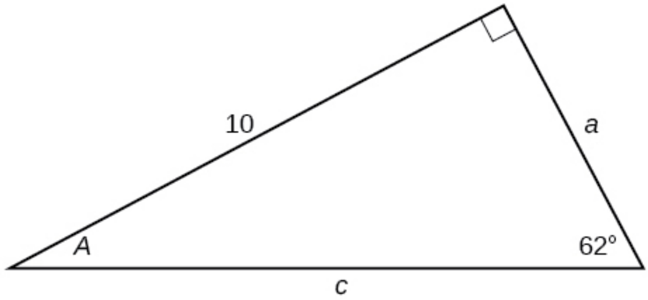


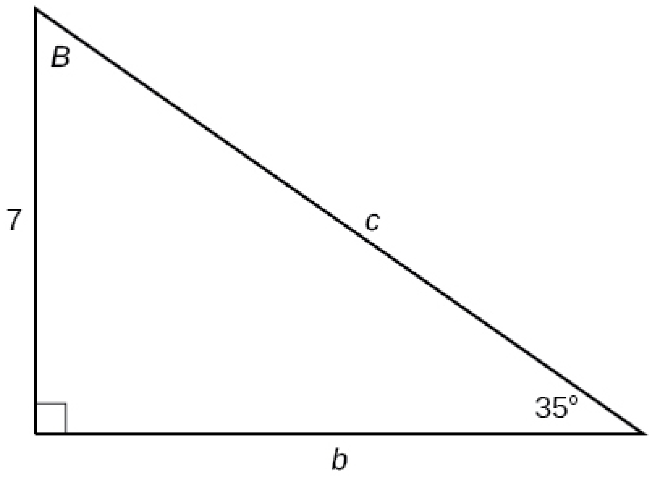


$a=15, b=15$

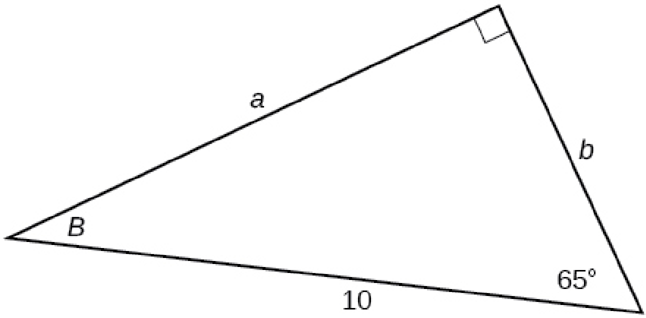
Technology

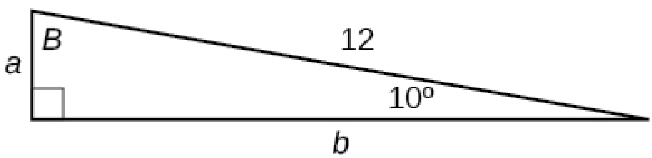
For the following exercises, use a calculator to find the length of each side to four decimal places.



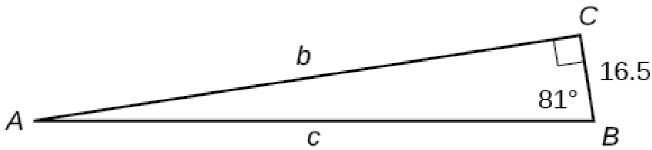


$b = 9.9970, c = 12.2041$





$a = 2.0838, b = 11.8177$



$b = 15, \measuredangle B = 15^\circ$

$a = 55.9808, c = 57.9555$

$c = 200, \measuredangle B = 5^\circ$

$c = 50, \measuredangle B = 21^\circ$

$a = 46.6790, b = 17.9184$

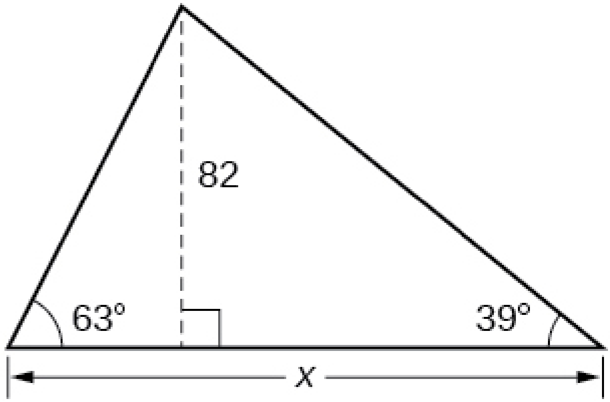
$a = 30, \measuredangle A = 27^\circ$

$b = 3.5, \measuredangle A = 78^\circ$

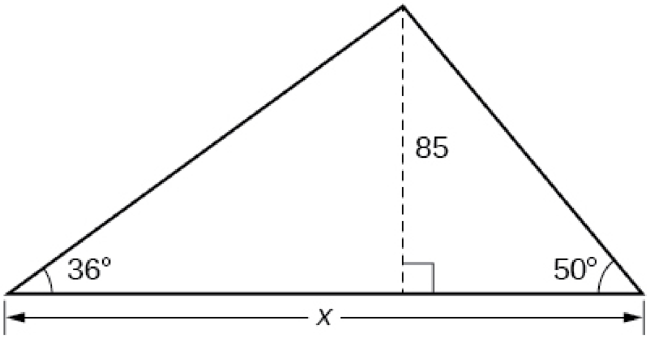
$a = 16.4662, c = 16.8341$

Extensions

Find x.

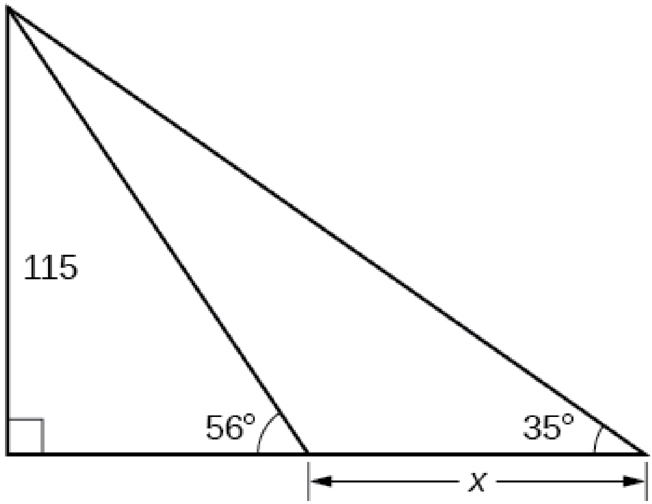


Find x.

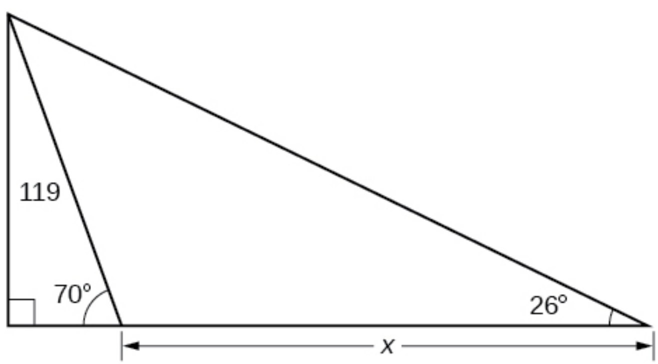


188.3159

Find x.



Find x.



200.6737

A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

498.3471 ft

A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the monument is 2°. How far is the person from the monument?

A 400-foot tall monument is located in the

distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

1060.09 ft

There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

27.372 ft

Real-World Applications

A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

22.6506 ft

The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

368.7633 ft

Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

Glossary

adjacent side
: in a right triangle, the side between a given angle and the right angle

angle of depression
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
: the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
: in a right triangle, the side most distant from a given angle

hypotenuse
: the side of a right triangle opposite the right angle

unit circle
: a circle with a center at (0,0) and radius 1

Unit Circle

In this section you will:

- Find function values for the sine and cosine of 30° or (π 6),45° or (π 4), and 60 ∘ or (π 3).
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

The Singapore Flyer is the world’s tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world’s tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a

mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Unit circle where the central angle is t radians

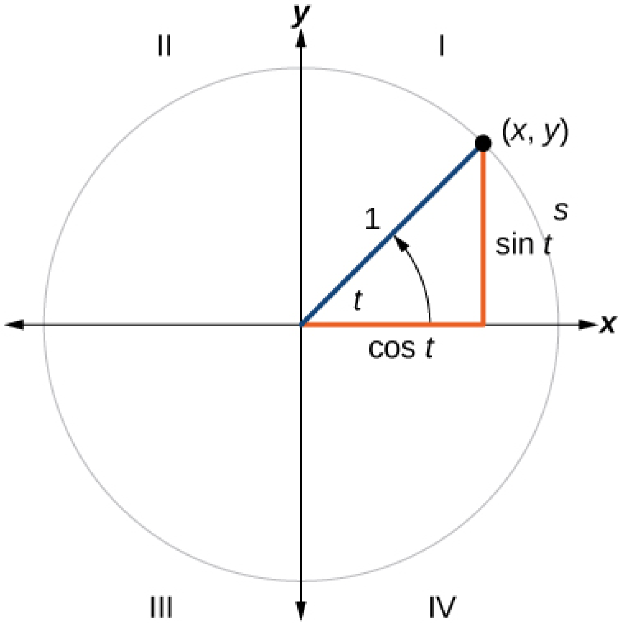
Finding Trigonometric Functions Using the Unit Circle

We have already defined the trigonometric functions in terms of right triangles. In this section, we will redefine them in terms of the unit circle. Recall that a unit circle is a circle centered at the origin with radius 1, as shown in [link]. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s=rt$, and knowing that $r=1$, we see that for a unit circle, $s=t$.

The *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x,y). The coordinates x and y will be

the outputs of the trigonometric functions f(t) = cost and f(t) = sint, respectively. This means x = cos t and y = sin t.



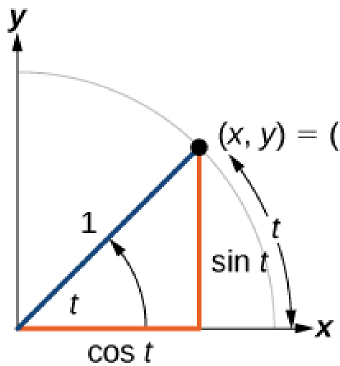
Unit Circle
A **unit circle** has a center at (0,0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t. Let (x,y) be the endpoint on the unit circle of an arc of arc length s. The (x,y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the

Unit Circle

The sine function relates a real number t to the *y*-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the *y*-value of the endpoint on the unit circle of an arc of length t. In [link], the sine is equal to y. Like all functions, the **sine function** has an input and an output. Its input is the measure of the angle; its output is the *y*-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the *x*-value of the endpoint on the unit circle of an arc of length t. In [link], the cosine is equal to x.



Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: sint is the same as sin(t) and cost is the same as cos(t). Likewise, cos 2 t is a commonly used shorthand notation for (cos(t)) 2 . Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the

extra parentheses when entering calculations into a calculator or computer.

Sine and Cosine Functions

If t is a real number and a point (x,y) on the unit circle corresponds to a central angle t, then

$\cos t = x$

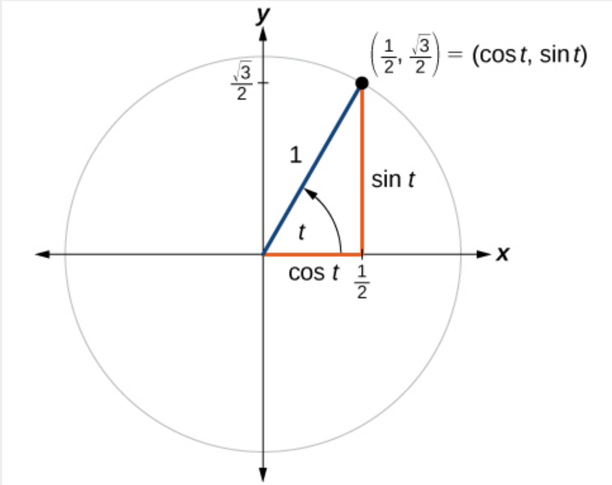
$\sin t = y$

Given a point *P* (x,y) on the unit circle corresponding to an angle of t, find the sine and cosine.

1. The sine of t is equal to the *y*-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the *x*-coordinate of point P: $\cos t = x$.

Finding Function Values for Sine and Cosine

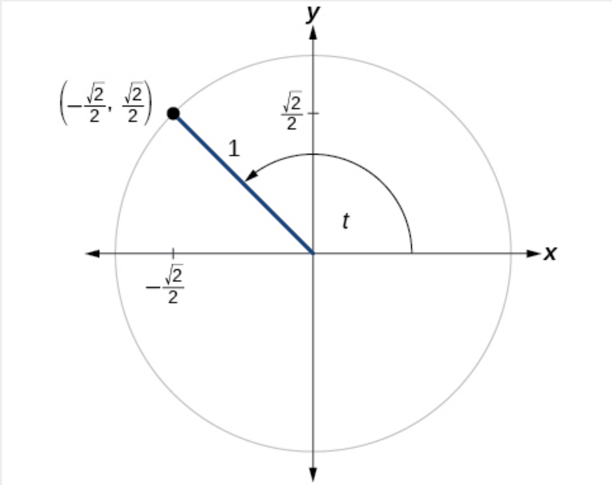
Point P is a point on the unit circle corresponding to an angle of t, as shown in [link]. Find $\cos(t)$ and $\sin(t)$.



We know that cost is the *x*-coordinate of the corresponding point on the unit circle and sint is the *y*-coordinate of the corresponding point on the unit circle. So:

x =cost = 1 2 y =sint = 3 2

A certain angle t corresponds to a point on the unit circle at (− 2 2 , 2 2) as shown in [link]. Find cost and sint.



$\cos(t) = -\ 2\ 2\ , \sin(t) = \ 2\ 2$

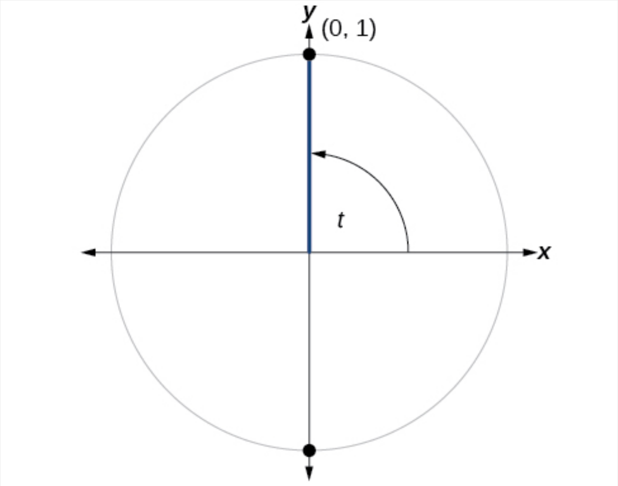
Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the *x*- or *y*-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Moving 90° counterclockwise around the unit circle from the positive *x*-axis brings us to the top of the circle, where the (x,y) coordinates are (0,1), as shown in [link].



We can then use our definitions of cosine and sine.

$$x = \cos t = \cos(90^\circ) = 0 \quad y = \sin t = \sin(90^\circ) = 1$$

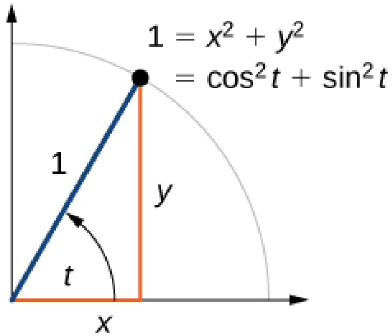
The cosine of 90° is 0; the sine of 90° is 1.

Find cosine and sine of the angle π.

$\cos(\pi) = -1, \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the Pythagorean Identity. See [link].



We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t,

$\cos^2 t + \sin^2 t = 1$

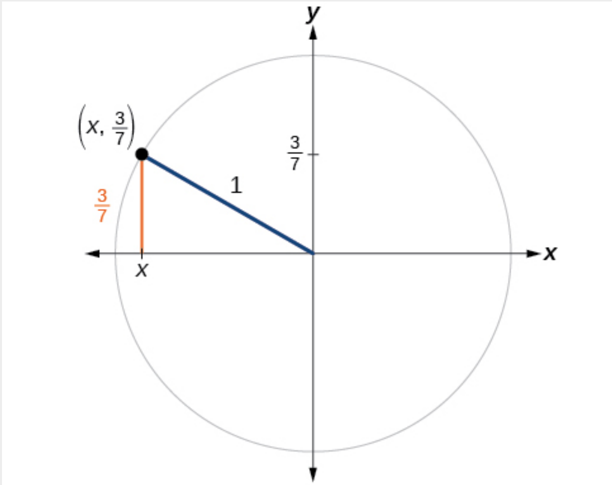
Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of sint into the Pythagorean Identity.
2. Solve for cost.
3. Choose the solution with the appropriate sign for the *x*-values in the quadrant where t is located.

Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find cos(t).

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See [link].



Substituting the known value for sine into the Pythagorean Identity,
cos 2 (t) + sin 2 (t) = 1 cos 2 (t) + 9 49 = 1 cos 2 (t) = 40 49 cos(t) = ± 40 49 = ± 40 7 = ± 2 10 7

Because the angle is in the second quadrant, we know the *x*-value is a negative real number, so the cosine is also negative.
cos(t) = − 2 10 7

If cos(t) = 24 25 and t is in the fourth quadrant, find sin(t).

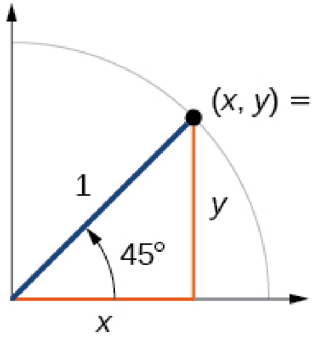
sin(t) = − 7 25

Finding Sines and Cosines of Special Angles

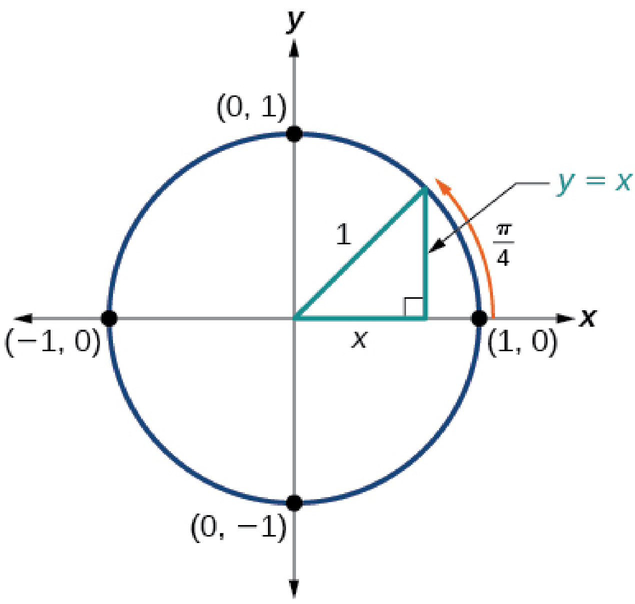
We have already learned some properties of the special angles, such as the conversion from radians to degrees, and we found their sines and cosines using right triangles. We can also calculate sines and cosines of the special angles using the Pythagorean Identity.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in [link]. A 45°–45°–90° triangle is an isosceles triangle, so the *x*- and *y*-coordinates of the corresponding point on the circle are the same. Because the *x*- and *y*-values are the same, the sine and cosine values will also be equal.



At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1 so the right triangle formed below the line $y = x$ has sides x and y ($y = x$), and radius $= 1$. See [link].



From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

We can then substitute $y=x$.

$x\ 2\ +\ x\ 2\ =1$

Next we combine like terms.

$2\ x\ 2\ =1$

And solving for x, we get

$x\ 2\ =\ 1\ 2\ x\ =\ \pm\ 1\ 2$

In quadrant I, $x=\ 1\ 2$.

At $t=\ \pi\ 4$ or 45 degrees,

$(\ x,y\ )\ =\ (\ x,x\ )=(\ 1\ 2\ ,\ 1\ 2\ )\ x\ =\ 1\ 2\ ,y=\ 1\ 2\ \cos t\ =\ 1\ 2\ ,\sin t=\ 1\ 2$

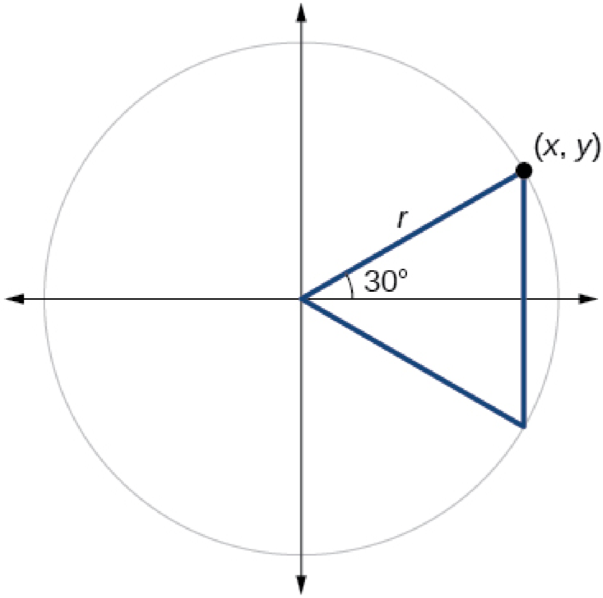
If we then rationalize the denominators, we get

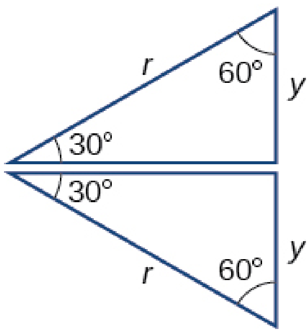
$\cos t\ =\ 1\ 2\ 2\ 2\ =\ 2\ 2\ \sin t\ =\ 1\ 2\ 2\ 2\ =\ 2\ 2$

Therefore, the (x,y) coordinates of a point on a circle of radius 1 at an angle of 45° are $(\ 2\ 2\ ,\ 2\ 2\ )$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\pi\ 6$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of −30°, as shown in [link]. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in [link].





Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r=2y$ or $y=\frac{1}{2}r$. Since $\sin t=y$,

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}r$$

And since $r=1$ in our unit circle,

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}(1)=\frac{1}{2}$$

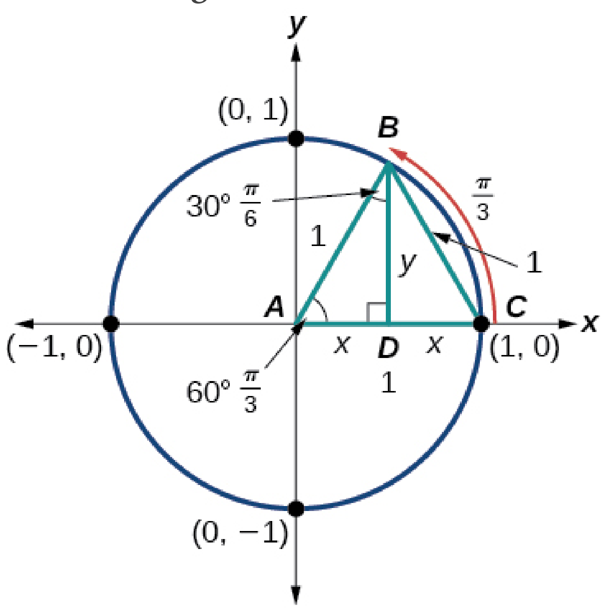
Using the Pythagorean Identity, we can find the

cosine value.

cos 2 (π 6)+ sin 2 (π 6) = 1 cos 2 (π 6)+ (1 2) 2 = 1 cos 2 (π 6) = 3 4

Use the square root property. cos(π 6) = ± 3 ± 4 = 3 2 Since y is positive, choose the positive root.

The (x,y) coordinates for the point on a circle of radius 1 at an angle of 30° are (3 2 , 1 2). At t= π 3 (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in [link]. Angle A has measure 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.



The measure of angle ABD is 30°. Angle ABC is double angle ABD, so its measure is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is 1 2 the radius, or 1 2 . Notice that AD is the *x*-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length 1 2 .

From the Pythagorean Theorem, we get

$x 2 + y 2 = 1$

Substituting $x = 1 2$, we get

$(1 2) 2 + y 2 = 1$

Solving for y, we get

$1 4 + y 2 = 1 \quad y 2 = 1 - 1 4 \quad y 2 = 3 4 \quad y = \pm 3 2$

Since $t = \pi 3$ has the terminal side in quadrant I where the *y*-coordinate is positive, we choose $y = 3 2$, the positive value.

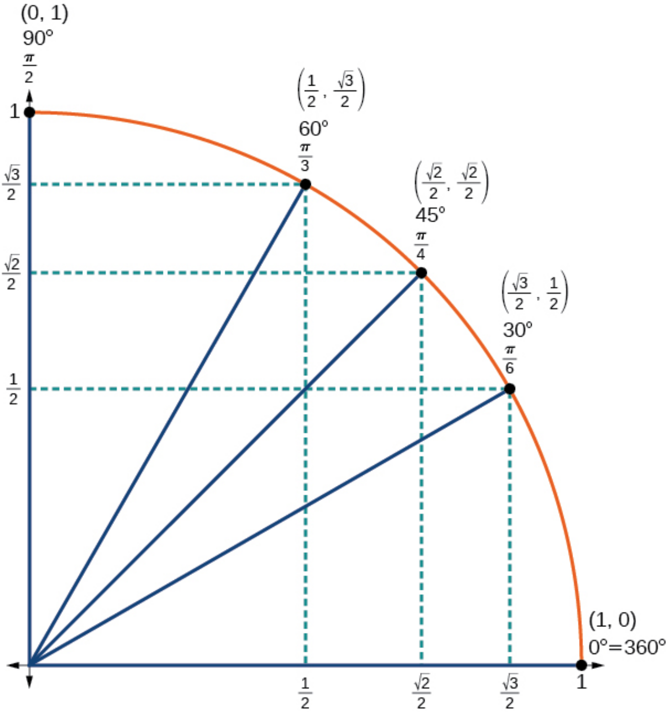
At $t = \pi 3$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are (1 2 , 3 2), so we can find the sine and cosine.

$(x,y) = (1 2 , 3 2) \quad x = 1 2 , y = 3 2 \quad \cos t = 1 2 , \sin t = 3 2$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. [link] summarizes these values.

Angle	0	π 6 , or 30°	π 4 , or 45°	π 3 , or 60°	π 2 , or 90°
Cosine	1	3 2	2 2	1 2	0
Sine	0	1 2	2 2	3 2	1

[link] shows the common angles in the first quadrant of the unit circle.



Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate cos(30)

on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Using a Graphing Calculator to Find Sine and Cosine

Evaluate cos(5π 3) using a graphing calculator or computer.

Enter the following keystrokes:

COS(5 × π ÷ 3) ENTER
$\cos\left(\frac{5\pi}{3} \right) = 0.5$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sine of 20°, for example, by including the conversion factor to radians as part of the input:

SIN(20 × π ÷ 180) ENTER

Evaluate $\sin(\frac{\pi}{3})$.

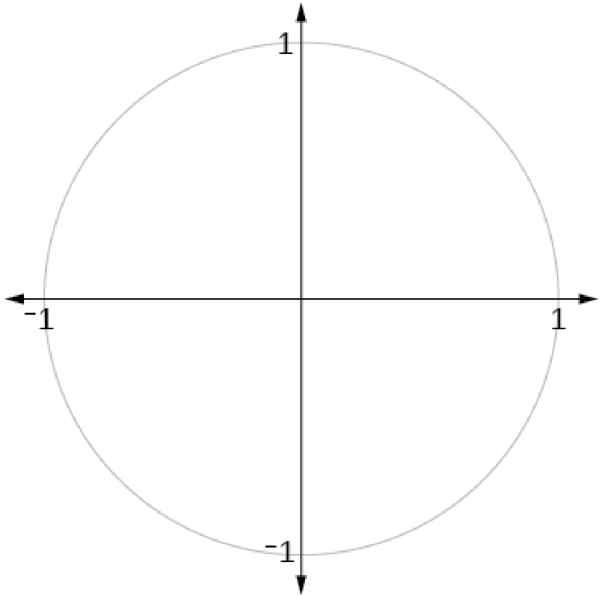
approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions?

Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in [link]. The bounds of the x-coordinate are $[-1,1]$. The bounds of the y-coordinate are also $[-1,1]$. Therefore, the range of both the sine and cosine functions is $[-1,1]$.



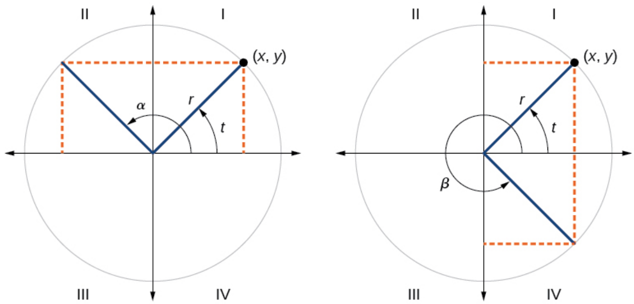
Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the *y*-coordinate on the unit circle, the other angle with the same sine will share the same *y*-value, but have the opposite *x*-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

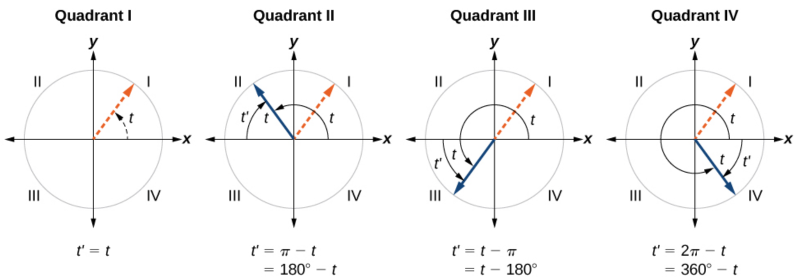
Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in [link], angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$\sin(t)=\sin(\alpha)$ and $\cos(t)=-\cos(\alpha)$ $\sin(t)=-\sin(\beta)$ and $\cos(t)=\cos(\beta)$



Recall that an angle’s reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and π 2 radians. As we can see from [link], for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.



Given an angle between 0 and 2π, find its reference angle.

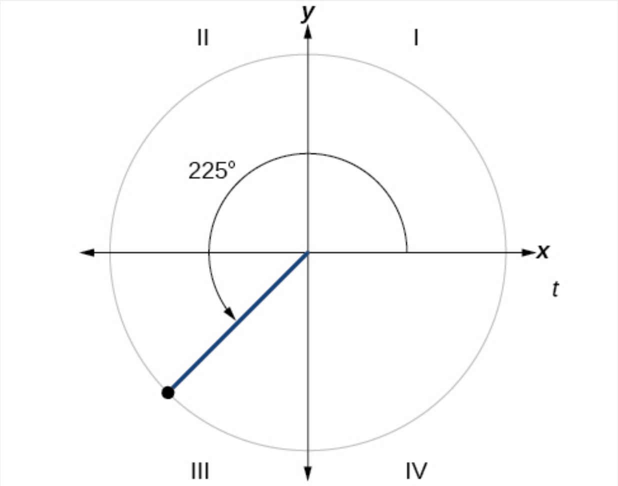
1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the

reference angle is $2\pi - t$ or $360^\circ - t$.

4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Finding a Reference Angle

Find the reference angle of 225° as shown in [link].



Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Find the reference angle of $\frac{5\pi}{3}$.
$\frac{\pi}{3}$

Special angles and coordinates of corresponding points on the unit circle

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x,y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate

Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the *x*-values in that quadrant. The sine will be positive or negative depending on the sign of the *y*-values in that quadrant.

Using Reference Angles to Find Cosine and Sine
Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of

the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.
2. Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

---

1. $150°$ is located in the second quadrant. The angle it makes with the x-axis is $180° - 150° = 30°$, so the reference angle is $30°$.

   This tells us that $150°$ has the same sine and cosine values as $30°$, except for the sign.
   $\cos(30°) = \frac{\sqrt{3}}{2}$ and $\sin(30°) = \frac{1}{2}$

   Since $150°$ is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\cos(150^\circ) = -\ 3\ 2$ and $\sin(150^\circ) = \ 1\ 2$

2. $5\pi\ 4$ is in the third quadrant. Its reference angle is $5\pi\ 4\ - \pi = \ \pi\ 4$. The cosine and sine of $\pi\ 4$ are both $2\ 2$. In the third quadrant, both x and y are negative, so: $\cos\ 5\pi\ 4\ = -\ 2\ 2$ and $\sin\ 5\pi\ 4\ = -\ 2\ 2$

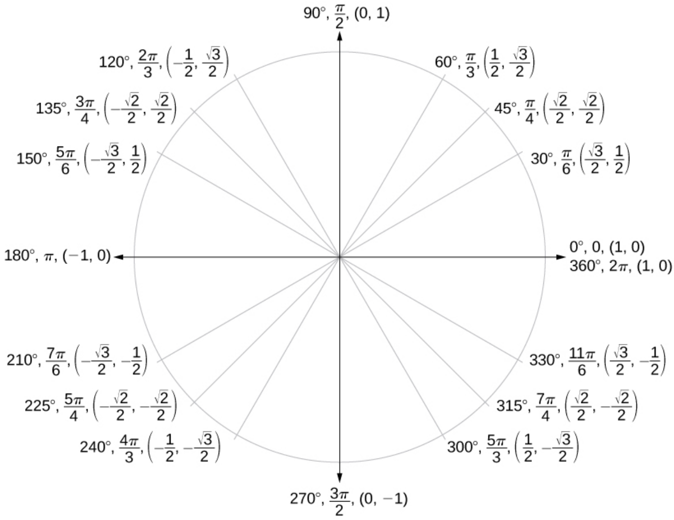
1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\ \pi\ 6$ to find $\cos(\ -\ \pi\ 6\ )$ and $\sin(\ -\ \pi\ 6\ )$.

1. $\cos(315^\circ) = \ 2\ 2$,$\sin(315^\circ) = \ –\ 2\ 2$
2. $\cos(\ -\ \pi\ 6\ ) = \ 3\ 2$,$\sin(\ -\ \pi\ 6\ ) = -\ 1\ 2$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown

in [link]. Take time to learn the (x,y) coordinates of all of the major angles in the first quadrant.



In addition to learning the values for special angles, we can use reference angles to find (x,y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$x = \cos t \; y = \sin t$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the *y*- and *x*-values of the quadrant.

Given the angle of a point on a circle and the radius of the circle, find the (x,y) coordinates

of the point.

1. Find the reference angle by measuring the smallest angle to the *x*-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $7\pi\ 6$.

We know that the angle $7\pi\ 6$ is in the third quadrant.

First, let’s find the reference angle by measuring the angle to the *x*-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$7\pi\ 6\ -\pi=\ \pi\ 6$

Next, we will find the cosine and sine of the reference angle.

$\cos(\ \pi\ 6\ )=\ 3\ 2$ $\sin(\ \pi\ 6\ )=\ 1\ 2$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$

Now we can calculate the (x,y) coordinates using the identities $x=\cos\theta$ and $y=\sin\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$ on the unit circle.

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit

- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos t = x$
Sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.

- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See [link].
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See [link].
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See [link].
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See [link].
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1,1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See [link].
- Reference angles can be used to find the sine

and cosine of the original angle. See [link].
- Reference angles can also be used to find the coordinates of a point on a circle. See [link].

Section Exercises

Verbal

Describe the unit circle.

The unit circle is a circle of radius 1 centered at the origin.

What do the *x*- and *y*-coordinates of the points on the unit circle represent?

Discuss the difference between a coterminal angle and a reference angle.

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

The sine values are equal.

Algebraic

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

$\sin(t) < 0$ and $\cos(t) < 0$

$\sin(t) > 0$ and $\cos(t) > 0$

I

$\sin(t) > 0$ and $\cos(t) < 0$

sin(t) > 0 and cos(t) > 0

IV

For the following exercises, find the exact value of each trigonometric function.

sin π 2

sin π 3

---

3 2

cos π 2

cos π 3

---

1 2

sin π 4

cos π 4

---

2 2

sin π 6

sinπ

0

sin 3π 2

cosπ

-1

cos0

cos π 6

3 2

sin0

Numeric

For the following exercises, state the reference angle for the given angle.

240°

60°

-170°

100°

80°

-315°

135°

45°

$5\pi\ 4$

$2\pi\ 3$

π 3

5π 6

-11π 3

π 3

-7π 4

$-\pi$ 8

π 8

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

225°

300°

60°, Quadrant IV, $\sin(300^\circ) = -\ 3\ 2$,$\cos(300^\circ) = \ 1\ 2$

320°

135°

45°, Quadrant II, $\sin(\ 135^\circ\ ) = \ 2\ 2$,$\cos(135^\circ) = -\ 2\ 2$

210°

120°

60°, Quadrant II, $\sin(\ 120^\circ\ ) = \ 3\ 2$,$\cos(120^\circ) = -\ 1\ 2$

250°

150°

30°, Quadrant II, $\sin(\ 150^\circ\ ) = \ 1\ 2$,$\cos(150^\circ) = -\ 3\ 2$

5π 4

7π 6

π 6 , Quadrant III, sin(7π 6)= − 1 2 ,cos(7π 6)= − 3 2

5π 3

3π 4

π 4 , Quadrant II, sin(3π 4)= 2 2 ,cos(4π 3)= − 2 2

4π 3

2π 3

π 3 , Quadrant II, sin(2π 3)= 3 2 ,cos(2π 3)= − 1 2

5π 6

7π 4

π 4 , Quadrant IV, sin(7π 4)= − 2 2 ,cos(7π 4)= 2 2

For the following exercises, find the requested value.

If cos(t)= 1 7 and t is in the fourth quadrant, find sin(t).

If cos(t)= 2 9 and t is in the first quadrant, find sin(t).

---

77 9

If sin(t)= 3 8 and t is in the second quadrant, find cos(t).

If sin(t)= − 1 4 and t is in the third quadrant, find cos(t).

− 15 4

Find the coordinates of the point on a circle

with radius 15 corresponding to an angle of 220°.

Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 120°.

$(-10,10 \sqrt{3})$

Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

(–2.778, 15.757)

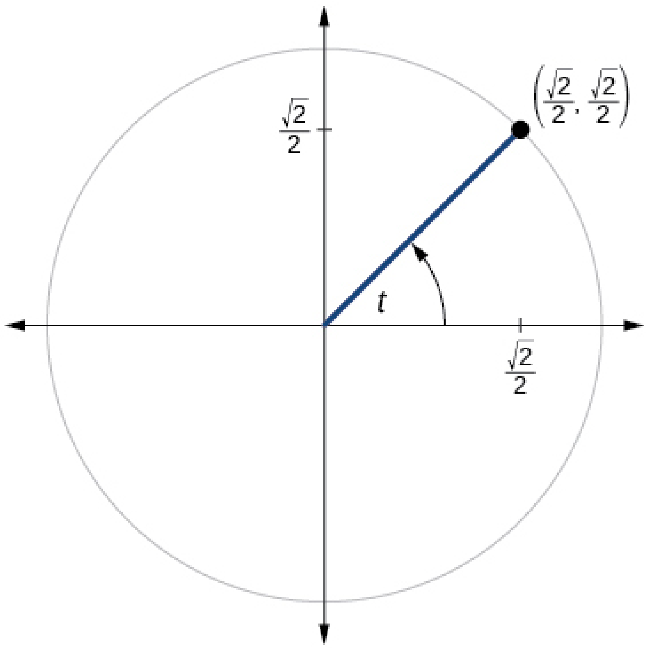
State the domain of the sine and cosine functions.

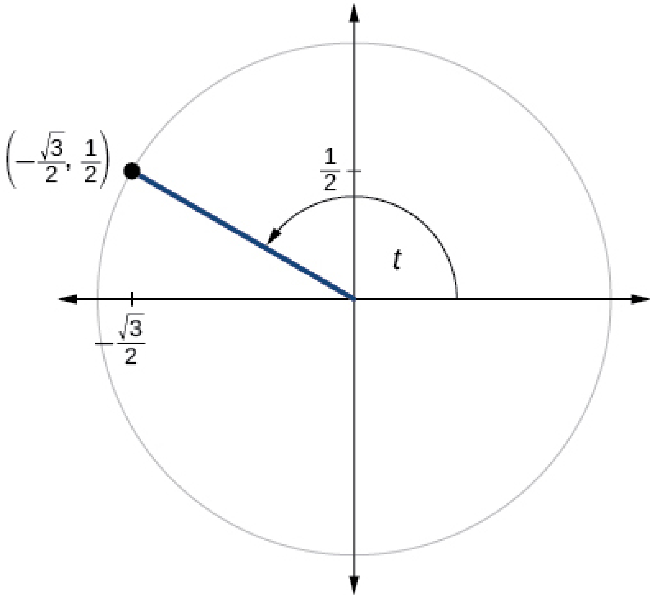
State the range of the sine and cosine functions.

[–1, 1]

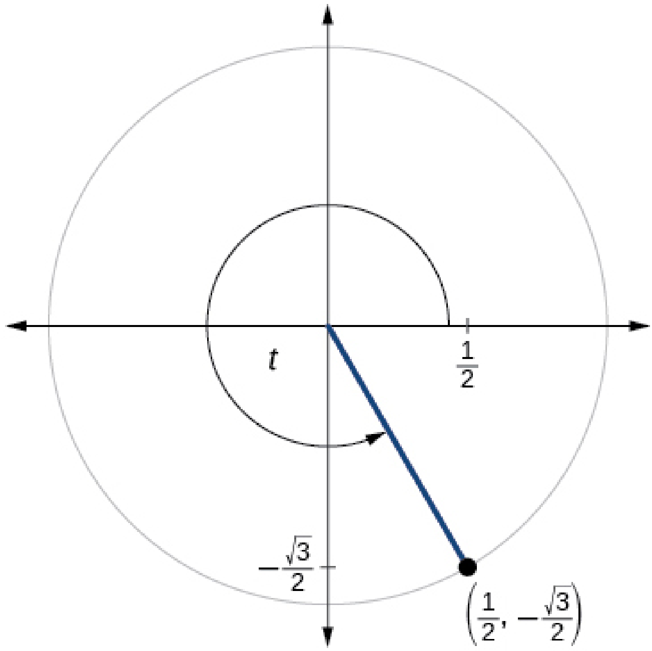
Graphical

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of t.

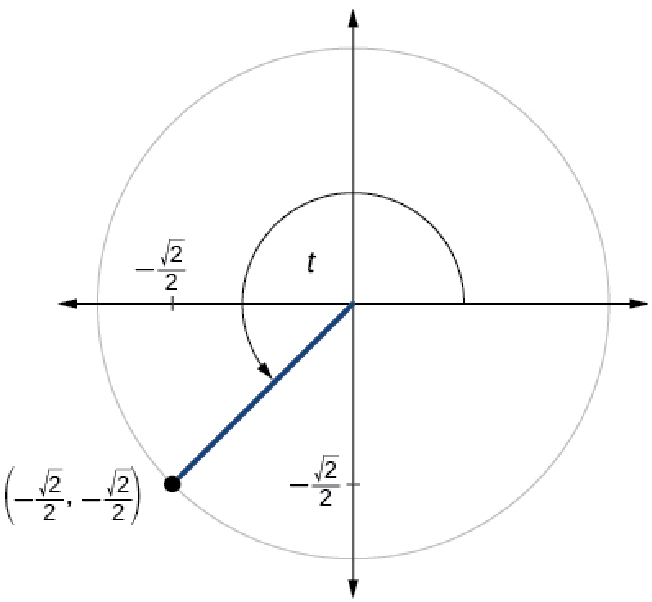




$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$

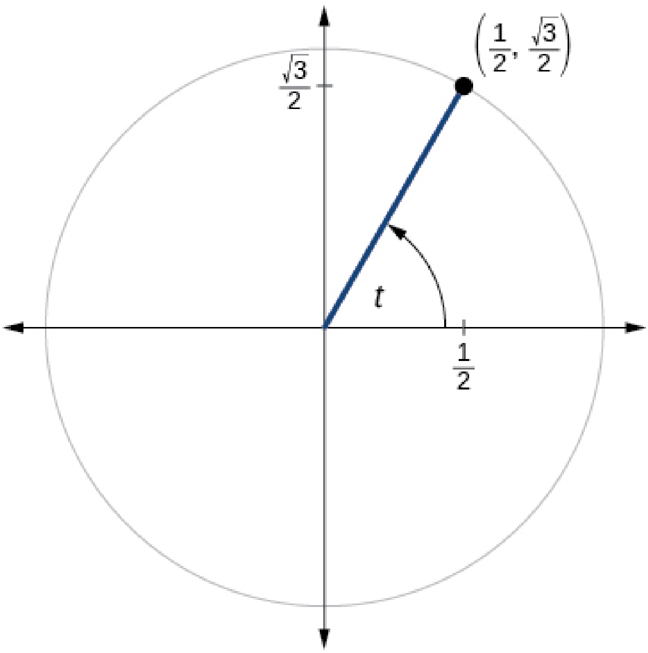


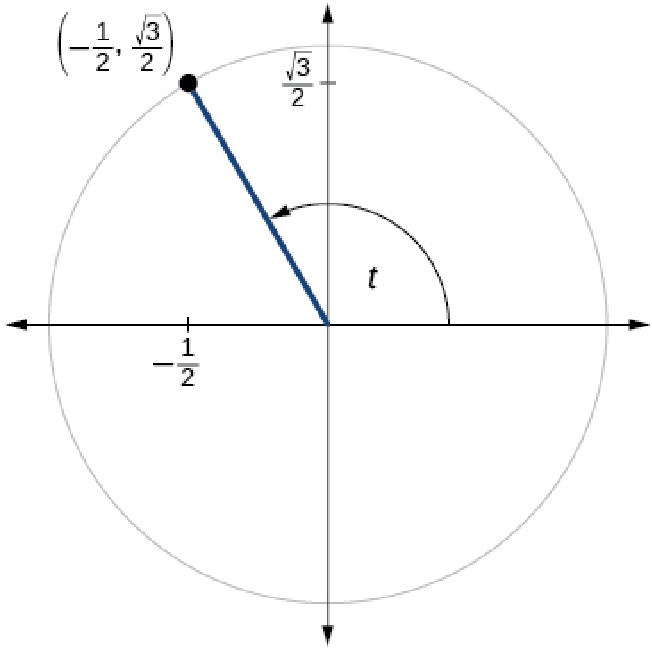
t
$\frac{1}{2}$
$-\frac{\sqrt{3}}{2}$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



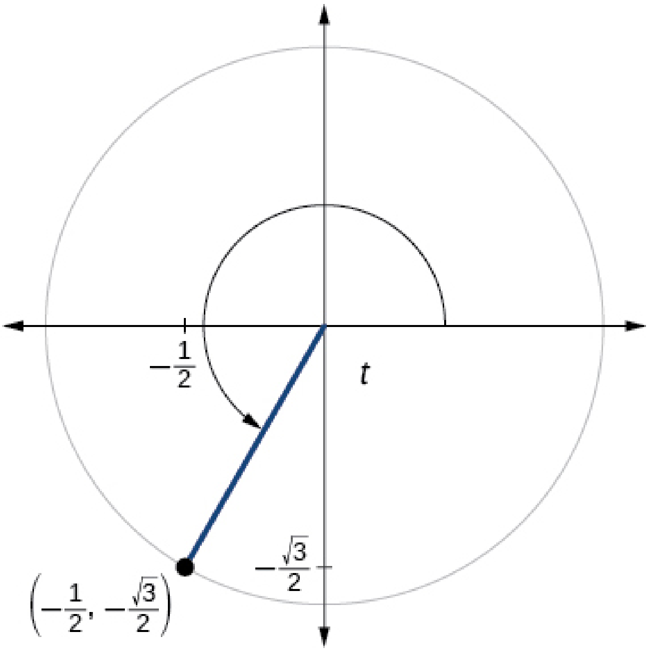
t
$-\frac{\sqrt{2}}{2}$
$-\frac{\sqrt{2}}{2}$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\sin t = -\ 2\ 2\ , \cos t = -\ 2\ 2$

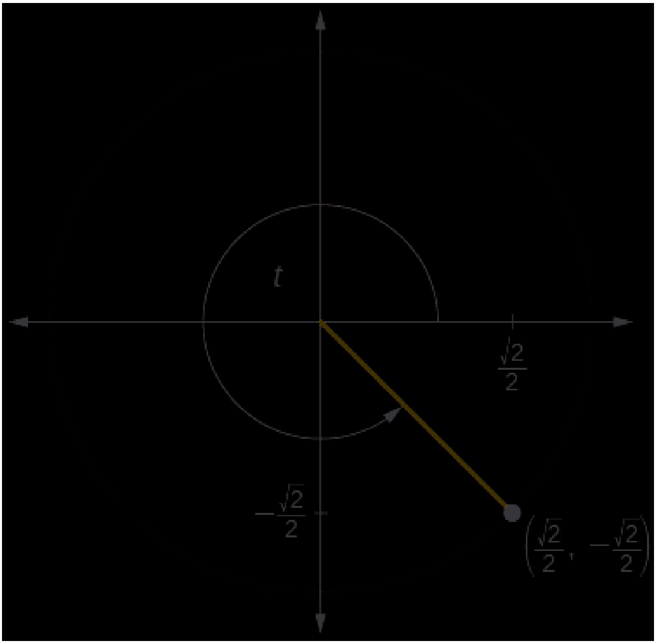




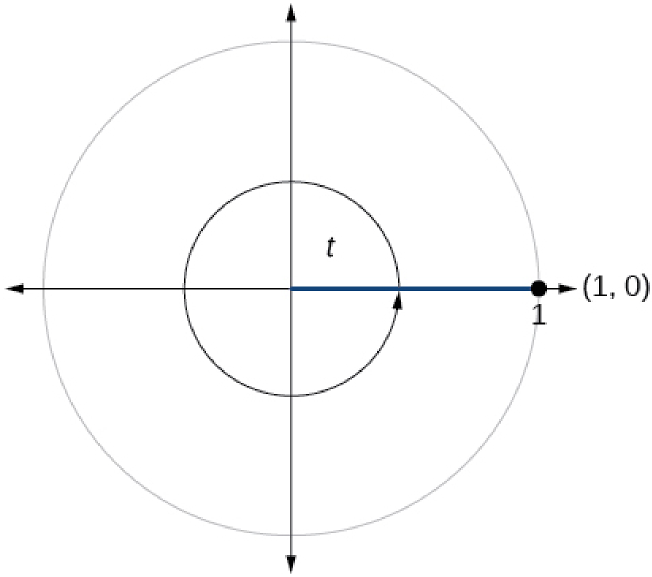
$\sin t = 3\ 2, \cos t = -\ 1\ 2$

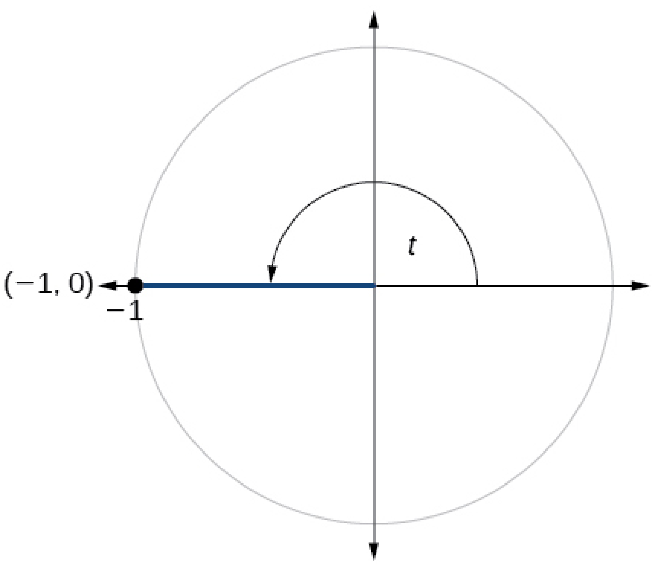


$-\frac{1}{2}$
t
$-\frac{\sqrt{3}}{2}$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

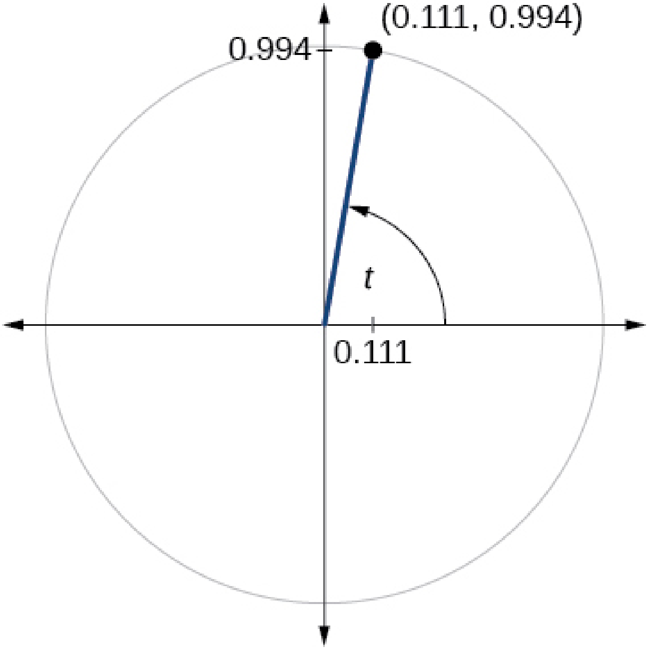


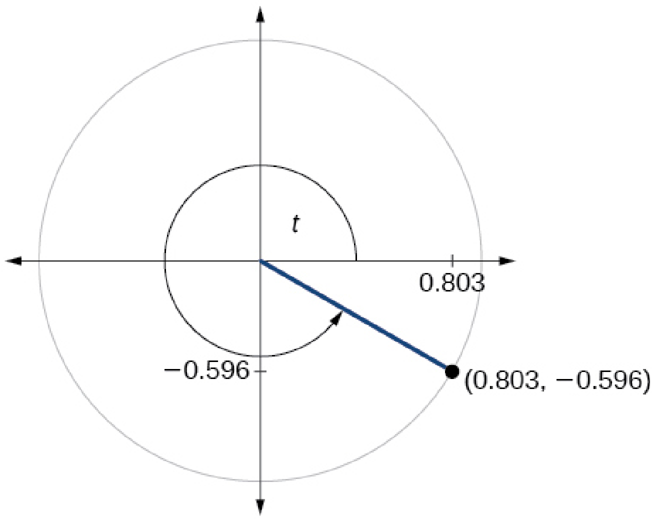
$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$


t
(1, 0)
1

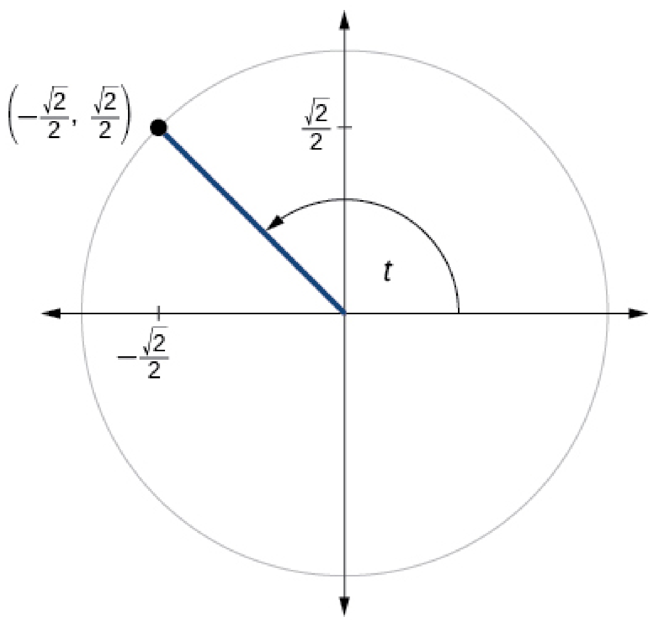

t
(−1, 0)
−1

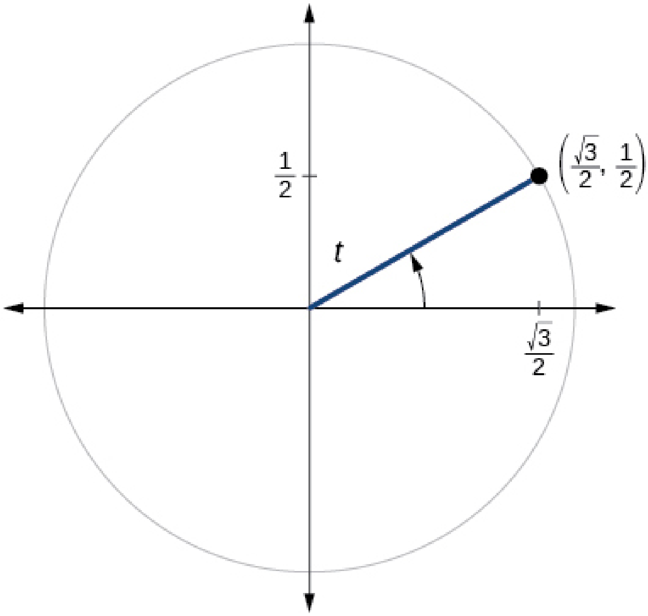
$\sin t = 0, \cos t = -1$



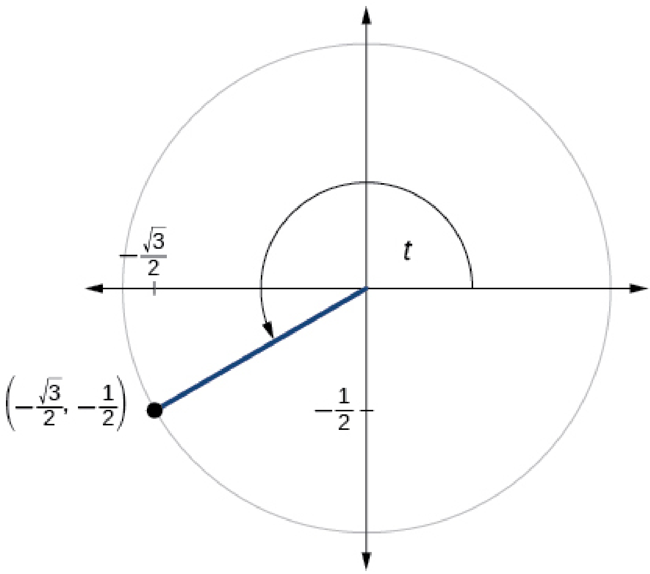


$\sin t = -0.596, \cos t = 0.803$

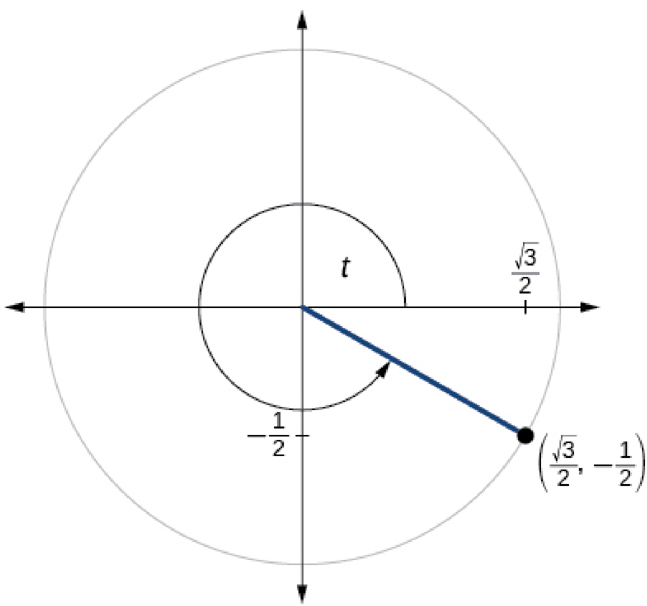




$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

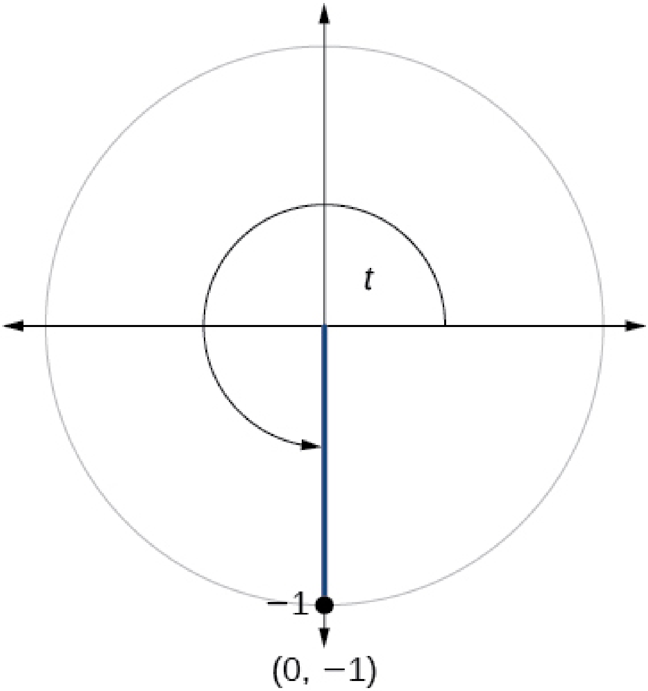


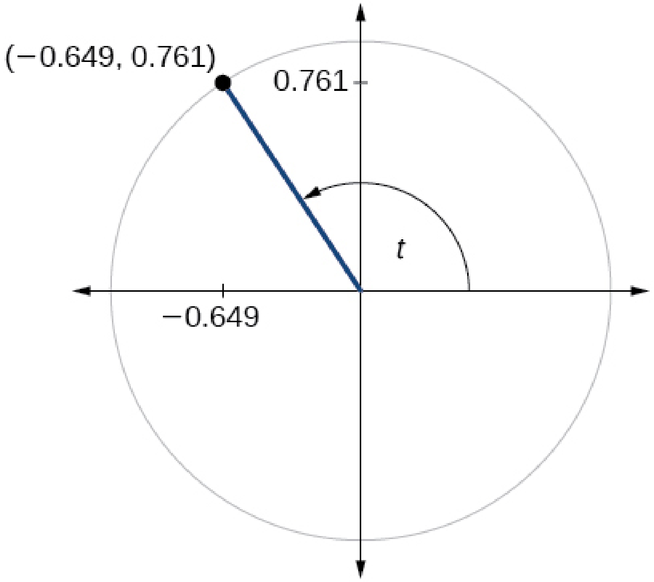
$-\frac{\sqrt{3}}{2}$
t
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$-\frac{1}{2}$



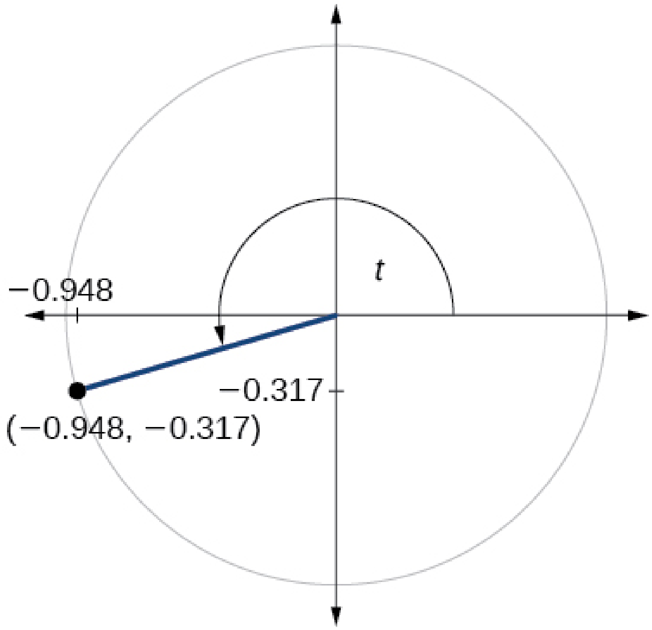
t
$\frac{\sqrt{3}}{2}$
$-\frac{1}{2}$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\sin t = - 1 \ 2 , \cos t = 3 \ 2$

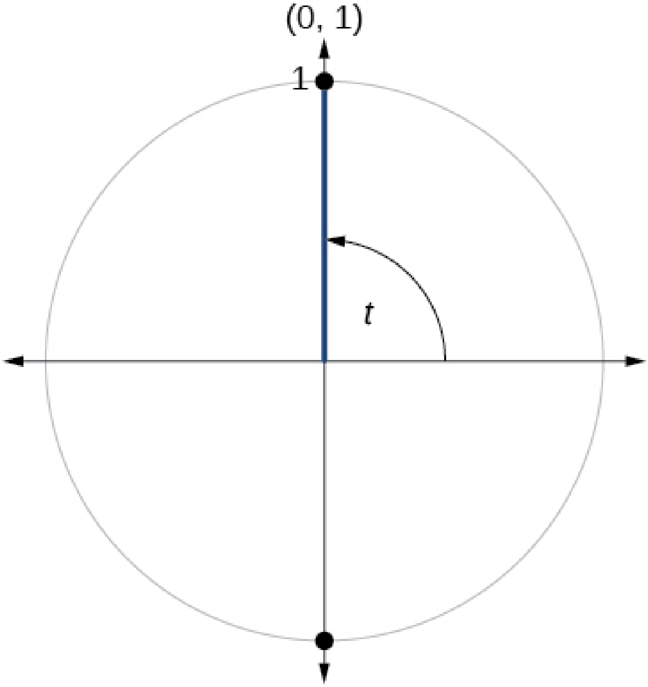




$$\sin t = 0.761, \cos t = -0.649$$



−0.948
t
−0.317
(−0.948, −0.317)



$\sin t = 1, \cos t = 0$

Technology

For the following exercises, use a graphing calculator to evaluate.

$\sin \frac{5\pi}{9}$

$\cos \frac{5\pi}{9}$

-0.1736 $\sin \pi$ 10 $\cos \pi$ 10
0.9511 $\sin 3\pi$ 4 $\cos 3\pi$ 4
-0.7071 $\sin 98^{\circ}$ $\cos 98^{\circ}$
-0.1392 $\cos 310^{\circ}$

$\sin 310^\circ$

-0.7660

Extensions

For the following exercises, evaluate.

sin(11π 3)cos(−5π 6)

sin(3π 4)cos(5π 3)

---

2 4

sin(− 4π 3)cos(π 2)

sin(−9π 4)cos(−π 6)

− 6 4

sin(π 6)cos(−π 3)

sin(7π 4)cos(−2π 3)

---

2 4

cos(5π 6)cos(2π 3)

cos(−π 3)cos(π 4)

---

2 4

sin(−5π 4)sin(11π 6)

sin(π)sin(π 6)

0

Real-World Applications

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point (0,1), that is, on the due north position. Assume the carousel revolves counter clockwise.

What are the coordinates of the child after 45 seconds?

What are the coordinates of the child after 90 seconds?

$(0,-1)$

What are the coordinates of the child after 125 seconds?

When will the child have coordinates $(0.707,-0.707)$ if the ride lasts 6 minutes? (There are multiple answers.)

37.5 seconds, 97.5 seconds, 157.5 seconds, 217.5 seconds, 277.5 seconds, 337.5 seconds

When will the child have coordinates $(-0.866,-0.5)$ if the ride lasts 6 minutes?

Glossary

cosine function
: the *x*-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity
: a corollary of the Pythagorean Theorem

stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function
: the y-value of the point on a unit circle corresponding to a given angle

The Other Trigonometric Functions

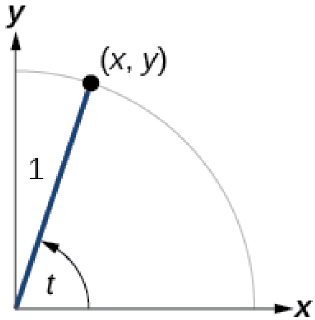
In this section you will:

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of π 3 , π 4 , and π 6 .
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is 1 12 or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

We can also define the remaining functions in terms of the unit circle with a point (x,y) corresponding to an angle of t, as shown in [link]. As with the sine and cosine, we can use the (x,y) coordinates to find the other functions.



The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In [link], the tangent of angle t is equal to y x ,x≠0. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as sint cost ,cost≠0. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In [link], the secant of angle t is equal to 1 cost = 1 x ,x≠0. The secant

function is abbreviated as sec.

- The **cotangent** function is the reciprocal of the tangent function. In [link], the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In [link], the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

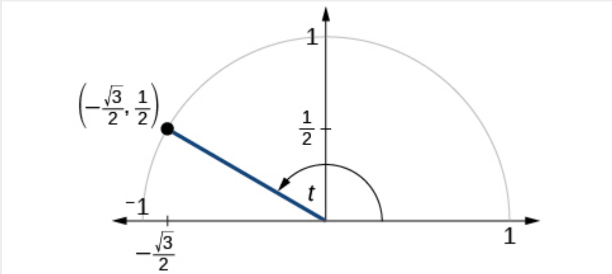
Tangent, Secant, Cosecant, and Cotangent Functions

If t is a real number and (x,y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$\tan t = \frac{y}{x}, x \neq 0$ $\sec t = \frac{1}{x}, x \neq 0$ $\csc t = \frac{1}{y}, y \neq 0$ $\cot t = \frac{x}{y}, y \neq 0$

Finding Trigonometric Functions from a Point on the Unit Circle

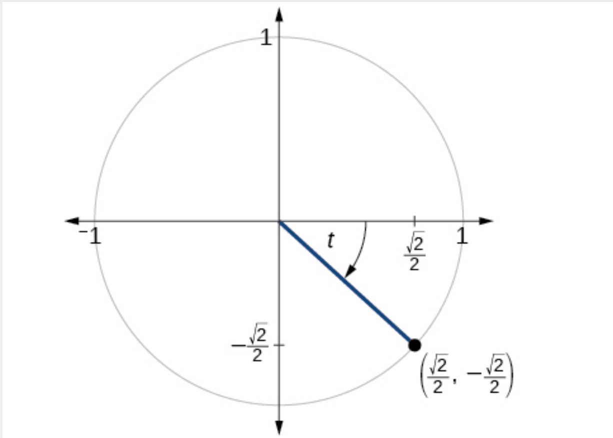
The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in [link]. Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.



Because we know the (x,y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

sin t =y = 1 2 cos t =x = − 3 2 tan t = y x = 1 2 − 3 2 = 1 2 (− 2 3) = − 1 3 = − 3 3 sec t = 1 x = 1 − 3 2 = − 2 3 = − 2 3 3 csc t = 1 y = 1 1 2 =2 cot t = x y = − 3 2 1 2 = − 3 2 (2 1) = − 3

The point (2 2 , − 2 2) is on the unit circle, as shown in [link]. Find sint,cost,tant,sect,csct, and cott.



sint= − 2 2 ,cost= 2 2 ,tant= −1,sect= 2 ,csct= − 2 ,cott= −1

Finding the Trigonometric Functions of an Angle

Find sint,cost,tant,sect,csct, and cott. when t= π 6 .

We have previously used the properties of equilateral triangles to demonstrate that sin π 6 = 1 2 and cos π 6 = 3 2 . We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

tan π 6 = sin π 6 cos π 6 = 1 2 3 2 = 1 3 = 3 3 sec π 6 = 1 cos π 6 = 1 3 2 = 2 3 = 2 3 3 csc π 6 = 1 sin π 6 = 1 1 2 =2 cot π 6 = cos π 6 sin π 6 = 3 2 1 2 = 3

Find sint,cost,tant,sect,csct, and cott. when t= π 3 .

sin π 3 = 3 2 cos π 3 = 1 2 tan π 3 = 3 sec π 3 =2 csc π 3 = 2 3 3 cot π 3 = 3 3

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in [link].

Angle	**0**	**π 6**	**π 4**	**π 3**	**π 2**

		,or 30°	,or 45°	,or 60°	,or 90°
Cosine	1	3 2	2 2	1 2	0
Sine	0	1 2	2 2	3 2	1
Tangent	0	3 3	1	3	Undefined
Secant	1	2 3 3	2	2	Undefined
Cosecant	Undefined	2	2	2 3 3	1
Cotangent	Undefined	3	1	3 3	0

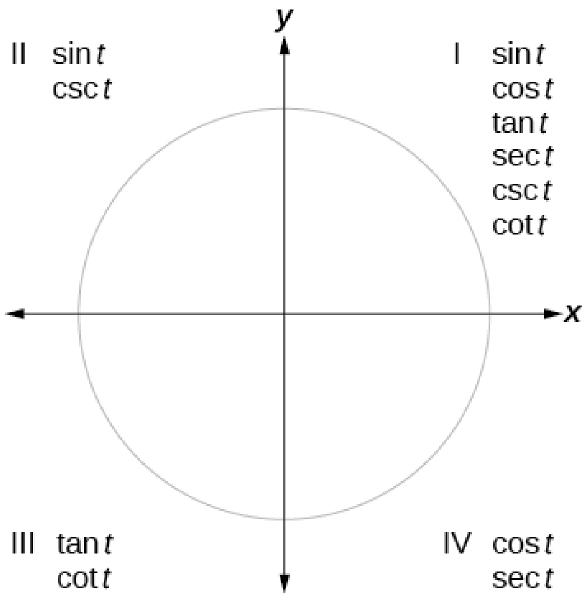
The trigonometric functions are each listed in the quadrants in which they are positive.

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by x- and y-values in the original quadrant. [link] shows which functions are positive in which quadrant.

To help remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of

the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.



Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side

of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\ 5\pi\ 6$.

The angle between this angle's terminal side and the *x*-axis is $\pi\ 6$, so that is the reference angle. Since $-\ 5\pi\ 6$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$\cos(-\ 5\pi\ 6) = -\ 3\ 2$,$\sin(-\ 5\pi\ 6) = -\ 1\ 2$,$\tan(-\ 5\pi\ 6) = 3\ 3$, $\sec(-\ 5\pi\ 6) = -\ 2\ 3\ 3$,$\csc(-\ 5\pi\ 6) = -2$,$\cot(-\ 5\pi\ 6) = 3$

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.
$\sin(-\frac{7\pi}{4})=\frac{\sqrt{2}}{2}$, $\cos(-\frac{7\pi}{4})=\frac{\sqrt{2}}{2}$, $\tan(-\frac{7\pi}{4})=1$, $\sec(-\frac{7\pi}{4})=\sqrt{2}$, $\csc(-\frac{7\pi}{4})=\sqrt{2}$, $\cot(-\frac{7\pi}{4})=1$

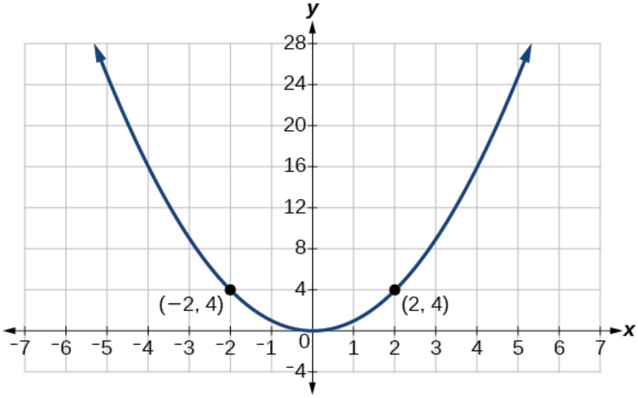
The function $f(x)=x^2$ is an even function. The function $f(x)=x^3$ is an odd function.

Using Even and Odd Trigonometric Functions

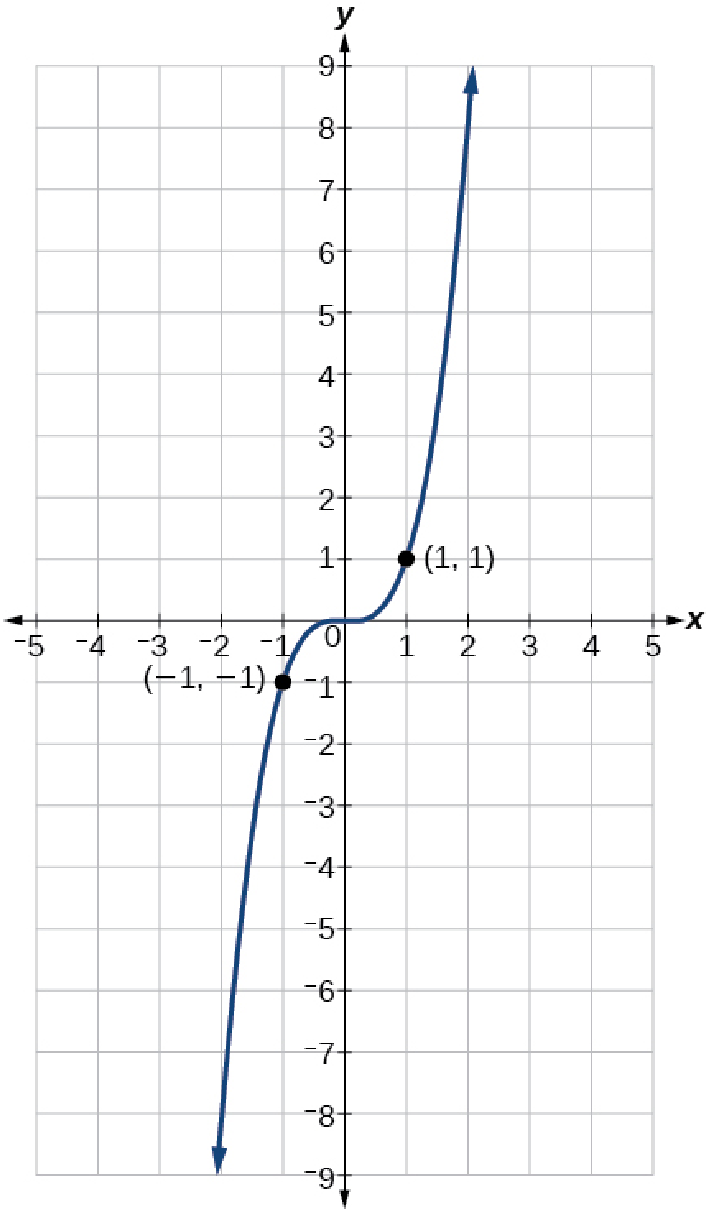
To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x)=x^2$, shown in [link]. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2=(-4)^2$, $(-5)^2=(5)^2$, and so on. So $f(x)=x^2$ is an even function, a function such that two inputs that are opposites have the same output. That means $f(-x$

$) = f(x)$.

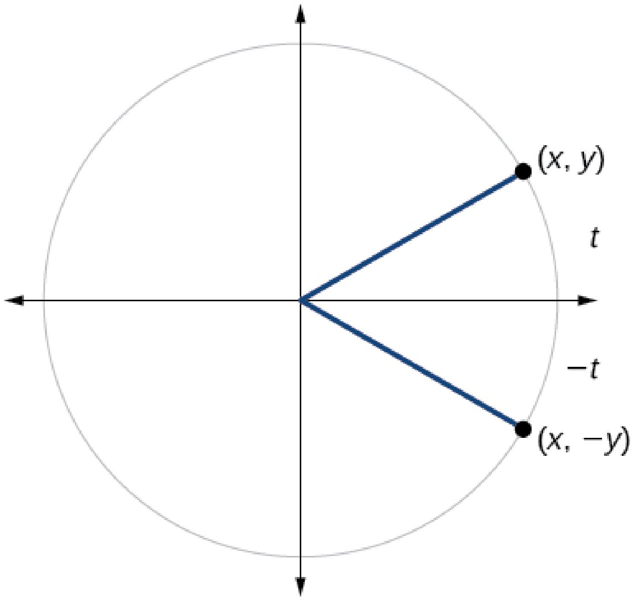


Now consider the function $f(x) = x^3$, shown in [link]. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an odd function, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.



We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in [link]. The sine of the

positive angle is y. The sine of the negative angle is −y. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in [link].



$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq \cot(-t)$

Even and Odd Trigonometric Functions

An even function is one in which $f(-x)=f(x)$.

An odd function is one in which $f(-x)=-f(x)$.

Cosine and secant are even:

$\cos(-t) = \cos t$ $\sec(-t) = \sec t$

Sine, tangent, cosecant, and cotangent are odd:

$\sin(-t) = -\sin t$ $\tan(-t) = -\tan t$ $\csc(-t) = -\csc t$ $\cot(-t) = -\cot t$

Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

If the cotangent of angle t is 3 , what is the cotangent of $-t$?

-3

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

Fundamental Identities

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$\tan t = \sin t \cos t$

$\sec t = 1 \cos t$

csct= 1 sint
cott= 1 tant = cost sint

Using Identities to Evaluate Trigonometric Functions

1. Given sin(45°)= 2 2 ,cos(45°)= 2 2 , evaluate tan(45°).
2. Given sin(5π 6)= 1 2 ,cos(5π 6)= − 3 2 , evaluate sec(5π 6).

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

1. tan(45°) = sin(45°) cos(45°) = 2 2 2 2 = 1
2. sec(5π 6) = 1 cos(5π 6) = 1 − 3 2 = −2 3 1 = −2 3 = − 2 3 3

Evaluate csc(7π 6).

-2

Using Identities to Simplify Trigonometric Expressions

Simplify $\sec t \tan t$.

We can simplify this by rewriting both functions in terms of sine and cosine.

sec t tan t = 1 cos t sin t cos t = 1 cos t · cos t sin t Multiply by the reciprocal. = 1 sin t =csc t Simplify and use the identity.

By showing that $\sec t \tan t$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$\sec t \tan t = \csc t$

Simplify $(\tan t)(\cos t)$.

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternate forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$.

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \qquad 1 + \tan^2 t = \sec^2 t$$

The other form is obtained by dividing both sides by $\sin^2 t$.

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \qquad \cot^2 t + 1 = \csc^2 t$$

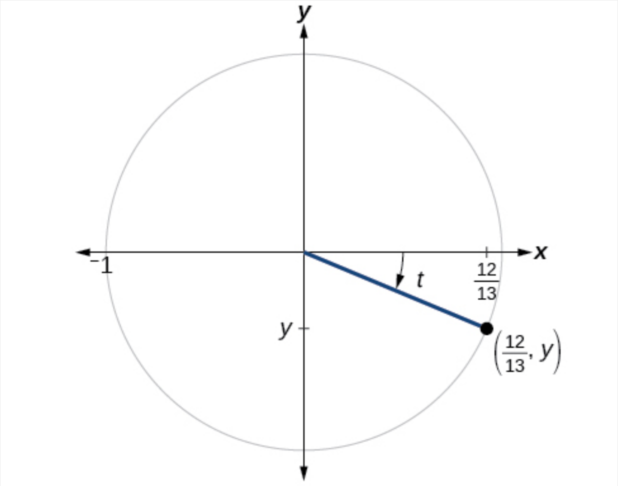
Alternate Forms of the Pythagorean Identity

$$1 + \tan^2 t = \sec^2 t$$

$$\cot^2 t + 1 = \csc^2 t$$

Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in [link], find the values of the other five trigonometric functions.



We can find the sine using the Pythagorean Identity, cos 2 t+ sin 2 t=1, and the remaining functions by relating them to sine and cosine.

(12 13) 2 + sin 2 t = 1 sin 2 t = 1− (12 13) 2 sin 2 t = 1− 144 169 sin 2 t = 25 169 sin t = ± 25 169 sin t = ± 25 169 sin t = ± 5 13

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, − 5 13 .

The remaining functions can be calculated using identities relating them to sine and cosine.

tan t = sin t cos t = − 5 13 12 13 =− 5 12

sec t = 1 cos t = 1 12 13 = 13 12 csc t = 1

sin t = 1 − 5 13 = -13 5 cot t = 1 tan t = 1 − 5 12 = − 12 5

If sec(t) = − 17 8 and 0 < t < π, find the values of the other five functions.

cost = − 8 17 ,sint = 15 17 ,tant = − 15 8 csct = 17 15 ,cott = − 8 15

As we discussed at the beginning of the chapter, a function that repeats its values in regular intervals is known as a periodic function. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π, will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and f(x) represents the number of days in February, then f(x + 4) = f(x). This pattern repeats over and over

through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

Period of a Function

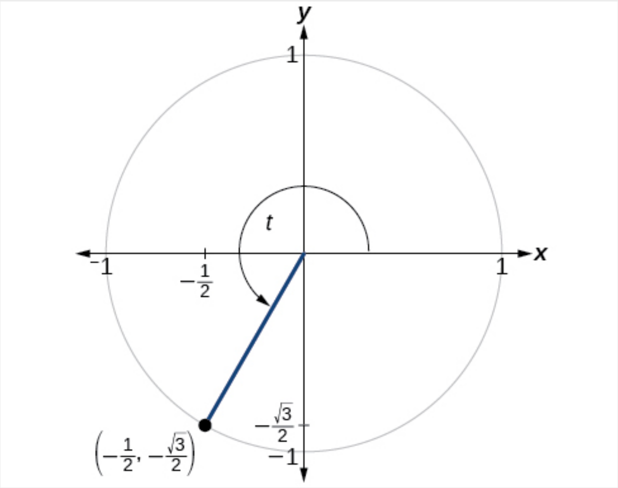
The **period** P of a repeating function f is the number representing the interval such that $f(x + P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

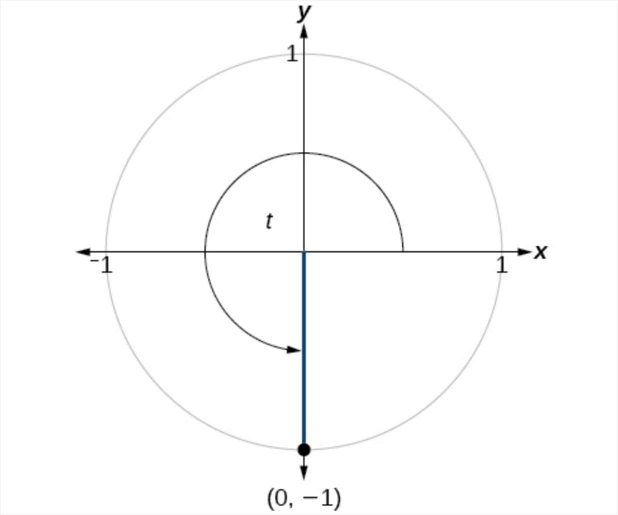
Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on [link].



sin t =y = − 3 2 cos t =x = − 1 2 tan t = sin t cos t = − 3 2 − 1 2 = 3 sec t = 1 cos t = 1 − 1 2 = −2 csc t = 1 sin t = 1 − 3 2 = − 2 3 3 cot t = 1 tan t = 1 3 = 3 3

Find the values of the six trigonometric functions of angle t based on [link].



$\sin t = -1, \cos t = 0, \tan t = \text{Undefined}$
$\sec t = \text{Undefined}, \csc t = -1, \cot t = 0$

Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t), \csc(t), \tan(t), \cot(t)$.

$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2$ $\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$ $\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$ $\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$

sin(t)= 2 2 andcos(t)= 2 2 ,findsec(t),csc(t),tan(t),andcot(t)

sect= 2 ,csct= 2 ,tant=1,cott=1

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that

case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor π 180 to convert the degrees to radians. To find the secant of 30°, we could press
(for a scientific calculator): 1 30 × π 180 COS or
(for a graphing calculator): 1 cos(30π 180)

Given an angle measure in radians, use a scientific calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1/
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

Given an angle measure in radians, use a graphing utility/calculator to find the cosecant.

- If the graphing utility has degree mode and radian mode, set it to radian mode.

- Enter: 1/
- Press the SIN key.
- Enter the value of the angle inside parentheses.
- Press the ENTER key.

Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

For a scientific calculator, enter information as follows:

$1/(5\times\pi/7)\text{ SIN } = \csc\left(\frac{5\pi}{7}\right) \approx 1.279$

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

Access these online resources for additional instruction and practice with other trigonometric

functions.

- Determing Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent

is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.

- The six trigonometric functions can be found from a point on the unit circle. See [link].
- Trigonometric functions can also be found from an angle. See [link].
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See [link].
- A function is said to be even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$ for all *x* in the domain of *f*.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See [link].
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See [link] and [link].
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See [link].
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f(x+P)=f(x)$ for any value of x.
- The values of trigonometric functions can be

found by mathematical analysis. See [link] and [link].
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See [link].

Section Exercises

Verbal

On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, a $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

What would you estimate the cosine of π degrees to be? Explain your reasoning.

For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

Describe the secant function.

Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the following exercises, find the exact value of each expression.

$\tan \frac{\pi}{6}$

$\sec \frac{\pi}{6}$

$\frac{2\sqrt{3}}{3}$

$\csc \frac{\pi}{6}$

cot π 6	
3	
tan π 4	
sec π 4	
2	
csc π 4	
cot π 4	
1	
tan π 3	
sec π 3	
2	

csc π 3

cot π 3

---

3 3

For the following exercises, use reference angles to evaluate the expression.

tan 5π 6

sec 7π 6

− 2 3 3

csc 11π 6

cot 13π 6

3

tan 7π 4

sec 3π 4

− 2

csc 5π 4

cot 11π 4

–1

tan 8π 3

sec 4π 3

-2

csc 2π 3

cot 5π 3

− 3 3

$\tan 225^\circ$

$\sec 300^\circ$

2

$\csc 150^\circ$

$\cot 240^\circ$

---

3 3

$\tan 330^\circ$

$\sec 120^\circ$

–2

$\csc 210^\circ$

$\cot 315^\circ$

–1

If sint= 3 4 , and t is in quadrant II, find cost,sect,csct,tant, and cott.

If cost= − 1 3 , and t is in quadrant III, find sint,sect,csct,tant, and cott.

---

sint= − 2 2 3 ,sect= −3,csct= − 3 2 4 ,tant=2 2 ,cott= 2 4

If tant= 12 5 , and 0≤t< π 2 , find sint,cost,sect,csct,andcott.

If sint= 3 2 and cost= 1 2 , find sect,csct,tant, and cott.

---

sect=2, csct= 2 3 3 , tant= 3 , cott= 3 3

If sin40° ≈0.643 and cos40° ≈0.766, find sec40°,csc40°,tan40°, and cot40°.

If sint= 2 2 , what is the sin(−t)?

$-\frac{\sqrt{2}}{2}$

If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

If $\sec t = 3.1$, what is the $\sec(-t)$?

3.1

If $\csc t = 0.34$, what is the $\csc(-t)$?

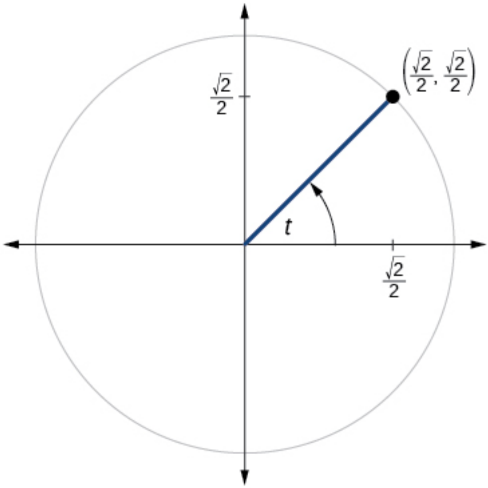
If $\tan t = -1.4$, what is the $\tan(-t)$?

1.4

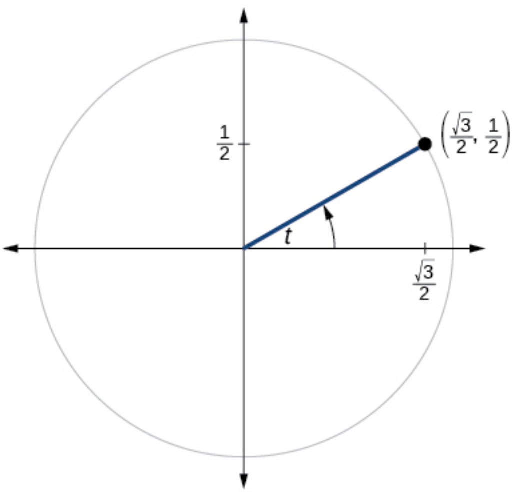
If $\cot t = 9.23$, what is the $\cot(-t)$?

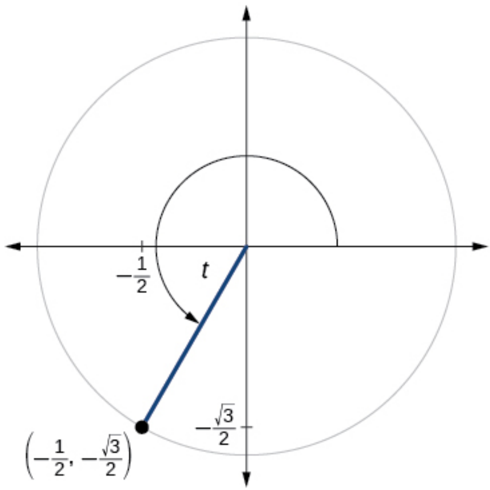
Graphical

For the following exercises, use the angle in the unit circle to find the value of the each of the six trigonometric functions.



$\sin t = 2\ 2$,$\cos t = 2\ 2$,$\tan t = 1$,$\cot t = 1$,$\sec t = 2$
,$\csc t = 2$





$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\frac{2\sqrt{3}}{3}$

Technology

For the following exercises, use a graphing calculator to evaluate to three decimal places.

$\csc \frac{5\pi}{9}$

$\cot \frac{4\pi}{7}$

–0.228

$\sec \frac{\pi}{10}$

tan 5π 8

–2.414

sec 3π 4

csc π 4

1.414

tan98°

cot33°

1.540

cot140°

sec310°

1.556

Extensions

For the following exercises, use identities to evaluate the expression.

If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

$\sin(t) \approx 0.79$

If $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

$\csc t \approx 1.16$

Determine whether the function $f(x) = 2\sin x\cos x$ is even, odd, or neither.

Determine whether the function $f(x) = 3\sin 2 x\cos x + \sec x$ is even, odd, or neither.

---

even

Determine whether the function $f(x) = \sin x - 2 \cos 2 x$ is even, odd, or neither.

Determine whether the function $f(x) = \csc 2 x + \sec x$ is even, odd, or neither.

---

even

For the following exercises, use identities to simplify the expression.

$\csc t \tan t$

$\frac{\sec t}{\csc t}$

$\frac{\sin t}{\cos t} = \tan t$

Real-World Applications

The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\frac{1}{600} d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the

period of the function.

The amount of sunlight in a certain city can be modeled by the function $h = 16\cos(1\ 500\ d)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

13.77 hours, period: 1000π

The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds. (a) Find the blood pressure after 15 seconds. (b) What are the maximum and minimum blood pressures?

The height of a piston, h, in inches, can be modeled by the equation $y = 3\sin x + 1$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

3.46 inches

The height of a piston, h, in inches, can be

modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

Chapter Review Exercises

Angles

For the following exercises, convert the angle measures to degrees.

$\frac{\pi}{4}$

$45°$

$-\frac{5\pi}{3}$

For the following exercises, convert the angle measures to radians.

$-210°$

$-\frac{7\pi}{6}$

$180°$

Find the length of an arc in a circle of radius 7 meters subtended by the central angle of $85°$.

10.385 meters

Find the area of the sector of a circle with diameter 32 feet and an angle of $3\pi 5$ radians.

For the following exercises, find the angle between $0°$ and $360°$ that is coterminal with the given angle.

$420°$

$60°$

$-80°$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal with the given angle.

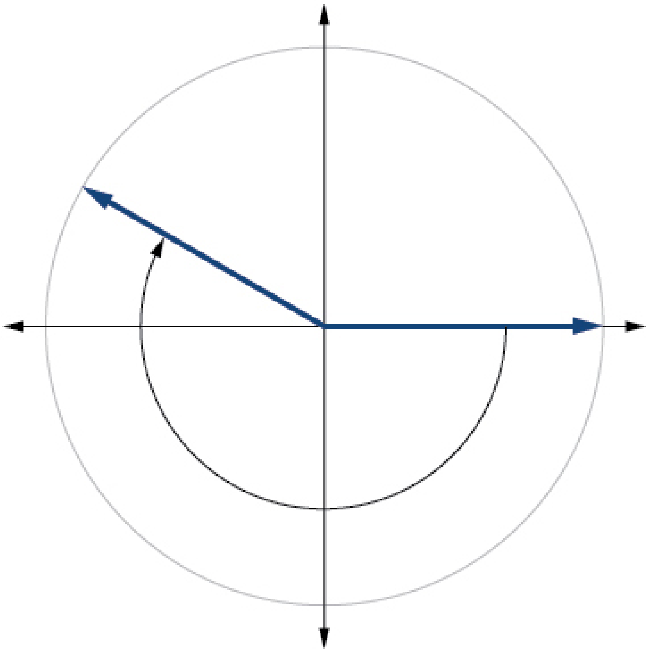
$-\ 20\pi\ 11$

2π 11

14π 5

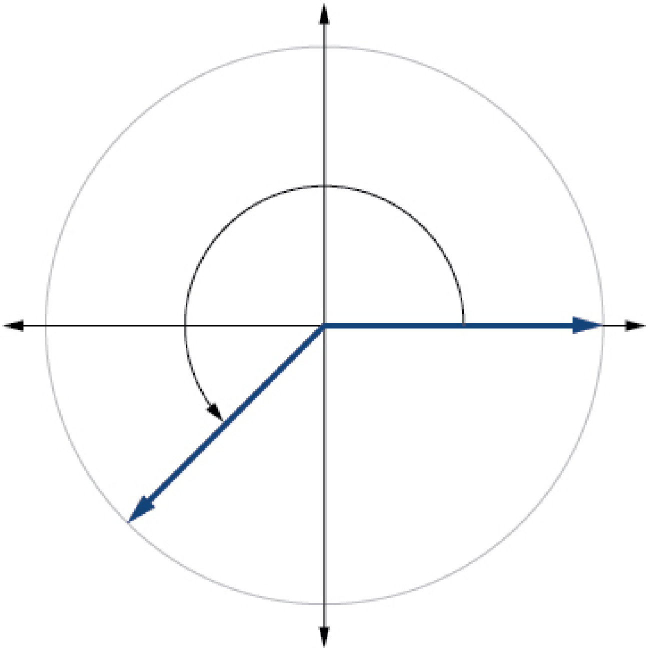
For the following exercises, draw the angle provided in standard position on the Cartesian plane.

-210°



75°

$\frac{5\pi}{4}$



$-\frac{\pi}{3}$

Find the linear speed of a point on the equator of the earth if the earth has a radius of 3,960 miles and the earth rotates on its axis every 24

hours. Express answer in miles per hour. Round to the nearest hundredth.

1036.73 miles per hour

A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour? Round to the nearest hundredth.

Right Triangle Trigonometry

For the following exercises, use side lengths to evaluate.

cos π 4

---

2 2

cot π 3

tan π 6

---

3 3

$\cos\left(\frac{\pi}{2}\right)=\sin(\_\_^\circ)$

$\csc(18^\circ)=\sec(\_\_^\circ)$

72°

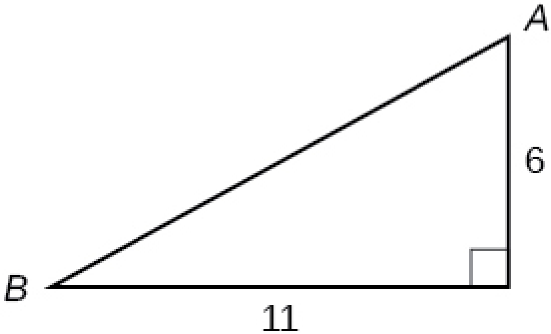
For the following exercises, use the given information to find the lengths of the other two sides of the right triangle.

$\cos B=\frac{3}{5}, a=6$

$\tan A=\frac{5}{9}, b=6$

$a=\frac{10}{3}, c=\frac{2\,106}{3}$

For the following exercises, use [link] to evaluate each trigonometric function.



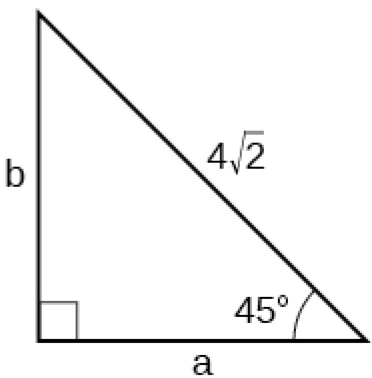
sinA

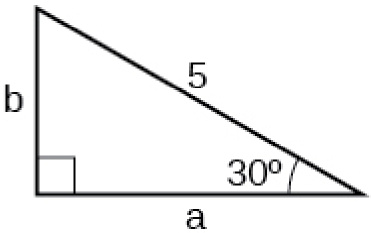
tanB

---

6 11

For the following exercises, solve for the unknown sides of the given triangle.





---

a= 5 3 2 ,b= 5 2

A 15-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building? Find the answer to four decimal places.

The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building. Find the answer to four decimal places.

369.2136 ft

Unit Circle

Find the exact value of sin π 3 .

Find the exact value of cos π 4 .

---

2 2

Find the exact value of cosπ.

State the reference angle for 300°.

60°

State the reference angle for 3π 4 .

Compute cosine of 330°.

---

3 2

Compute sine of 5π 4 .

State the domain of the sine and cosine functions.

---

all real numbers

State the range of the sine and cosine functions.

The Other Trigonometric Functions

For the following exercises, find the exact value of the given expression.

cos π 6

---

3 2

tan π 4

csc π 3

---

2 3 3

sec π 4

For the following exercises, use reference angles to evaluate the given expression.

sec $11\pi\ 3$

2

sec$315°$

If $\sec(t) = -2.5$, what is the $\sec(-t)$?

–2.5

If $\tan(t) = -0.6$, what is the $\tan(-t)$?

If $\tan(t) = 1\ 3$, find $\tan(t-\pi)$.

$1\ 3$

If $\cos(t) = 2\ 2$, find $\sin(t+2\pi)$. There are two possible solutions.

Which trigonometric functions are even?

---

cosine, secant

Which trigonometric functions are odd?

Chapter Practice Test

Convert $\frac{5\pi}{6}$ radians to degrees.

150°

Convert -620° to radians.

Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of 30°.

6.283 centimeters

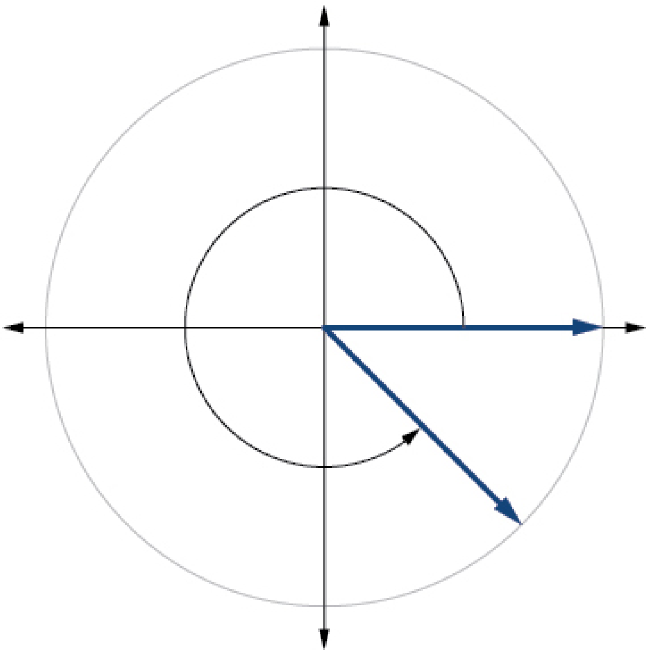
Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

Find the angle between 0° and 360° that is coterminal with 375°.

15°

Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

Draw the angle 315° in standard position on the Cartesian plane.



Draw the angle $-\frac{\pi}{6}$ in standard position on

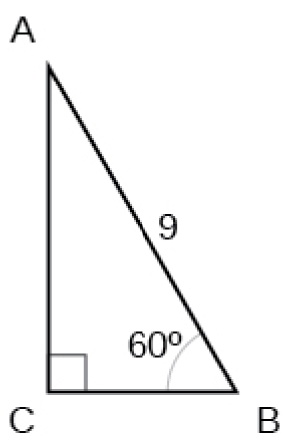
the Cartesian plane.

A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

3.351 feet per second, 2π 75 radians per second

Find the missing sides of the triangle ABC:$\sin B =$ 3 4 ,$c = 12$.

Find the missing sides of the triangle.



a= 9 2 ,b= 9 3 2

The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2000 feet from the base of the building. Using this information, find the height of the building.

Find the exact value of sin π 6 .

---

1 2

Compute sine of 240°.

State the domain of the sine and cosine functions.

---

real numbers

State the range of the sine and cosine functions.

Find the exact value of cot π 4 .

1

Find the exact value of $\tan \pi\ 3$.

Use reference angles to evaluate $\csc 7\pi\ 4$.

$-\ 2$

Use reference angles to evaluate $\tan 210^\circ$.

If $\csc t = 0.68$, what is the $\csc(-t)$?

–0.68

If $\cos t =\ 3\ 2$, find $\cos(t - 2\pi)$.

Find the missing angle: $\cos(\ \pi\ 6\ ) = \sin(\ \_\_\ )$

$\pi\ 3$

Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = 1 \; y$, $y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = x \; y$, $y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x+P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = 1 \; x$, $x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = y \; x$, $x \neq 0$

Graphs of the Sine and Cosine Functions

In this section, you will:

- Graph variations of $y=\sin(x)$ and $y=\cos(x)$.
- Use phase shifts of sine and cosine curves.

Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a

rainbow.

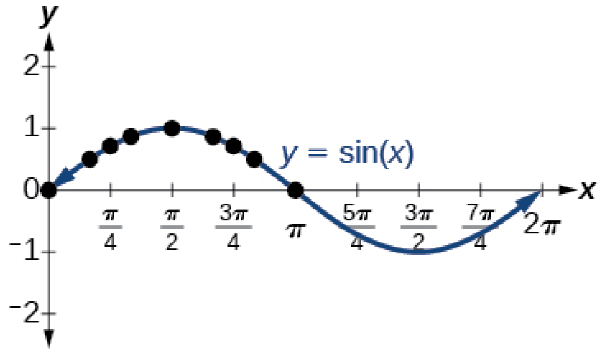
Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.
The sine function Plotting values of the sine function The cosine functionOdd symmetry of the sine function Even symmetry of the cosine function

Graphing Sine and Cosine Functions

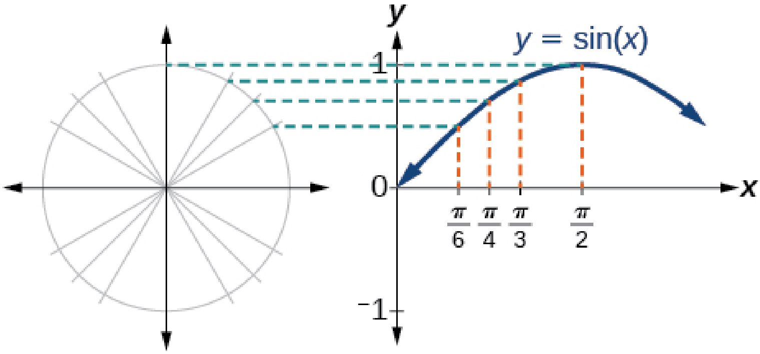
Recall that the sine and cosine functions relate real number values to the *x*- and *y*-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. [link] lists some of the values for the sine function on a unit circle.

x	0	π 6	π 4	π 3	π 2	2π 3	3π 4	5π 6	π
sin(x)	0	1 2	2 2	3 2	1	3 2	2 2	1 2	0

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function. See [link].



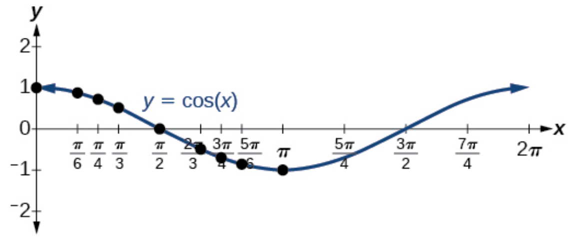
Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See [link].



Now let’s take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. [link] lists some of the values for the cosine function on a unit circle.

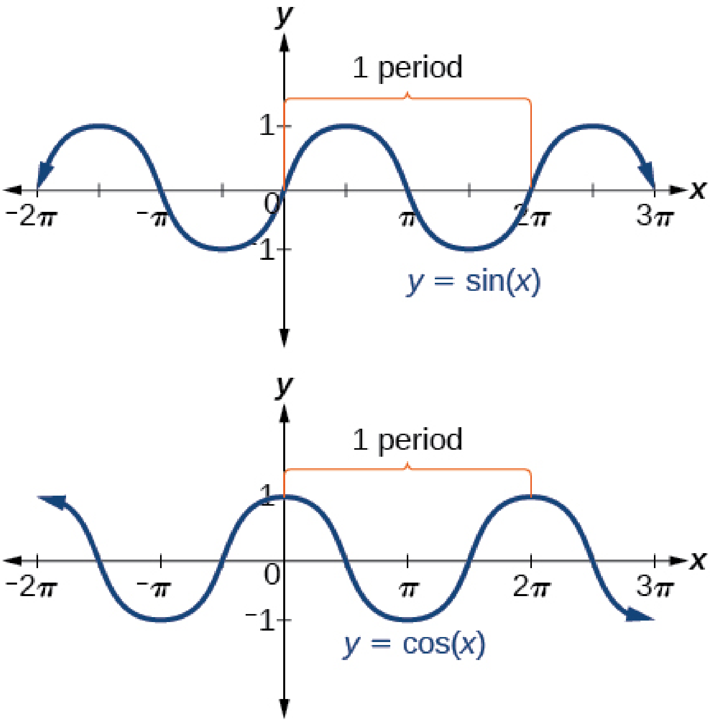
x	0	π 6	π 4	π 3	π 2	2π 3	3π 4	5π 6	π
cos(x)	1	3 2	2 2	1 2	0	− 1 2	− 2 2	− 3 2	−1

As with the sine function, we can plots points to create a graph of the cosine function as in [link].

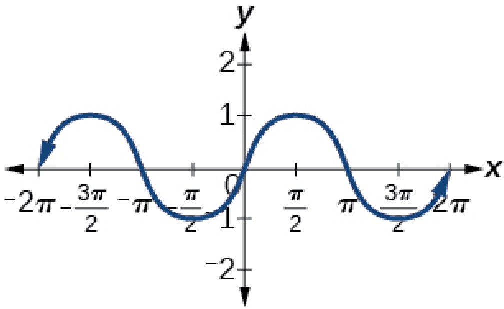


Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[\ -1,1\ ]$.

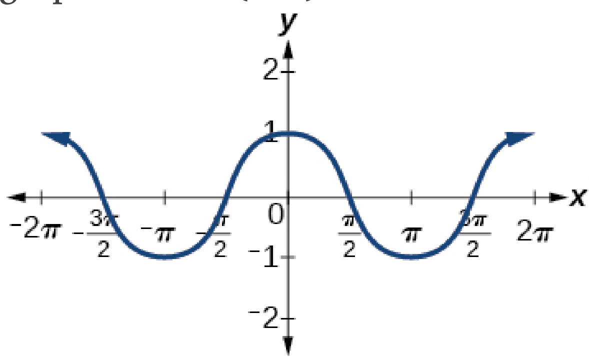
In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific horizontal shift, *P*, results in a function equal to the original function: $f(\ x+P\ )=f(\ x\ )$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P>0$ the period of the function. [link] shows several periods of the sine and cosine functions.



Looking again at the sine and cosine functions on a domain centered at the *y*-axis helps reveal symmetries. As we can see in [link], the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin x$. Now we can clearly see this property from the graph.



[link] shows that the cosine function is symmetric about the *y*-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos x$.



Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.

- The graph of $y = \sin x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos x$ is symmetric about the y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

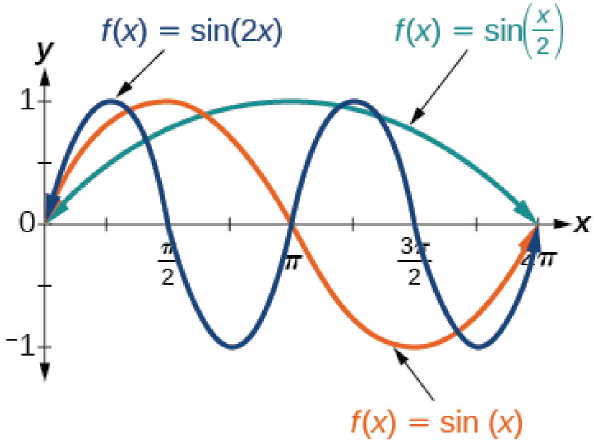
$y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by

$P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π, which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in [link] how the period is indirectly related to $|B|$.



Period of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx)$$

$$y = A\cos(Bx)$$

The period is $\frac{2\pi}{|B|}$.

Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x)=\sin\left(\frac{\pi}{6}x\right)$.

Let's begin by comparing the equation to the general form $y=A\sin(Bx)$.

In the given equation, $B=\frac{\pi}{6}$, so the period will be

$$P=\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{6}}=2\pi\cdot\frac{6}{\pi}=12$$

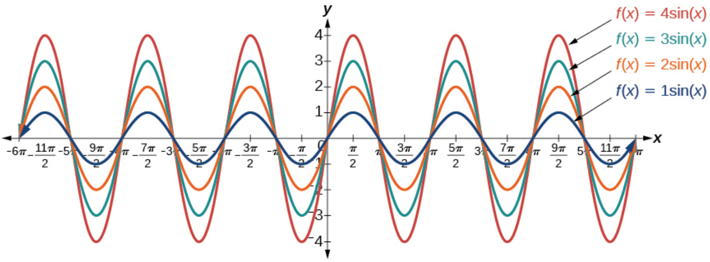
Determine the period of the function $g(x)=\cos\left(\frac{x}{3}\right)$.

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable

A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the horizontal **midline** of the graph, which is the line $y=D$; because $D=0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A|>1$, the function is stretched. For example, the amplitude of $f(x)=4\sin x$ is twice the amplitude of $f(x)=2\sin x$. If $|A|<1$, the function is compressed. [link] compares several sine functions with different amplitudes.



Amplitude of Sinusoidal Functions

If we let $C=0$ and $D=0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y=A\sin(Bx) \text{ and } y=A\cos(Bx)$$

The **amplitude** is $|A|$, which is the vertical height from the **midline** . In addition, notice in the example that

$$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Identifying the Amplitude of a Sine or Cosine Function

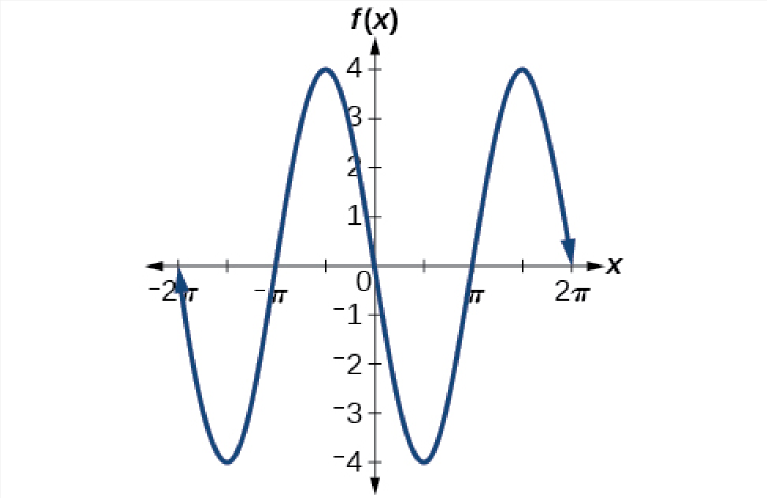
What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the *x*-axis of the sine function, as shown in [link].



What is the amplitude of the sinusoidal function f(x)= 1 2 sin(x)? Is the function stretched or compressed vertically?

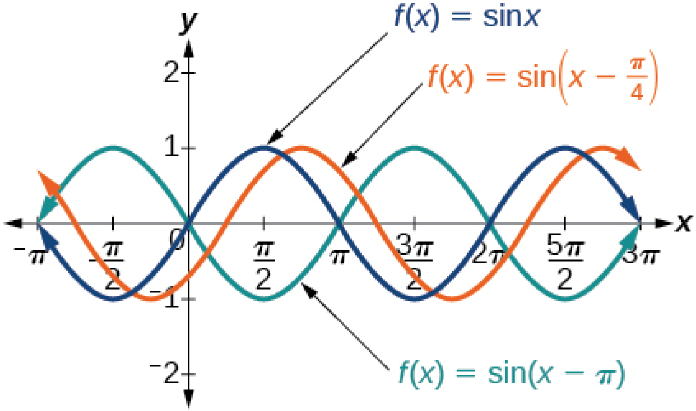
1 2 compressed

Analyzing Graphs of Variations of $y = \sin x$ and $y = \cos x$

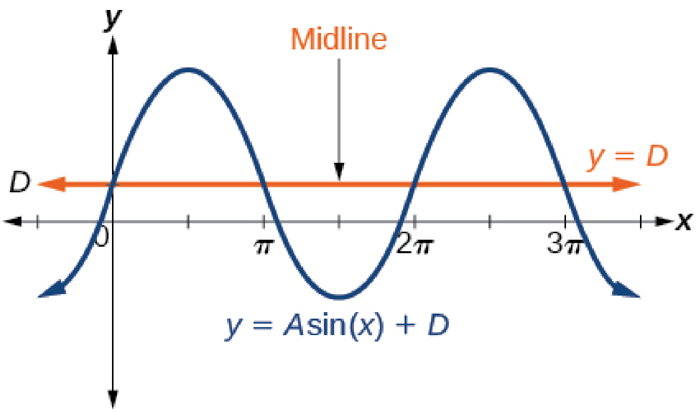
Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$y=A\sin(Bx-C)+D$ and $y=A\cos(Bx-C)+D$ or $y=A\sin\left(B\left(x-\frac{C}{B}\right)\right)+D$ and $y=A\cos\left(B\left(x-\frac{C}{B}\right)\right)+D$

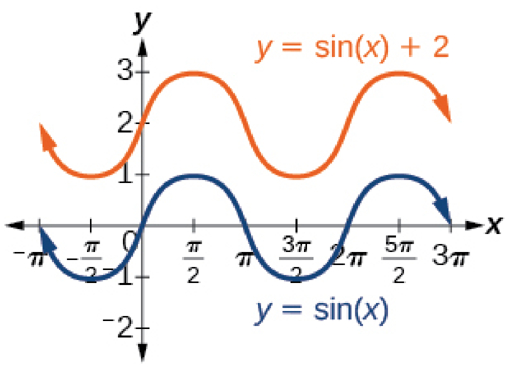
The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C>0$, the graph shifts to the right. If $C<0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. [link] shows that the graph of $f(x)=\sin(x-\pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x)=\sin\left(x-\frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.



While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See [link]. The function $y=\cos(x)+D$ has its midline at $y=D$.



Any value of D other than zero shifts the graph up or down. [link] compares $f(x)=\sin(x)$ with $f(x)=\sin(x)+2$, which is shifted 2 units up on a graph.



Variations of Sine and Cosine Functions

Given an equation in the form $f(\ x\ )=A\sin(\ Bx-C\ )+D$ or $f(\ x\ )=A\cos(\ Bx-C\ )+D$, $C\ B$ is the **phase shift** and D is the vertical shift.

Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x)=\sin\left(x+\frac{\pi}{6}\right)-2$.

Let’s begin by comparing the equation to the general form $y=A\sin(Bx-C)+D$.

In the given equation, notice that $B=1$ and $C=-\frac{\pi}{6}$. So the phase shift is

$$\frac{C}{B}=-\frac{\frac{\pi}{6}}{1}=-\frac{\pi}{6}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x)=\sin\left(x+\frac{\pi}{6}\right)-2$ can be rewritten as $f(x)=\sin\left(x-\left(-\frac{\pi}{6}\right)\right)-2$. If the value of C is negative, the shift is to the left.

Determine the direction and magnitude of the phase shift for $f(x)=3\cos\left(x-\frac{\pi}{2}\right)$.

$\frac{\pi}{2}$; right

Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x)=\cos(x)-3$.

Let’s begin by comparing the equation to the general form $y=A\cos(Bx-C)+D$.

In the given equation, $D=-3$ so the shift is 3 units downward.

Determine the direction and magnitude of the vertical shift for $f(x)=3\sin(x)+2$.

2 units up

Given a sinusoidal function in the form $f(x)=A\sin(Bx-C)+D$, **identify the midline,**

amplitude, period, and phase shift.

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

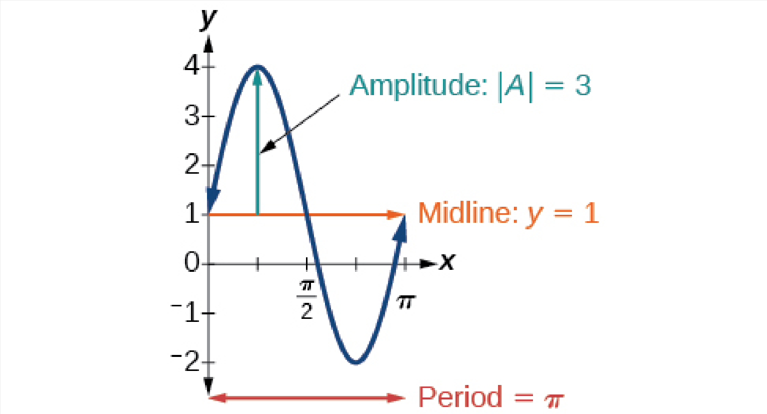
Next, $B = 2$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\frac{C}{B} = \frac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y=1$, and the amplitude is 3. See [link].

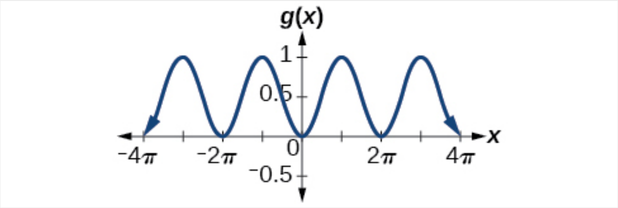


Determine the midline, amplitude, period, and phase shift of the function $y=\frac{1}{2}\cos\left(\frac{x}{3}-\frac{\pi}{3}\right)$.

midline: $y=0$; amplitude: $|A|=\frac{1}{2}$; period: $P=\frac{2\pi}{|B|}=6\pi$; phase shift: $\frac{C}{B}=\pi$

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in [link].



To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$y = A\sin(Bx - C) + D$ $y = A\cos(Bx - C) + D$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

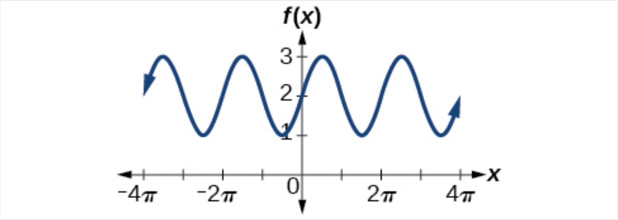
The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the

amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the *x*-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B=1$; and the graph is not shifted horizontally, so $C=0$.

Putting this all together,
$g(x) = -0.5\cos(x) + 0.5$

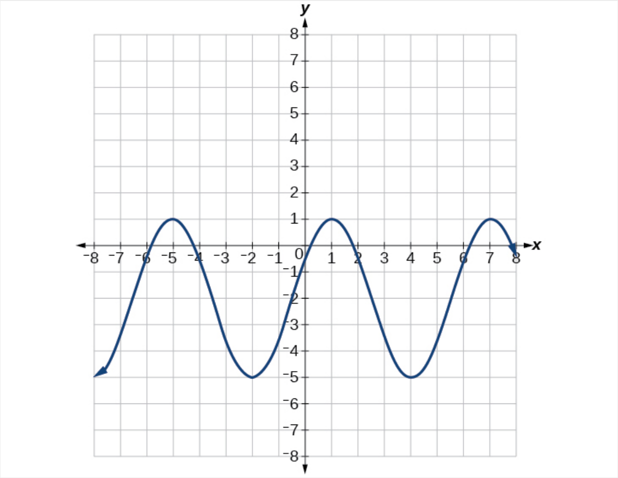
Determine the formula for the sine function in [link].



$f(x) = \sin(x) + 2$

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in [link].



With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D=-2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3$.

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7$, or from the distance between the lowest points. Therefore, $P=\frac{2\pi}{|B|}=6$. Using the positive value for B, we find that

$$B=\frac{2\pi}{P}=\frac{2\pi}{6}=\frac{\pi}{3}$$

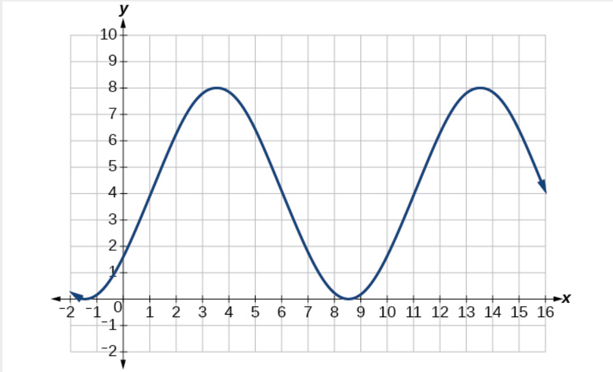
So far, our equation is either $y=3\sin\left(\frac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\frac{\pi}{3}x-C\right)-2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes $y=3\cos\left(\frac{\pi}{3}x-\frac{\pi}{3}\right)-2$ or $y=-3\cos\left(\frac{\pi}{3}x+\frac{2\pi}{3}\right)-2$

Again, these functions are equivalent, so both yield the same graph.

Write a formula for the function graphed in [link].



two possibilities: $y=4\sin\left(\frac{\pi}{5}x-\frac{\pi}{5}\right)+4$ or $y=-4\sin\left(\frac{\pi}{5}x+\frac{4\pi}{5}\right)+4$

Graphing Variations of $y=\sin x$ and $y=\cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations $y=A\sin(Bx-C)+D$ and $y=A\cos(Bx-C)+D$,

we will let $C=0$ and $D=0$ and work with a

simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $| A |$.
2. Identify the period, $P = 2\pi \ | B |$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \pi \ 2| B |$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \pi \ | B |$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = 3\pi \ 2| B |$ with $y = –A$.
7. The curve returns again to the *x*-axis at $x = 2\pi \ | B |$.

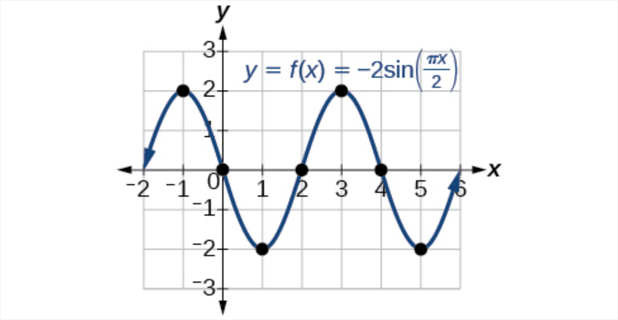
Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin(\pi x \ 2)$.

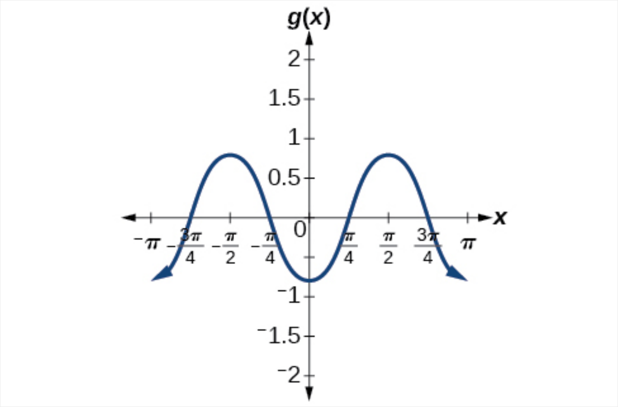
Let’s begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A=-2$, so the amplitude is 2.
  $|A|=2$
- *Step 2.* The equation shows that $B=\frac{\pi}{2}$, so the period is
  $P=\frac{2\pi}{\frac{\pi}{2}}=2\pi\cdot\frac{2}{\pi}=4$
- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4–7.* The *x*-intercepts are at the beginning of one period, $x=0$, the horizontal midpoints are at $x=2$ and at the end of one period at $x=4$.

The quarter points include the minimum at $x=1$ and the maximum at $x=3$. A local minimum will occur 2 units below the midline, at $x=1$, and a local maximum will occur at 2 units above the midline, at $x=3$. [link] shows the graph of the function.



Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.



midline: $y=0$; amplitude: $|A|=0.8$; period: $P=\frac{2\pi}{|B|}=\pi$; phase shift: $\frac{C}{B}=0$ or none

Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form $y=A\sin(Bx-C)+D$ or $y=A\cos(Bx-C)+D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P=\frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $f(x)=A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Graphing a Transformed Sinusoid

Sketch a graph of $f(x)=3\sin\left(\frac{\pi}{4}x-\frac{\pi}{4}\right)$.

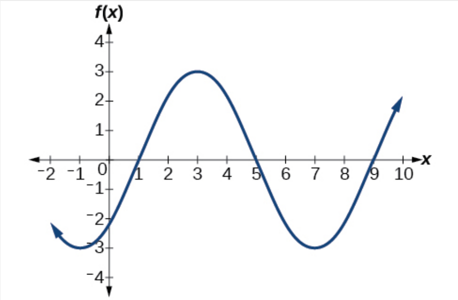
- *Step 1.* The function is already written in general form: $f(x)=3\sin\left(\frac{\pi}{4}x-\frac{\pi}{4}\right)$. This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A|=|3|=3$. The amplitude is 3.
- *Step 3.* Since $|B|=\left|\frac{\pi}{4}\right|=\frac{\pi}{4}$, we determine the period as follows.

  $$P=\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{4}}=2\pi\cdot\frac{4}{\pi}=8$$

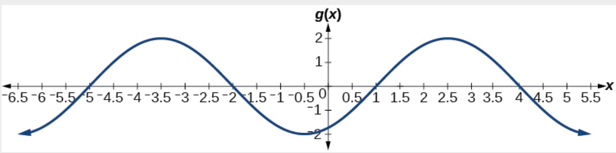
  The period is 8.
- *Step 4.* Since $C=\frac{\pi}{4}$, the phase shift is

  $$\frac{C}{B}=\frac{\frac{\pi}{4}}{\frac{\pi}{4}}=1.$$

  The phase shift is 1 unit.
- *Step 5.* [link] shows the graph of the function. A horizontally compressed, vertically stretched, and horizontally shifted sinusoid



Draw a graph of $g(x)=-2\cos\left(\frac{\pi}{3}x+\frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.



midline: $y=0$; amplitude: $|A|=2$; period: $P=\frac{2\pi}{|B|}=6$; phase shift: $\frac{C}{B}=-\frac{1}{2}$

Identifying the Properties of a Sinusoidal Function

Given $y=-2\cos\left(\frac{\pi}{2}x+\pi\right)+3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

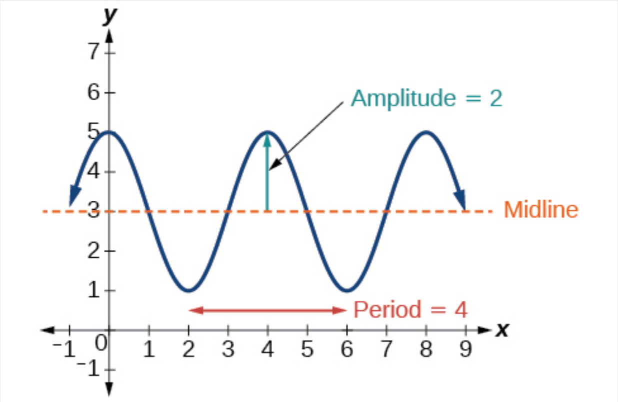
Begin by comparing the equation to the general form and use the steps outlined in [link].

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the *x*-axis.

[link] shows one cycle of the graph of the function.



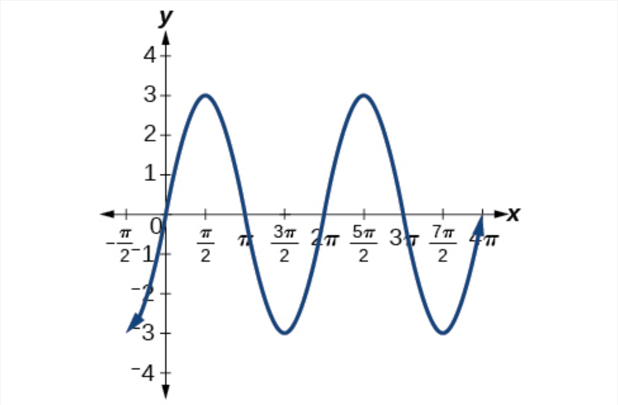
Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the *y*-coordinate of the point as a function of the angle of rotation.

Recall that, for a point on a circle of radius *r*, the *y*-coordinate of the point is $y=r\sin(x)$, so in this case, we get the equation $y(x)=3\sin(x)$. The constant 3 causes a vertical stretch of the *y*-values of the function by a factor of 3, which we can see in the graph in [link].

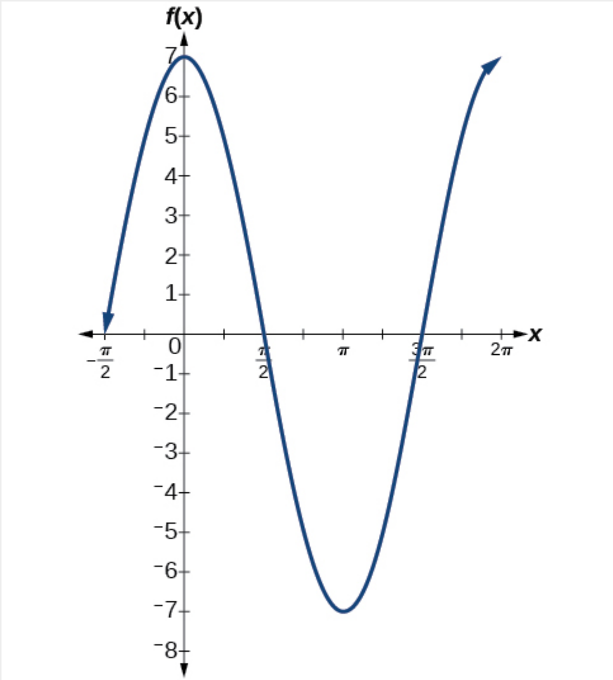


Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3,0)$ for $x=2\pi,4\pi,6\pi,\ldots$ Because the outputs of the graph will now oscillate between –3 and 3, the amplitude of the sine wave is 3.

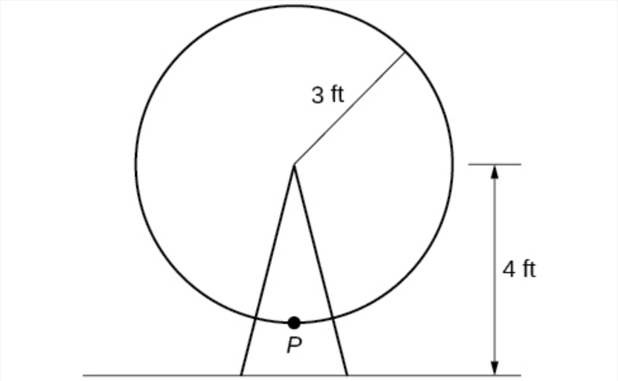
What is the amplitude of the function $f(x)=7\cos(x)$? Sketch a graph of this function.

7

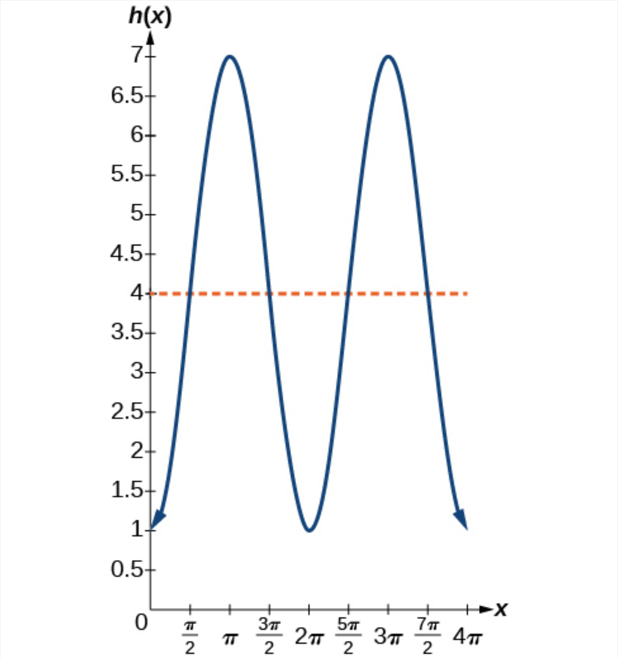


Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled *P*, as shown in [link]. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.



Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in [link].



Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let’s use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

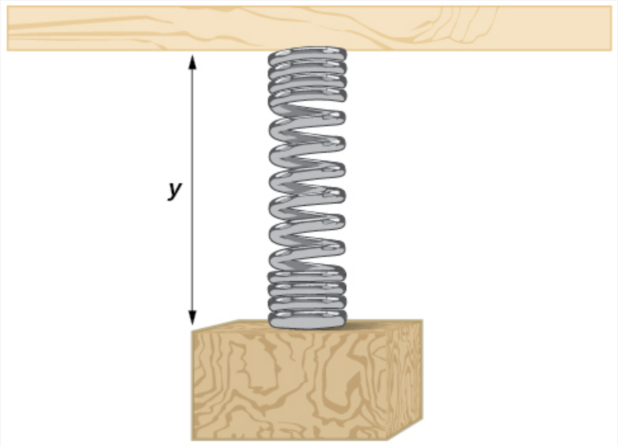
Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

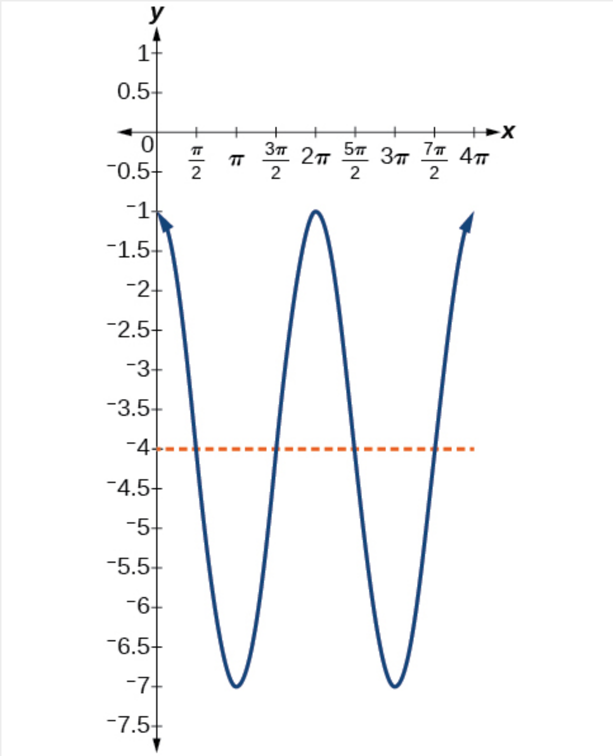
$$y=-3\cos(x)+4$$

A weight is attached to a spring that is then hung from a board, as shown in [link]. As the spring oscillates up and down, the position y of the weight relative to the board ranges from

–1 in. (at time $x = 0$) to –7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.



$y = 3\cos(x) - 4$



Determining a Rider’s Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider’s height above ground as a function of time in minutes.

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5+2=69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A=67.5$
- Midline: 69.5, so $D=69.5$
- Period: 30, so $B=\frac{2\pi}{30}=\frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider’s height would be
$y=-67.5\cos\left(\frac{\pi}{15}t\right)+69.5$

where t is in minutes and y is measured in meters.

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x)=A\sin(Bx-C)+D$ $f(x)=A\cos(Bx-C)+D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.

- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$. See [link].
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See [link].
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See [link].
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See [link].
- Combinations of variations of sinusoidal functions can be detected from an equation. See [link].
- The equation for a sinusoidal function can be determined from a graph. See [link] and [link].
- A function can be graphed by identifying its amplitude and period. See [link] and [link].
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See [link].
- Sinusoidal functions can be used to solve real-world problems. See [link], [link], and [link].

Section Exercises

Verbal

Why are the sine and cosine functions called periodic functions?

The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the *x*-axis.

How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

For the equation $A\cos(Bx+C)+D$, what constants affect the range of the function and how do they affect the range?

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph

vertically.

How does the range of a translated sine function relate to the equation $y = A\sin(Bx + C) + D$?

How can the unit circle be used to construct the graph of $f(t) = \sin t$?

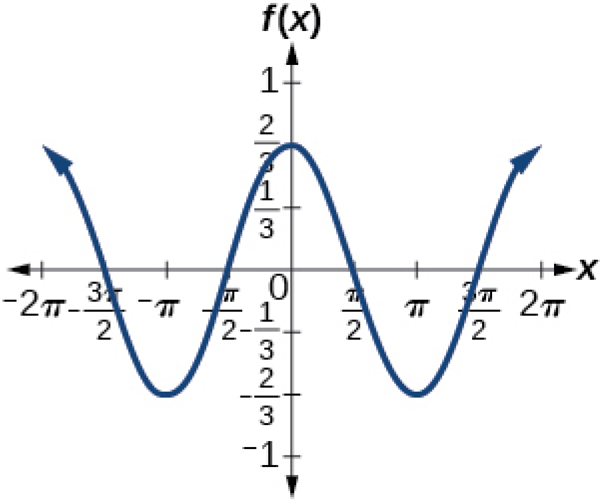
At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the *y*-coordinate of the point.

Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum *y*-values and their corresponding *x*-values on one period for $x > 0$. Round answers to two decimal places if necessary.

$f(x) = 2\sin x$

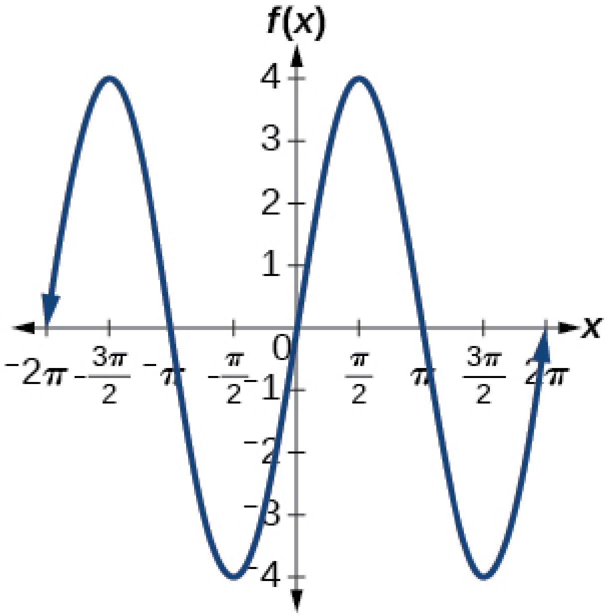
$f(x) = \ 2\ 3\ \cos x$



amplitude: 2 3 ; period: 2π; midline: $y=0$; maximum: $y=$ 2 3 occurs at $x=0$; minimum: $y=-$ 2 3 occurs at $x=\pi$; for one period, the graph starts at 0 and ends at 2π

$f(x)=-3\sin x$

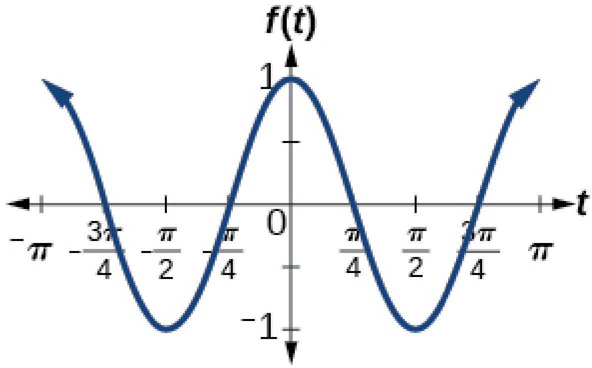
$f(x)=4\sin x$



amplitude: 4; period: 2π; midline: $y=0$; maximum $y=4$ occurs at $x=\frac{\pi}{2}$; minimum: $y=-4$ occurs at $x=\frac{3\pi}{2}$; one full period occurs from $x=0$ to $x=2\pi$

$f(x)=2\cos x$

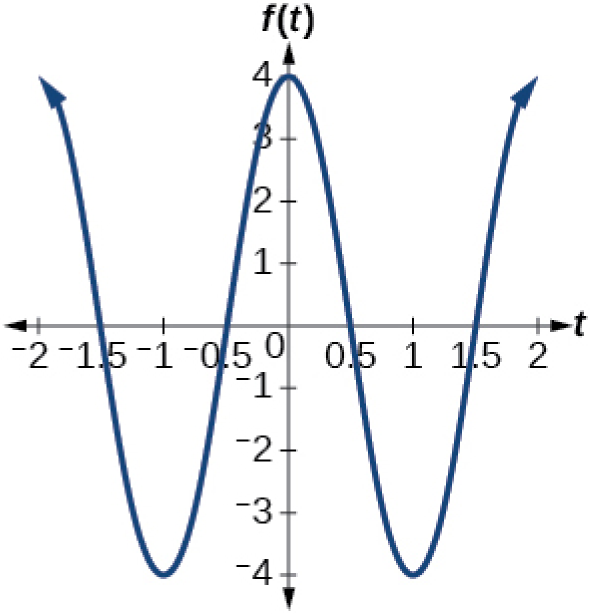
$f(x)=\cos(2x)$



amplitude: 1; period: π; midline: $y = 0$; maximum: $y = 1$ occurs at $x = \pi$; minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$; one full period is graphed from $x = 0$ to $x = \pi$

$f(x) = 2\sin\left(\frac{1}{2}x\right)$

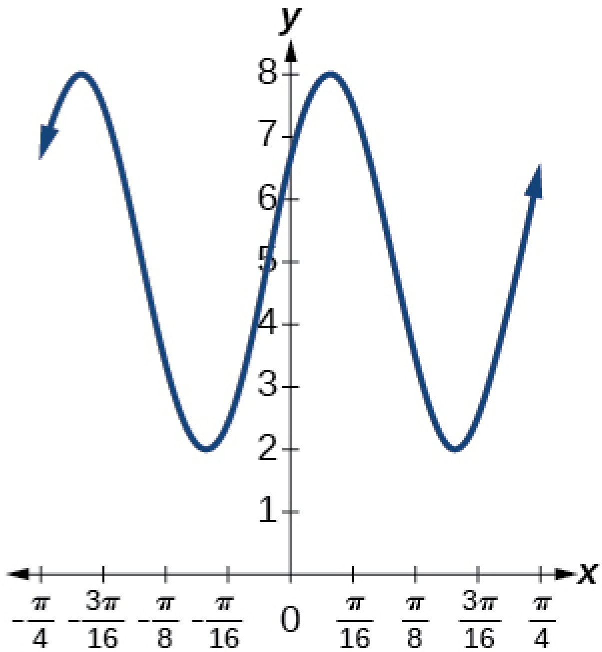
$f(x) = 4\cos(\pi x)$



amplitude: 4; period: 2; midline: $y = 0$; maximum: $y = 4$ occurs at $x = 0$; minimum: $y = -4$ occurs at $x = 1$

$f(x) = 3\cos(\ 6\ 5\ x\ )$

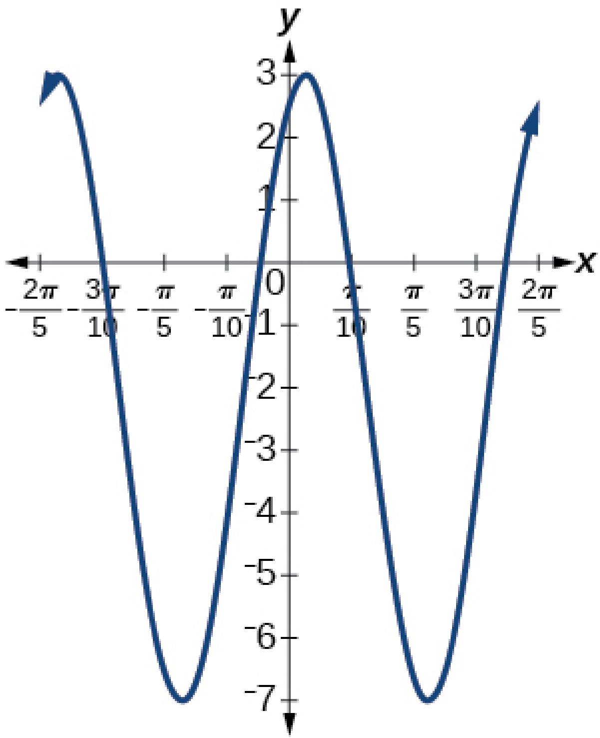
$y = 3\sin(8(x + 4)) + 5$



amplitude: 3; period: π 4 ; midline: $y=5$; maximum: $y=8$ occurs at $x=0.12$; minimum: $y=2$ occurs at $x=0.516$; horizontal shift: -4; vertical translation 5; one period occurs from $x=0$ to $x=$ π 4

$y=2\sin(3x-21)+4$

$y=5\sin(5x+20)-2$

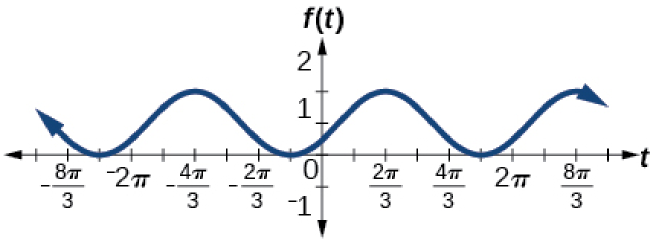


amplitude: 5; period: 2π 5 ; midline: $y=-2$; maximum: $y=3$ occurs at $x=0.08$; minimum: $y=-7$ occurs at $x=0.71$; phase shift: -4; vertical translation: -2; one full period can be graphed on $x=0$ to $x=$ 2π 5

For the following exercises, graph one full period of each function, starting at $x=0$. For each function, state the amplitude, period, and midline. State the maximum and minimum *y*-values and their corresponding *x*-values on one period for $x>0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

$f(t)=2\sin\left(t-\frac{5\pi}{6}\right)$

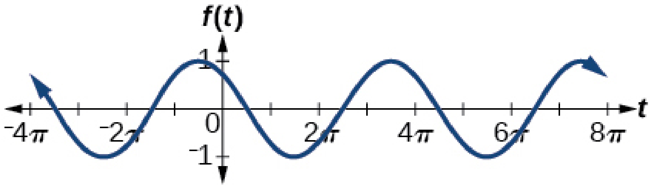
$f(t)=-\cos\left(t+\frac{\pi}{3}\right)+1$



amplitude: 1 ; period: 2π; midline: $y=1$; maximum: $y=2$ occurs at $x=2.09$; maximum: $y=2$ occurs at $t=2.09$; minimum: $y=0$ occurs at $t=5.24$; phase shift: $-\frac{\pi}{3}$; vertical translation: 1; one full period is from $t=0$ to $t=2\pi$

$f(t)=4\cos\left(2\left(t+\frac{\pi}{4}\right)\right)-3$

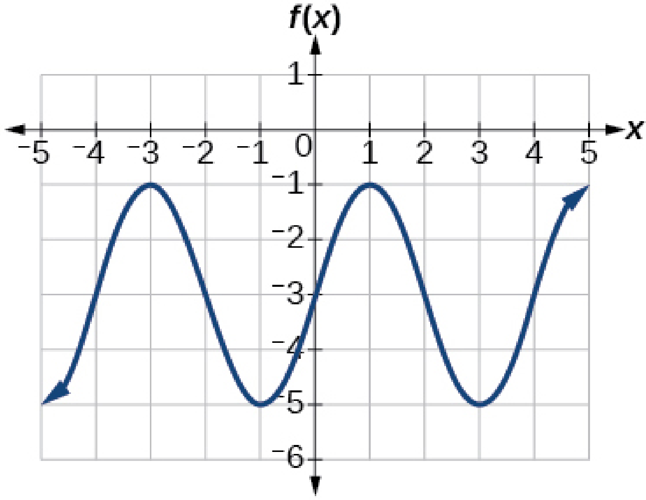
$f(t)=-\sin\left(\frac{1}{2}t+\frac{5\pi}{3}\right)$



amplitude: 1; period: 4π; midline: $y=0$; maximum: $y=1$ occurs at $t=11.52$; minimum: $y=-1$ occurs at $t=5.24$; phase shift: $-\frac{10\pi}{3}$; vertical shift: 0

$f(x)=4\sin\left(\frac{\pi}{2}(x-3)\right)+7$

Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in [link].



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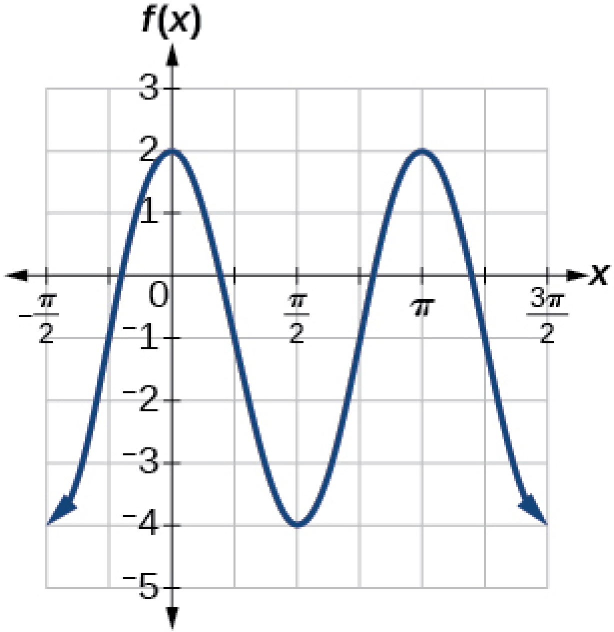
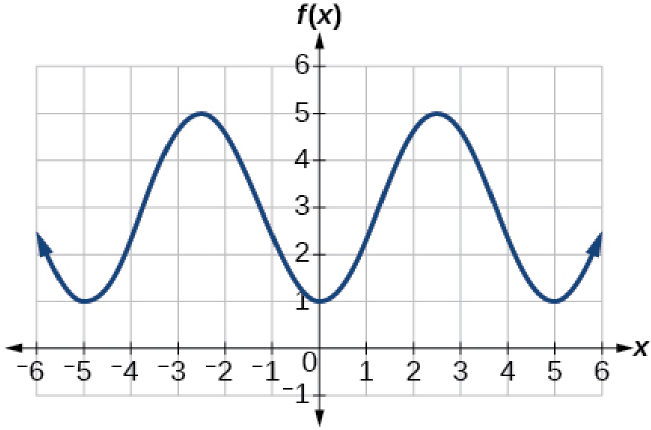
amplitude: 2; midline: $y=-3$; period: 4; equation: $f(x)=2\sin\left(\frac{\pi}{2}x\right)-3$

Determine the amplitude, period, midline, and an equation involving cosine for the graph

shown in [link].

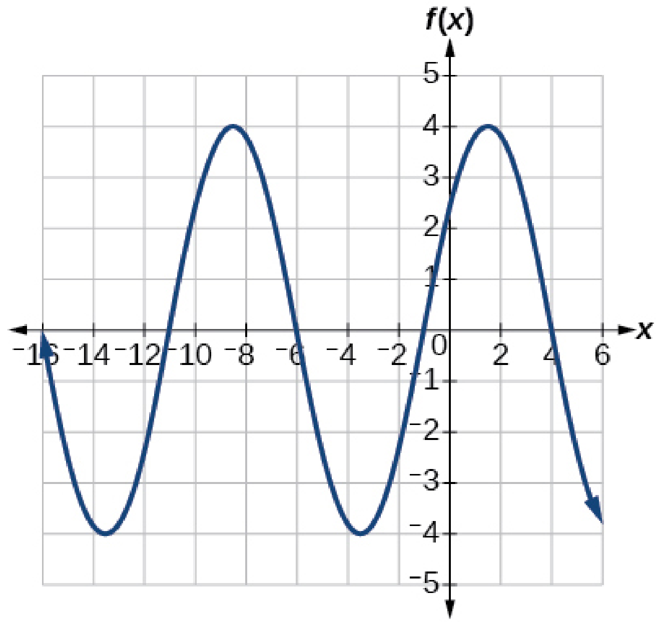


Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [link].

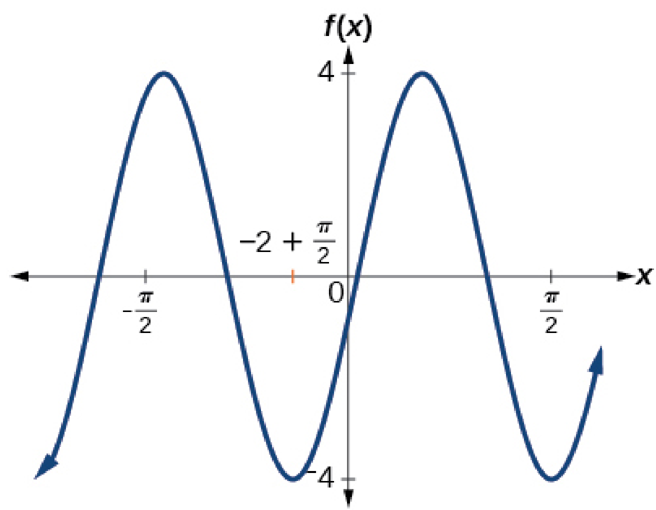


amplitude: 2; period: 5; midline: $y = 3$; equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [link].

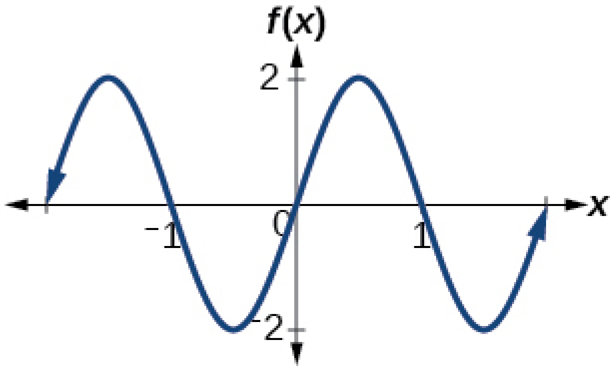


Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [link].

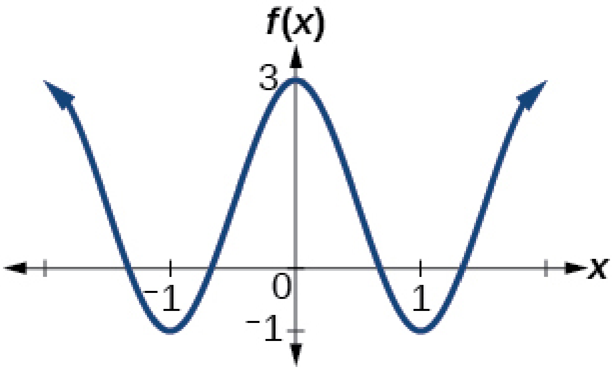


amplitude: 4; period: 2; midline: $y=0$; equation: $f(x)=-4\cos\left(\pi\left(x-\frac{\pi}{2}\right)\right)$

Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [link].

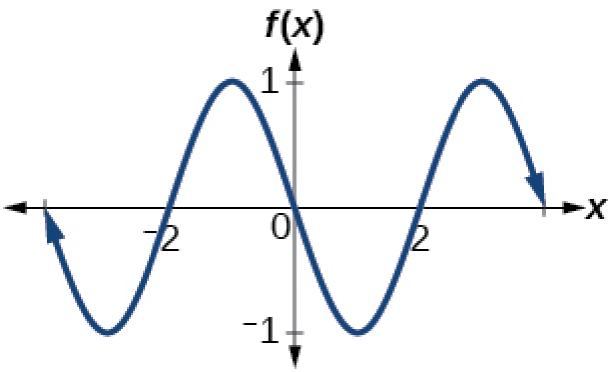


Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in [link].



amplitude: 2; period: 2; midline $y=1$; equation: $f(x)=2\cos(\pi x)+1$

Determine the amplitude, period, midline, and an equation involving sine for the graph shown in [link].



Algebraic

For the following exercises, let $f(x)=\sin x$.

On $[0,2\pi)$, solve $f(x)=0$.

$0,\pi$

On $[0,2\pi)$, solve $f(x)=\frac{1}{2}$.

Evaluate $f(\frac{\pi}{2})$.

$\sin(\frac{\pi}{2}) = 1$

On $[0, 2\pi), f(x) = \frac{\sqrt{2}}{2}$. Find all values of x.

On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what *x*-value(s)?

$\frac{\pi}{2}$

On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what *x*-value(s)?

Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to ________.

$f(x) = \sin x$ is symmetric

For the following exercises, let $f(x) = \cos x$.

On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$.

On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

$\frac{\pi}{3}, \frac{5\pi}{3}$

On $[0, 2\pi)$, find the *x*-intercepts of $f(x) = \cos x$.

On $[0, 2\pi)$, find the *x*-values at which the function has a maximum or minimum value.

Maximum: 1 at $x = 0$; minimum: -1 at $x = \pi$

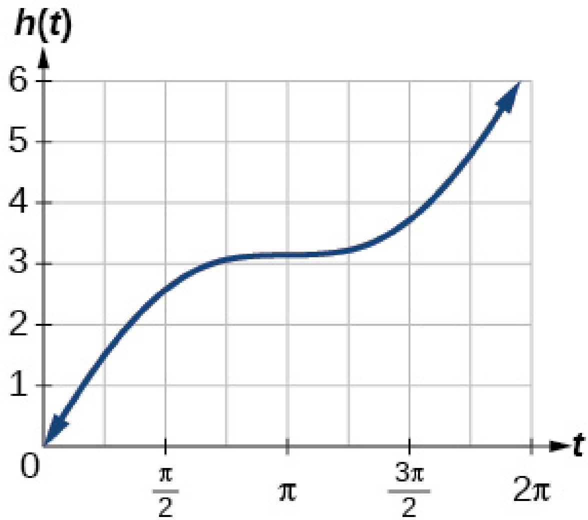
On $[0, 2\pi)$, solve the equation $f(x) = \frac{\sqrt{3}}{2}$.

Technology

Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

A linear function is added to a periodic sine function. The graph does not have an amplitude because as the linear function increases without bound the combined function $h(x) = x + \sin x$ will increase without bound as well. The graph is bounded between the graphs of $y = x + 1$ and

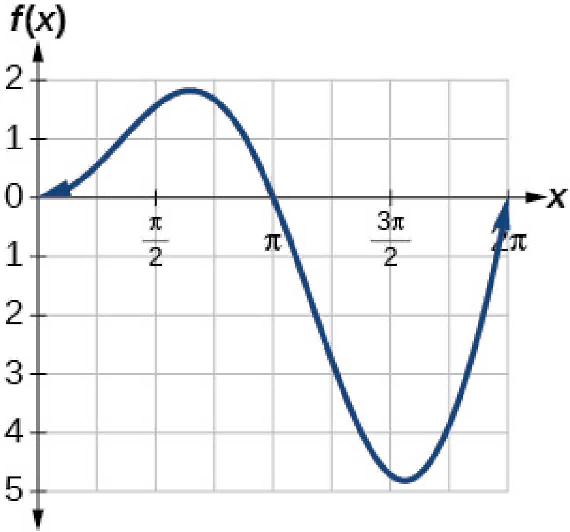
y=x-1 because sine oscillates between −1 and 1.



Graph h(x)=x+sinx on [−100,100]. Did the graph appear as predicted in the previous exercise?

Graph f(x)=xsinx on [0,2π] and verbalize how the graph varies from the graph of f(x)=sinx.

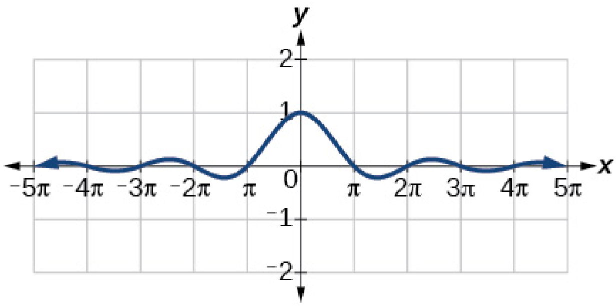
There is no amplitude because the function is not bounded.



Graph $f(x)=x\sin x$ on the window $[\,-10,10\,]$ and explain what the graph shows.

Graph $f(x)=\ \sin x\ x$ on the window $[\,-5\pi,5\pi\,]$ and explain what the graph shows.

The graph is symmetric with respect to the y-axis and there is no amplitude because the function’s bounds decrease as $|x|$ grows. There appears to be a horizontal asymptote at $y=0$.



Real-World Applications

A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function h(t) gives a person's height in meters above the ground *t* minutes after the wheel begins to turn.

1. Find the amplitude, midline, and period of h(t).
2. Find a formula for the height function h(t).
3. How high off the ground is a person after 5 minutes?

Glossary

amplitude
: the vertical height of a function; the constant A appearing in the definition of a sinusoidal function

midline
: the horizontal line $y=D$, where D appears in the general form of a sinusoidal function

periodic function
: a function $f(x)$ that satisfies $f(x+P)=f(x)$ for a specific constant P and any value of x

phase shift
: the horizontal displacement of the basic sine or cosine function; the constant $C B$

sinusoidal function
: any function that can be expressed in the form $f(x)=A\sin(Bx-C)+D$ or $f(x)=A\cos(Bx-C)+D$

Graphs of the Other Trigonometric Functions

In this section, you will:

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Graph of the tangent function

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent.} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even.} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd.} \\ &= -\tan x && \text{Definition of tangent.} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in [link].

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan(x)	undefined	-3	-1	$-\frac{3}{3}$	0	$\frac{3}{3}$	1	3	undefined

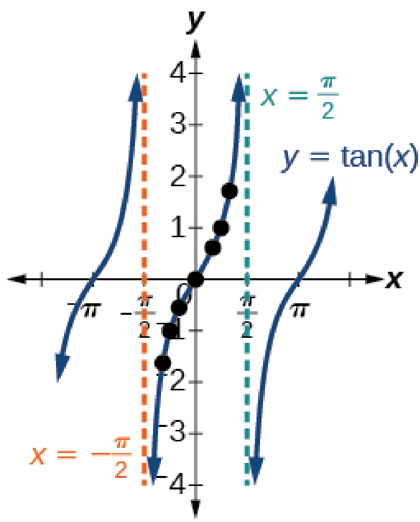
These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in [link].

x	1.3	1.5	1.55	1.56
tanx	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in [link].

x	-1.3	-1.5	-1.55	-1.56
tanx	-3.6	-14.1	-48.1	-92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x=0$. For example, $\cos\left(\frac{\pi}{2}\right)=0$ and $\cos\left(\frac{3\pi}{2}\right)=0$. At these values, the tangent function is undefined, so the graph of $y=\tan x$ has discontinuities at $x=\frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. [link] represents the graph of $y=\tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.



Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the tangent

function can be described by a general equation.
$y = A\tan(Bx)$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

Features of the Graph of $y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \pi \, |B|$.
- The domain is all real numbers x, where $x \neq \pi \, 2|B| + \pi \, |B| \, k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \pi \, 2|B| + \pi \, |B| \, k$, where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x)=A\tan(Bx)$. We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P=\frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x=-\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x=\frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right)=A\tan\left(B\frac{P}{4}\right)=A\tan\left(B\frac{\pi}{4B}\right)=A$$

because $\tan\left(\frac{\pi}{4}\right)=1$.

Given the function $f(x)=A\tan(Bx)$, graph one period.

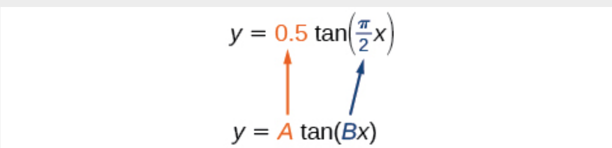
1. Identify the stretching factor, $|A|$.

2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

First, we identify A and B.

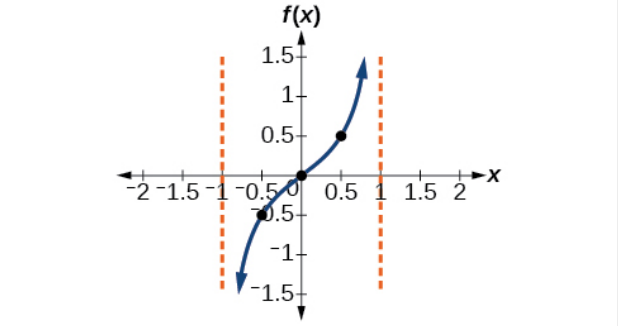


Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin,

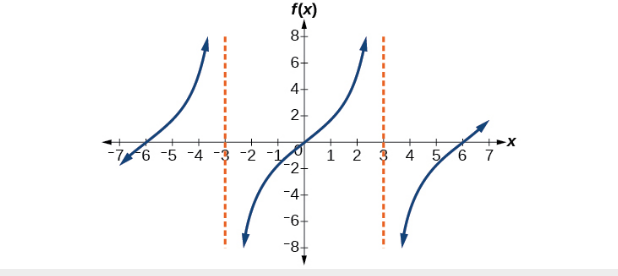
we have

$f(0.5) = 0.5\tan\left(\frac{0.5\pi}{2}\right) = 0.5\tan\left(\frac{\pi}{4}\right) = 0.5$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$, and $(-0.5, -0.5)$. The only inflection point is at the origin. [link] shows the graph of one period of the function.



Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.



Graphing One Period of a Shifted Tangent Function

Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.
$f(x)=A\tan(Bx-C)+D$

The graph of a transformed tangent function is different from the basic tangent function tanx in several ways:

Features of the Graph of $y = A\tan(Bx-C)+D$

- The stretching factor is $|A|$.
- The period is $\pi\ |B|$.
- The domain is $x\neq\ C\ B\ +\ \pi\ |B|\ k$, where k is an integer.
- The range is $(-\infty,\infty)$.
- The vertical asymptotes occur at $x=\ C\ B\ +\ \pi\ 2|B|\ k$, where k is an odd integer.
- There is no amplitude.
- $y=A\tan(Bx\text{-}C)+D$ is an odd function because it is the quotient of odd and even functions (sine and cosine respectively).

Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

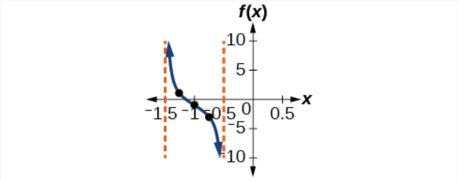
1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.* $A = -2$, so the stretching factor is $|A| = 2$.

- *Step 3.* $B=\pi$, so the period is $P=\frac{\pi}{|B|}=\frac{\pi}{\pi}=1$.
- *Step 4.* $C=-\pi$, so the phase shift is $\frac{C}{B}=\frac{-\pi}{\pi}=-1$.
- *Step 5-7.* The asymptotes are at $x=-\frac{3}{2}$ and $x=-\frac{1}{2}$ and the three recommended reference points are $(-1.25,1)$, $(-1,-1)$, and $(-0.75,-3)$. The graph is shown in [link].



Analysis

Note that this is a decreasing function because $A<0$.

How would the graph in [link] look different if we made $A=2$ instead of -2?

It would be reflected across the line $y=-1$, becoming an increasing function.

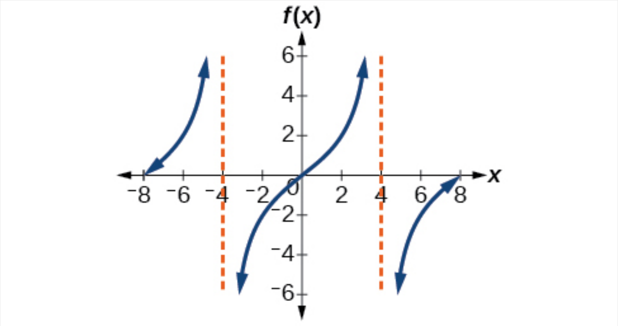
Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x)=A\tan(\ \pi\ P\ x\ )$.
3. Determine a convenient point $(x,f(x))$ on the given graph and use it to determine A.

Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in [link].

A stretched tangent function



The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from –4 to 4, so the period is $P=8$. Since $P=\ \pi\ |\ B\ |$, we

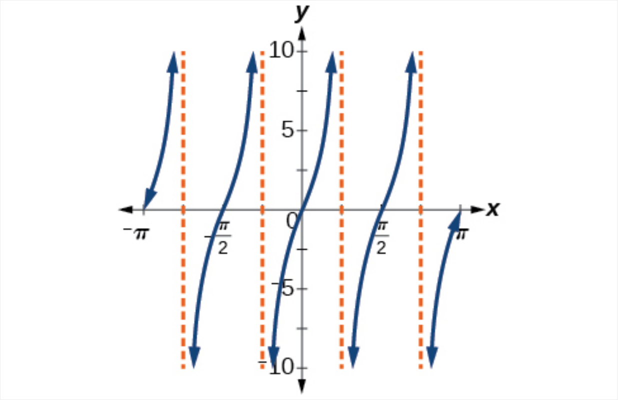
have $B=\frac{\pi}{P}=\frac{\pi}{8}$.

- *Step 2.* The equation must have the form $f(x)=A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A, we can use the point $(2,2)$.
  $2=A\tan\left(\frac{\pi}{8}\cdot 2\right)=A\tan\left(\frac{\pi}{4}\right)$

Because $\tan\left(\frac{\pi}{4}\right)=1$, $A=2$.

This function would have a formula $f(x)=2\tan\left(\frac{\pi}{8}x\right)$.

Find a formula for the function in [link].



$g(x)=4\tan(2x)$

Graph of the secant function, $f(x)=\sec x=\frac{1}{\cos x}$The graph of the cosecant function, $f(x)=\csc x=\frac{1}{\sin x}$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

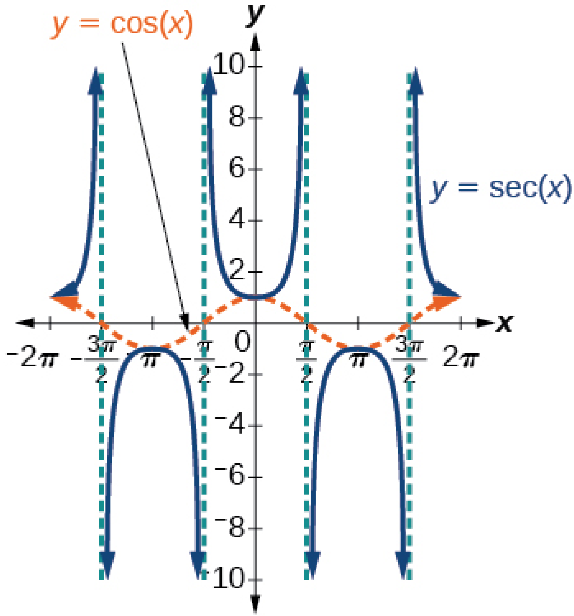
The secant was defined by the reciprocal identity $\sec x=\frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y=\sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See [link]. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the *x*-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant

and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x)=\sec x$.



As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

Features of the Graph of $y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{2|B|}k$, where k is an odd integer.

- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

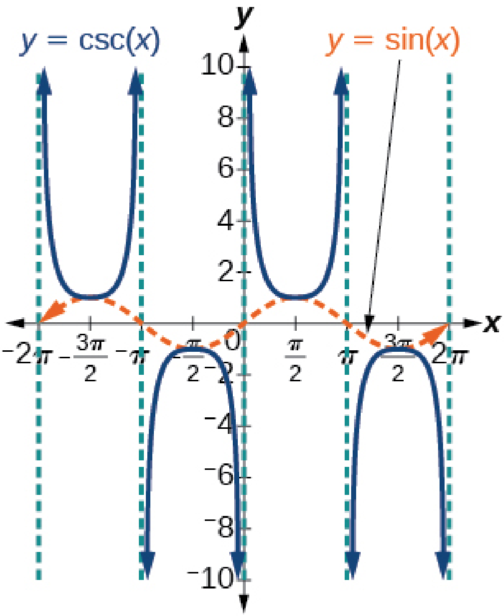
Similar to the secant, the cosecant is defined by the reciprocal identity $\csc x = \frac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See [link]. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the *x*-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in [link], is similar to the graph of secant.



Features of the Graph of $y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $2\pi |B|$.
- The domain is $x \neq \pi |B| k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \pi |B| k$, where k is an integer.

- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D$$

$$y = A\csc(Bx - C) + D$$

Features of the Graph of $y = A\sec(Bx - C) + D$

- The stretching factor is | A |.
- The period is 2π | B | .
- The domain is x≠ C B + π 2| B | k, where k is an odd integer.
- The range is (− ∞,−| A |+D]∪[| A |+D,∞).
- The vertical asymptotes occur at x= C B + π 2| B | k, where k is an odd integer.
- There is no amplitude.
- y=Asec(Bx-C)+D is an even function because cosine is an even function.

Features of the Graph of $y = A\csc(Bx - C) + D$

- The stretching factor is | A |.
- The period is 2π | B | .
- The domain is x≠ C B + π | B | k, where k is an integer.
- The range is (− ∞,−| A |+D]∪[| A |+D,∞).
- The vertical asymptotes occur at x= C B + π |B| k, where k is an integer.
- There is no amplitude.
- y=Acsc(Bx-C)+D is an odd function because sine is an odd function.

Given a function of the form y=Asec(Bx),

graph one period.

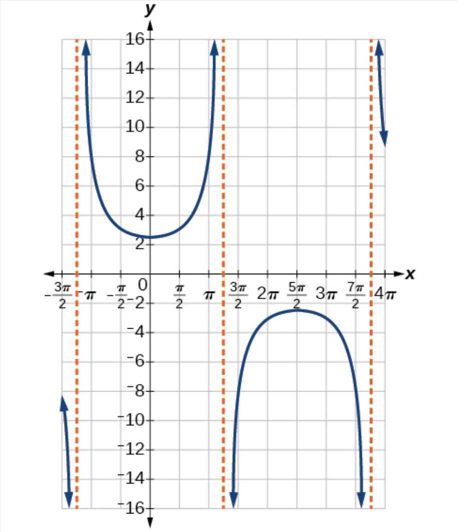
1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos x$ and $y = \sec x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$.

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \frac{2\pi}{0.4} = 5\pi$. The period is 5π units.

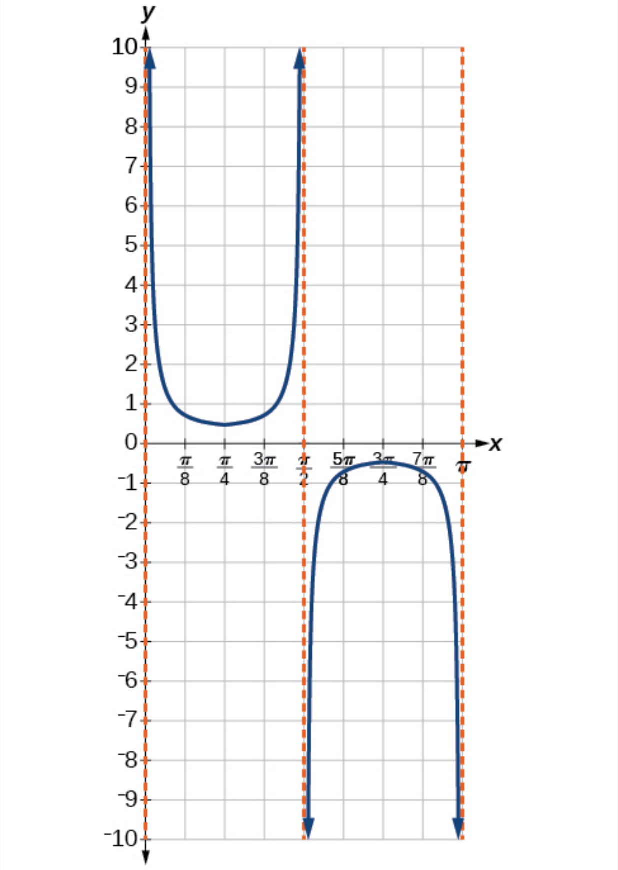
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at ($0, 2.5$) and the local maximum at ($2.5\pi, -2.5$). [link] shows the graph.



Graph one period of $f(x) = -2.5\sec(0.4x)$.

This is a vertical reflection of the preceding

graph because A is negative.



Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x)=A\sec(Bx-C)+D$ *is* $(-\infty, -|A|+D] \cup [|A|+D, \infty)$.

Given a function of the form $f(x)=A\sec(Bx-C)+D$, graph one period.

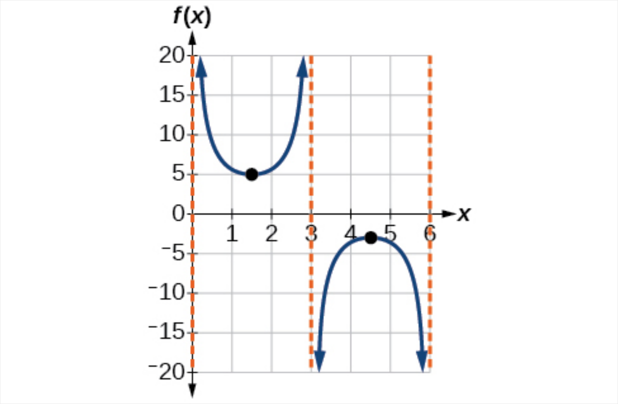
1. Express the function given in the form $y=A\sec(Bx-C)+D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y=A\sec(Bx)$, but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x=\frac{C}{B}+\frac{\pi}{2|B|}k$, where k is an odd integer.

Graphing a Variation of the Secant Function

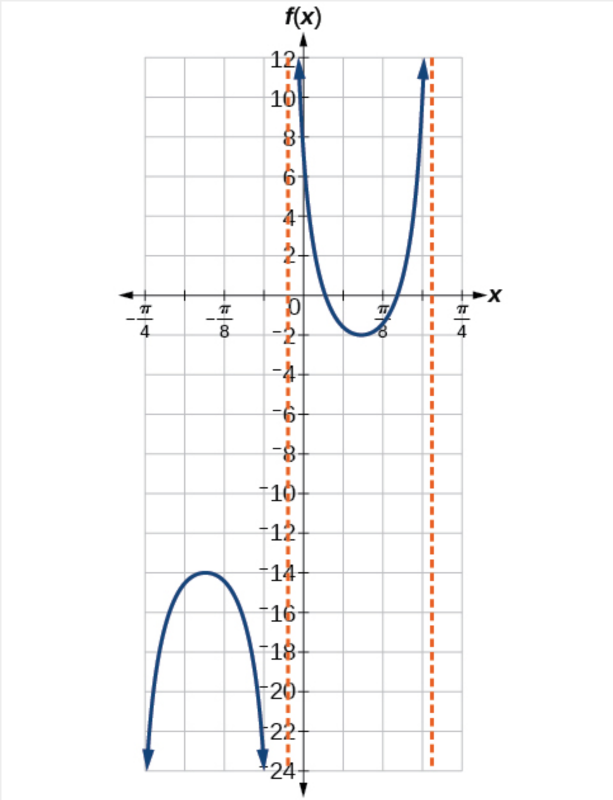
Graph one period of $y=4\sec\left(\frac{\pi}{3}x-\frac{\pi}{2}\right)+1$.

- *Step 1.* Express the function given in the form $y=4\sec\left(\frac{\pi}{3}x-\frac{\pi}{2}\right)+1$.
- *Step 2.* The stretching/compressing factor is $|A|=4$.
- *Step 3.* The period is
  $\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{3}}=\frac{2\pi}{1}\cdot\frac{3}{\pi}=6$
- *Step 4.* The phase shift is
  $\frac{C}{B}=\frac{\frac{\pi}{2}}{\frac{\pi}{3}}=\frac{\pi}{2}\cdot\frac{3}{\pi}=1.5$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0, x = 3,$ and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. [link] shows the graph.



Graph one period of $f(x) = -6\sec(4x+2) - 8$.



The domain of $\csc x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$be$x \neq$ $C + k\pi$ B ?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Given a function of the form $y=A\csc(Bx)$, graph one period.

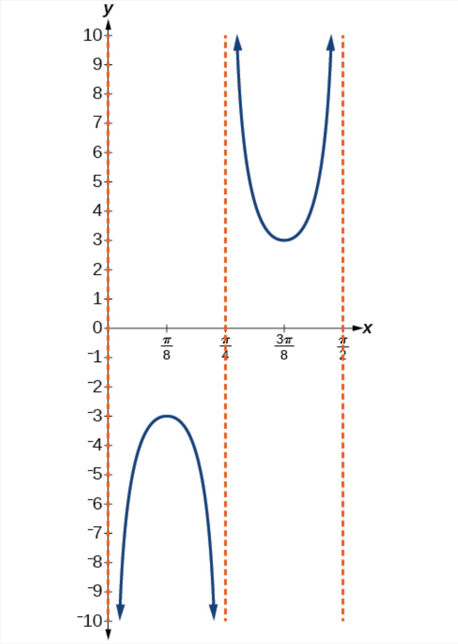
1. Express the function given in the form $y=A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P=\frac{2\pi}{|B|}$.
4. Draw the graph of $y=A\sin(Bx)$.
5. Use the reciprocal relationship between $y=\sin x$ and $y=\csc x$ to draw the graph of $y=A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Graphing a Variation of the Cosecant Function

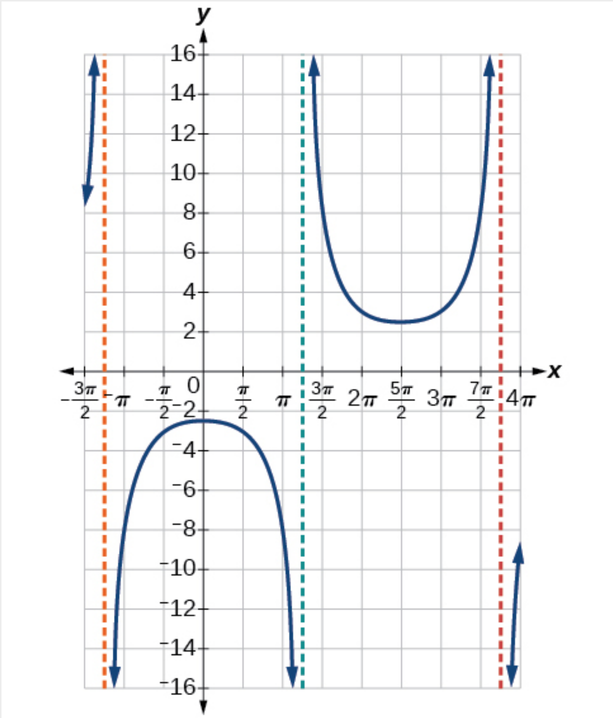
Graph one period of $f(x)=-3\csc(4x)$.

- *Step 1.* The given function is already written in the general form, $y=A\csc(Bx)$.
- *Step 2.* $|A|=|-3|=3$, so the stretching factor is 3.
- *Step 3.* $B=4$, so $P=\frac{2\pi}{4}=\frac{\pi}{2}$. The period is $\frac{\pi}{2}$ units.

- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.
- *Steps 6–7.* Sketch three asymptotes at $x=0, x=\frac{\pi}{4}$, and $x=\frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. [link] shows the graph.



Graph one period of $f(x) = 0.5\csc(2x)$.



Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $| A |$.
3. Identify B and determine the period, $2\pi \mid B \mid$.
4. Identify C and determine the phase shift, C B .

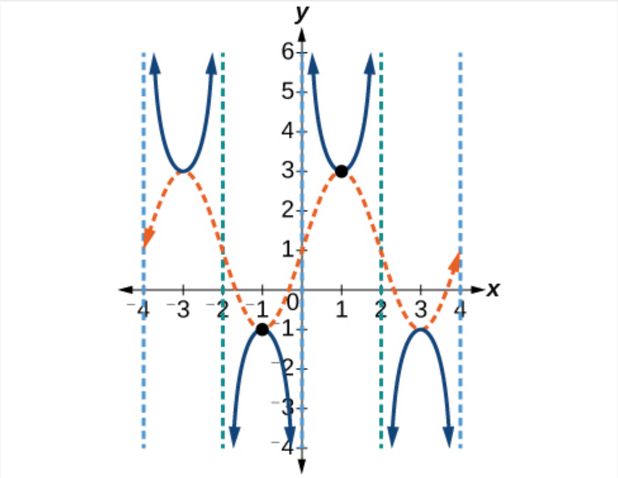
5. Draw the graph of $y=A\csc(Bx)$ but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x=\frac{C}{B}+\frac{\pi}{|B|}k$, where k is an integer.

Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y=2\csc\left(\frac{\pi}{2}x\right)+1$. What are the domain and range of this function?

- *Step 1.* Express the function given in the form $y=2\csc\left(\frac{\pi}{2}x\right)+1$.
- *Step 2.* Identify the stretching/compressing factor, $|A|=2$.
- *Step 3.* The period is $\frac{2\pi}{|B|}=\frac{2\pi}{\frac{\pi}{2}}=\frac{2\pi}{1}\cdot\frac{2}{\pi}=4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}}=0$.
- *Step 5.* Draw the graph of $y=A\csc(Bx)$ but shift it up $D=1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x=0, x=2, x=4$.

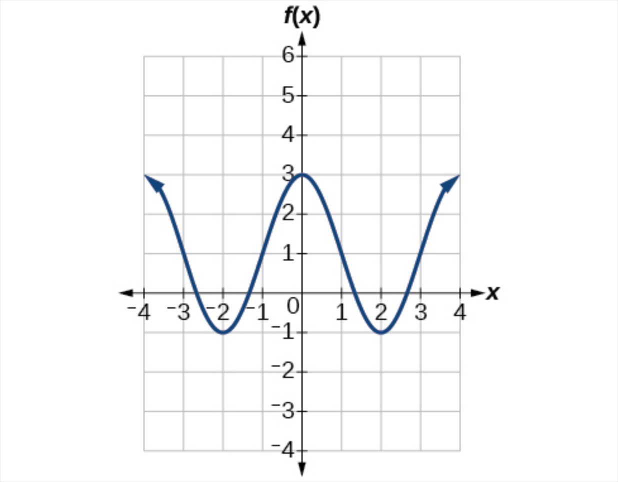
The graph for this function is shown in [link].
A transformed cosecant function

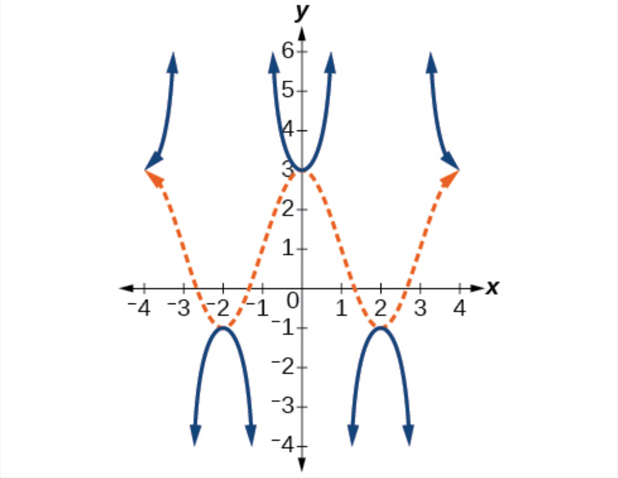


Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x)=2\sin(\frac{\pi}{2}x)+1$, shown as the orange dashed wave.

Given the graph of $f(x)=2\cos(\frac{\pi}{2}x)+1$ shown in [link], sketch the graph of $g(x)=2\sec(\frac{\pi}{2}x)+1$ on the same axes.





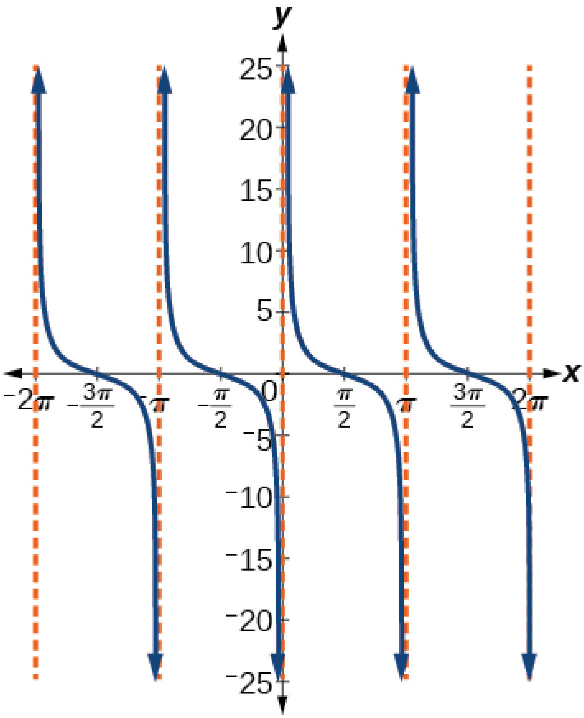
The cotangent function

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at $0, \pi$, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See [link]. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.



Features of the Graph of $y = A\cot(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \pi\ |B|$.
- The domain is $x \neq \pi\ |B|\ k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \pi\ |B|\ k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D$$

> **Features of the Graph of $y = A\cot(Bx - C) + D$**
>
> - The stretching factor is $|A|$.
> - The period is $\frac{\pi}{|B|}$.
> - The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
> - The range is $(-\infty, \infty)$.
> - The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
> - There is no amplitude.
> - $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

> **Given a modified cotangent function of the form $f(x) = A\cot(Bx)$, graph one period.**
>
> 1. Express the function in the form $f(x) = A\cot($

Bx).
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P=\frac{\pi}{|B|}$.
4. Draw the graph of $y=A\tan(Bx)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y=A\cot(Bx)$.
7. Sketch the asymptotes.

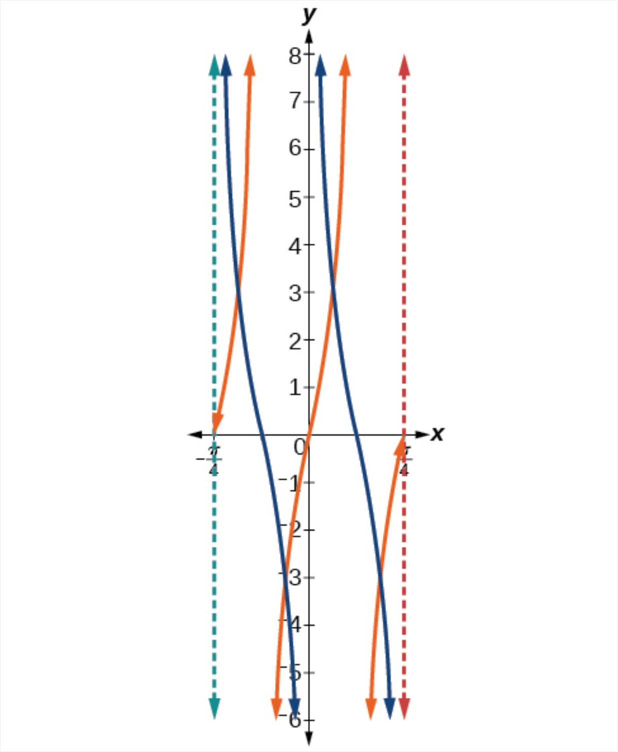
Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y=3\cot(4x)$, and then sketch a graph.

- *Step 1.* Expressing the function in the form $f(x)=A\cot(Bx)$ gives $f(x)=3\cot(4x)$.
- *Step 2.* The stretching factor is $|A|=3$.
- *Step 3.* The period is $P=\frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y=3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16},3\right)$ and $\left(\frac{3\pi}{16},-3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y=3\cot(4x)$.

- *Step 7.* Sketch the asymptotes, $x=0, x=\frac{\pi}{4}$.

The orange graph in [link] shows $y=3\tan(4x)$ and the blue graph shows $y=3\cot(4x)$.



Given a modified cotangent function of the form $f(x)=A\cot(Bx-C)+D$, graph one period.

1. Express the function in the form $f(x)=A\cot($

$Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

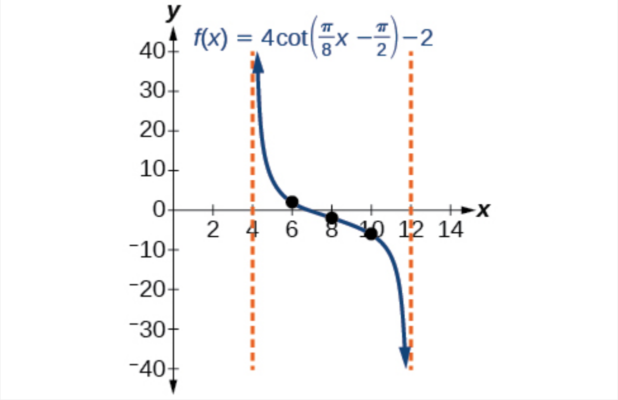
- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$, so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$, so the phase shift is $\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6,2)$, $(8,-2)$, and $(10,-6)$.

We use the reciprocal relationship of tangent and cotangent to draw f(x)=4cot(π 8 x− π 2)−2.

- *Step 8.* The vertical asymptotes are x=4 and x=12.

The graph is shown in [link].

One period of a modified cotangent function



Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let’s return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police

car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

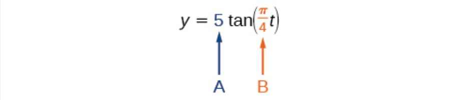
Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y=5\tan\left(\frac{\pi}{4}t\right)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

1. Find and interpret the stretching factor and period.
2. Graph on the interval $[0,5]$.
3. Evaluate $f(1)$ and discuss the function's value at that input.

---

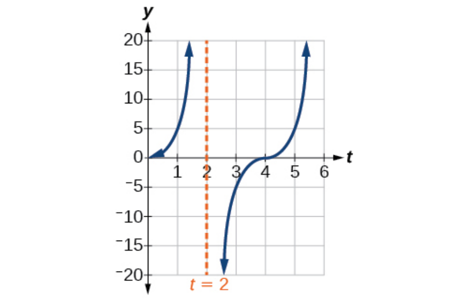
1. We know from the general form of $y=A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.



We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is π π 4 = π 1 · 4 π = 4. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

2. To graph the function, we draw an asymptote at $t=2$ and use the stretching factor and period. See [link]



3. period: $f(1)=5\tan(\frac{\pi}{4}(1))=5(1)=5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See [link], [link], and [link].
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See [link] and [link].
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See [link] and [link].
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See [link] and [link].
- Real-world scenarios can be solved using graphs of trigonometric functions. See [link].

Section Exercises

Verbal

Explain how the graph of the sine function can be used to graph $y=\csc x$.

Since $y=\csc x$ is the reciprocal function of $y=\sin x$, you can plot the reciprocal of the coordinates on the graph of $y=\sin x$ to obtain the *y*-coordinates of $y=\csc x$. The *x*-intercepts of the graph $y=\sin x$ are the vertical asymptotes for the graph of $y=\csc x$.

How can the graph of $y=\cos x$ be used to construct the graph of $y=\sec x$?

Explain why the period of $\tan x$ is equal to π.

Answers will vary. Using the unit circle, one can show that $\tan(x+\pi)=\tan x$.

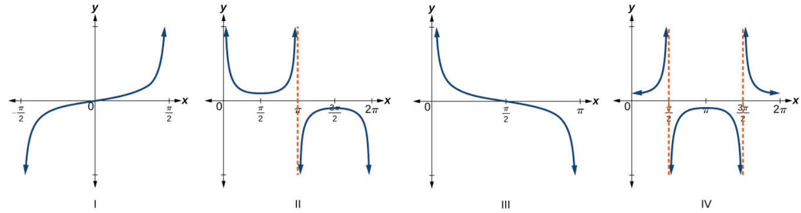
Why are there no intercepts on the graph of $y=\csc x$?

How does the period of $y=\csc x$ compare with the period of $y=\sin x$?

The period is the same: 2π.

Algebraic

For the following exercises, match each trigonometric function with one of the following graphs.



$f(x) = \tan x$

$f(x) = \sec x$

IV

$f(x) = \csc x$

$f(x) = \cot x$

III

For the following exercises, find the period and horizontal shift of each of the functions.

$f(x) = 2\tan(4x - 32)$

$h(x) = 2\sec(\pi 4 (x + 1))$

---

period: 8; horizontal shift: 1 unit to left

$m(x) = 6\csc(\pi 3 x + \pi)$

If $\tan x = -1.5$, find $\tan(-x)$.

1.5

If $\sec x = 2$, find $\sec(-x)$.

If $\csc x = -5$, find $\csc(-x)$.

5

If $x\sin x = 2$, find $(-x)\sin(-x)$.

For the following exercises, rewrite each expression such that the argument x is positive.

$\cot(-x)\cos(-x)+\sin(-x)$

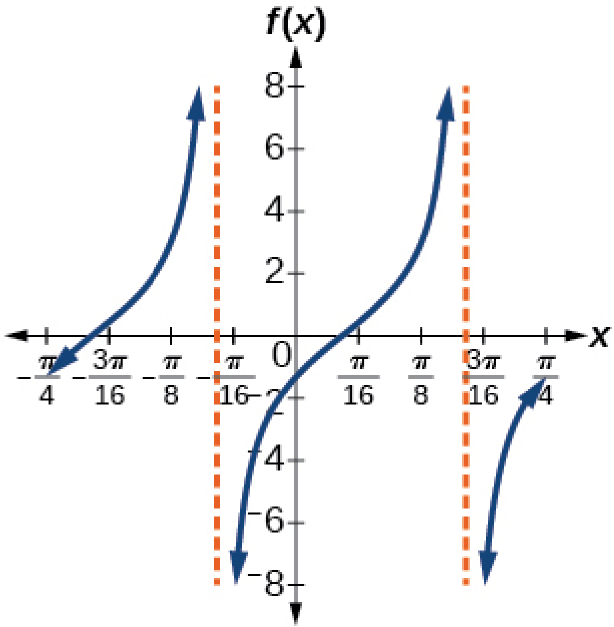
$-\cot x\cos x-\sin x$

$\cos(-x)+\tan(-x)\sin(-x)$

Graphical

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

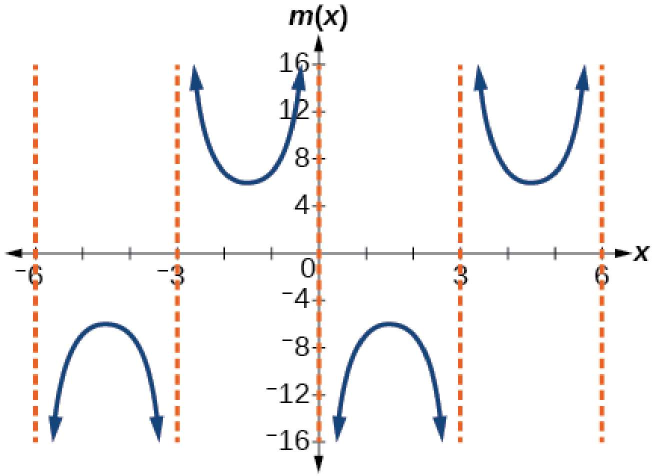
$f(x)=2\tan(4x-32)$



stretching factor: 2; period: $\frac{\pi}{4}$; asymptotes: $x=\frac{1}{4}\left(\frac{\pi}{2}+\pi k\right)+8$, where k is an integer

$h(x)=2\sec\left(\frac{\pi}{4}(x+1)\right)$

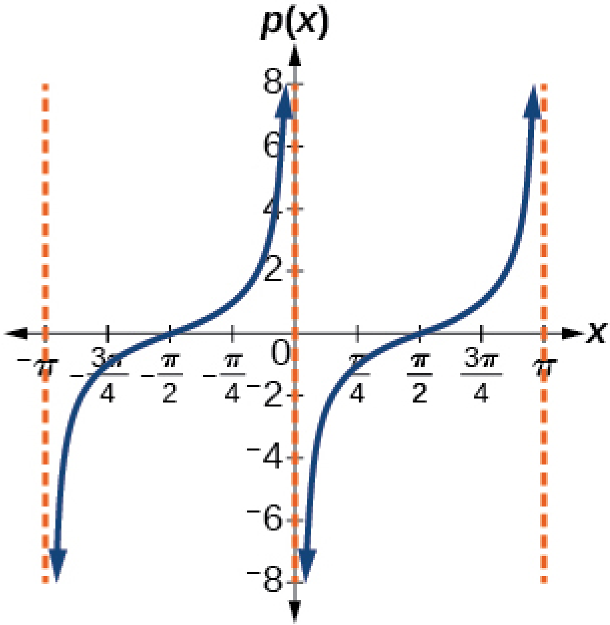
$m(x)=6\csc\left(\frac{\pi}{3}x+\pi\right)$



stretching factor: 6; period: 6; asymptotes: $x=3k$, where k is an integer

$j(x)=\tan\left(\frac{\pi}{2}x\right)$

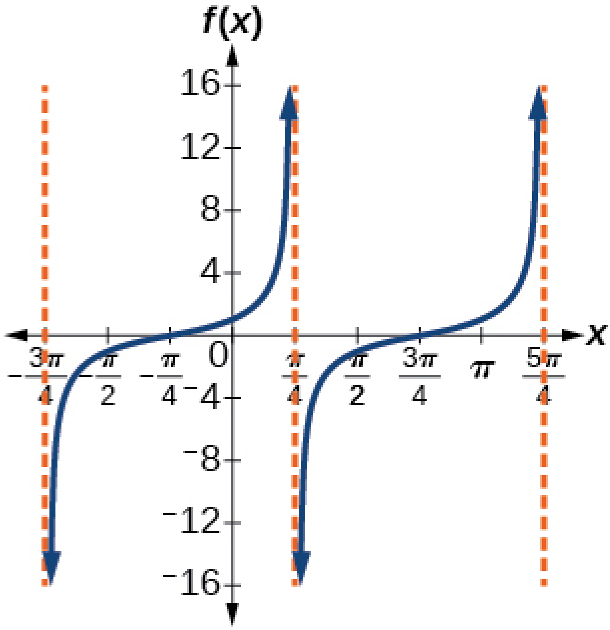
$p(x)=\tan\left(x-\frac{\pi}{2}\right)$



stretching factor: 1; period: π; asymptotes: $x = \pi k$, where k is an integer

$f(x) = 4\tan(x)$

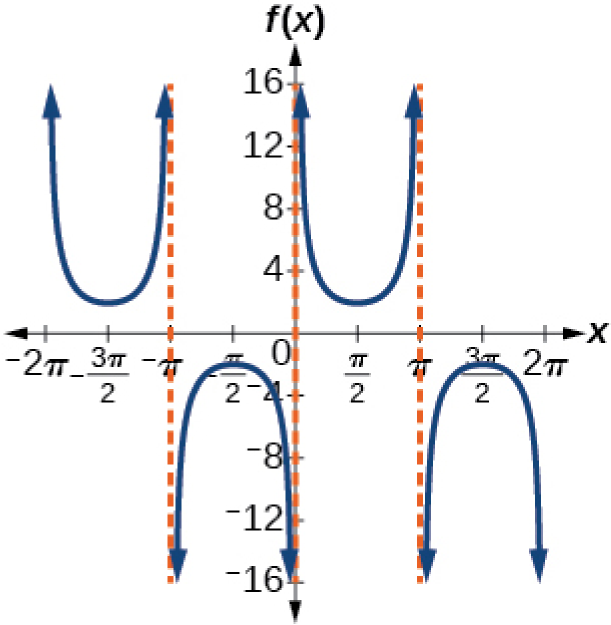
$f(x) = \tan\left(x + \frac{\pi}{4} \right)$



Stretching factor: 1; period: π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

$f(x) = \pi \tan(\pi x - \pi) - \pi$

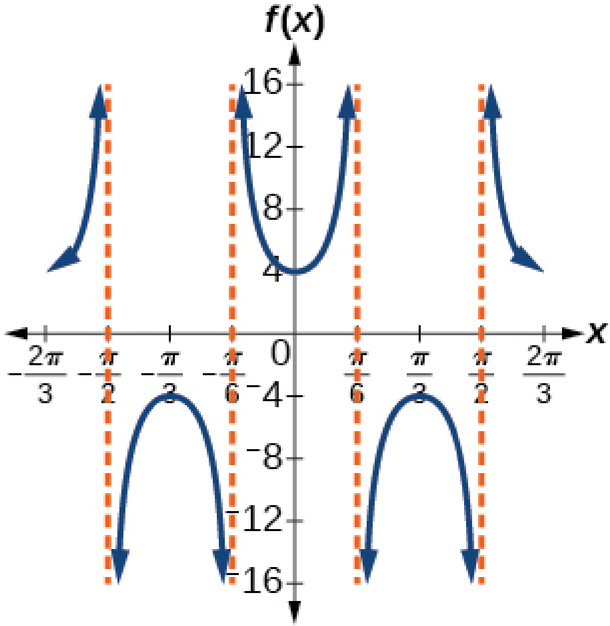
$f(x) = 2\csc(x)$



stretching factor: 2; period: 2π; asymptotes: $x = \pi k$, where k is an integer

$f(x) = -\frac{1}{4}\csc(x)$

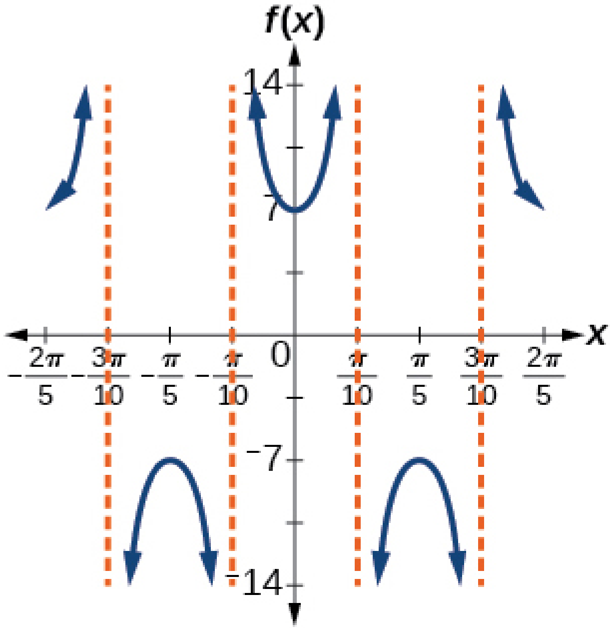
$f(x) = 4\sec(3x)$



stretching factor: 4; period: $\frac{2\pi}{3}$; asymptotes: $x=\frac{\pi}{6}k$, where k is an odd integer

$f(x)=-3\cot(2x)$

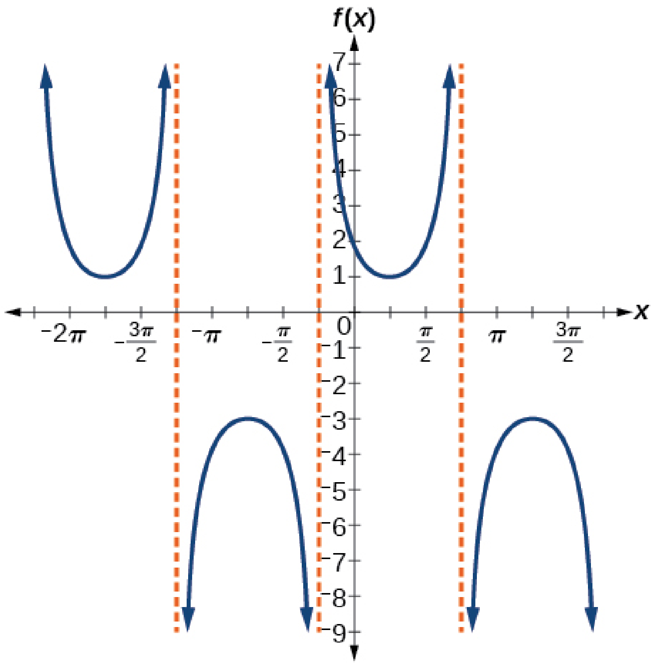
$f(x)=7\sec(5x)$



stretching factor: 7; period: $\frac{2\pi}{5}$; asymptotes: $x=\frac{\pi}{10}k$, where k is an odd integer

$f(x)=\frac{9}{10}\csc(\pi x)$

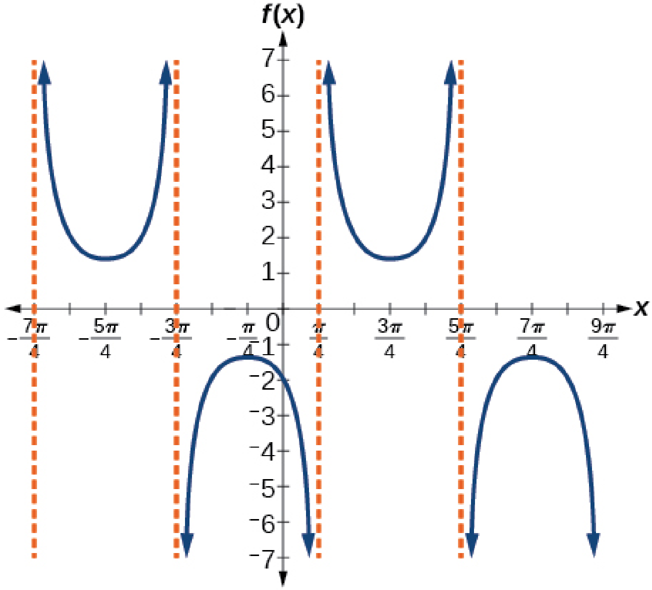
$f(x)=2\csc\left(x+\frac{\pi}{4}\right)-1$



stretching factor: 2; period: 2π; asymptotes: $x = -\frac{\pi}{4} + \pi k$, where k is an integer

$f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

$f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$



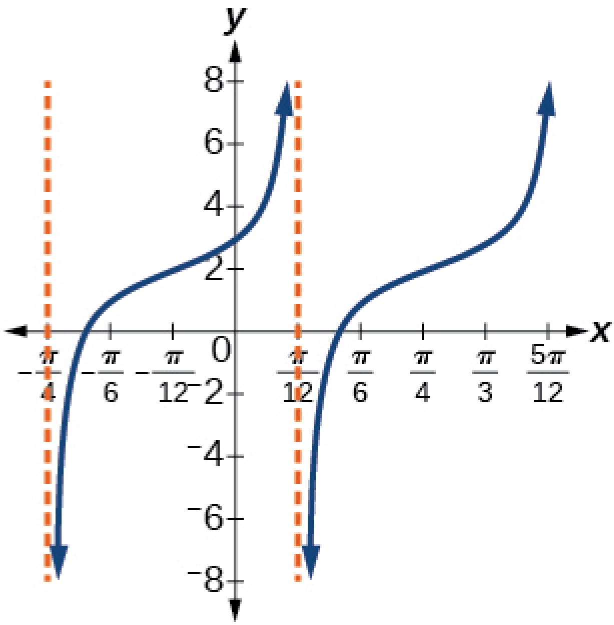
stretching factor: $\frac{7}{5}$; period: 2π; asymptotes: $x=\frac{\pi}{4}+\pi k$, where k is an integer

$f(x)=5\left(\cot\left(x+\frac{\pi}{2}\right)-3\right)$

For the following exercises, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

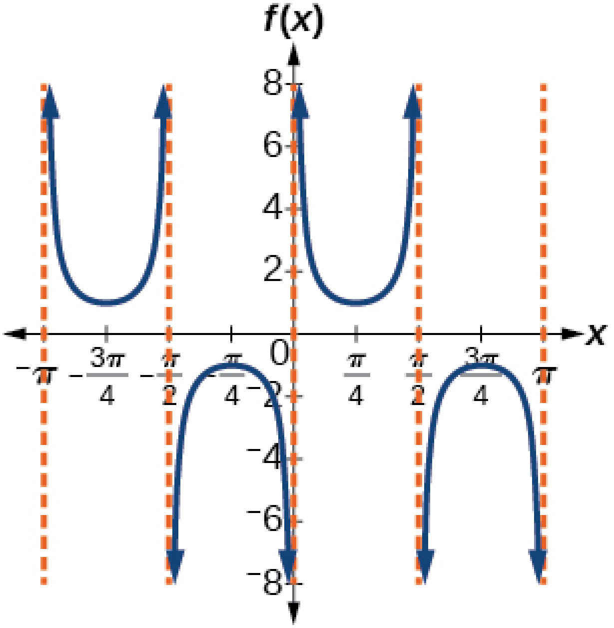
A tangent curve, $A=1$, period of $\frac{\pi}{3}$; and phase shift $(h,k)=\left(\frac{\pi}{4},2\right)$

$y=\tan\left(3\left(x-\frac{\pi}{4}\right)\right)+2$

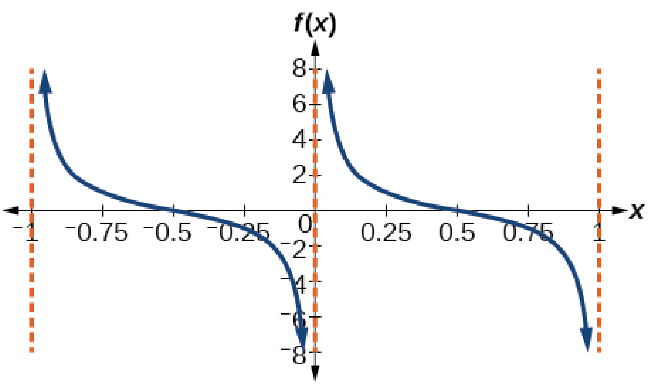


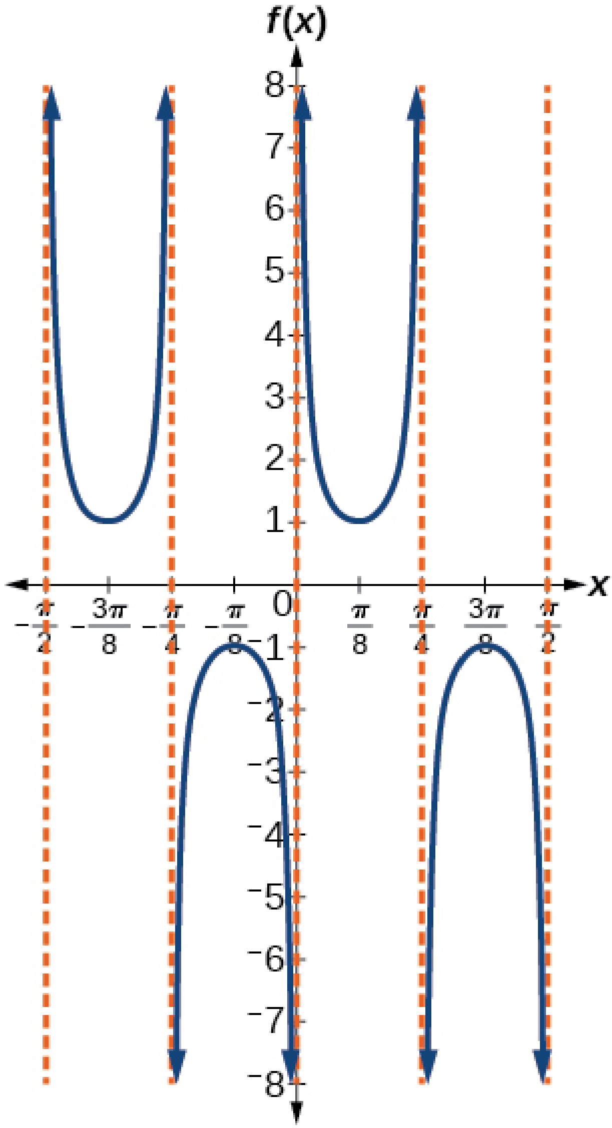
A tangent curve, $A=-2$, period of $\frac{\pi}{4}$, and phase shift $(h,k)=\left(-\frac{\pi}{4},-2\right)$

For the following exercises, find an equation for the graph of each function.


f(x)
8
6
4
2
x
−π
−3π/4
−π/2
−π/4
0
π/4
π/2
3π/4
π
−2
−4
−6
−8

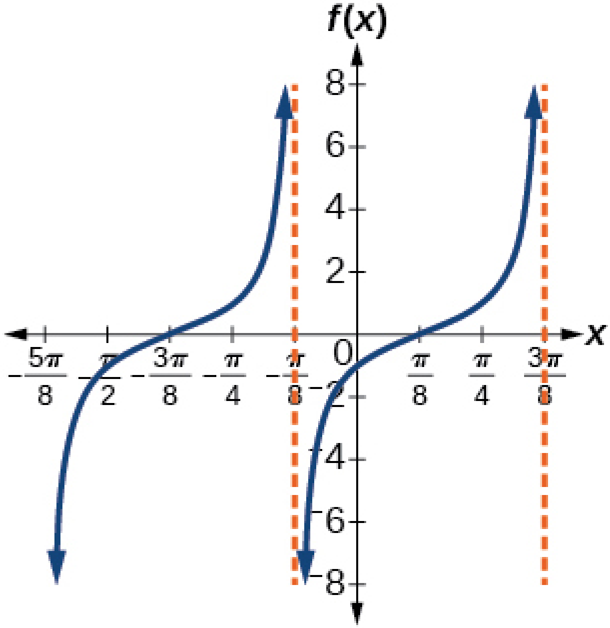
$f(x) = \csc(2x)$

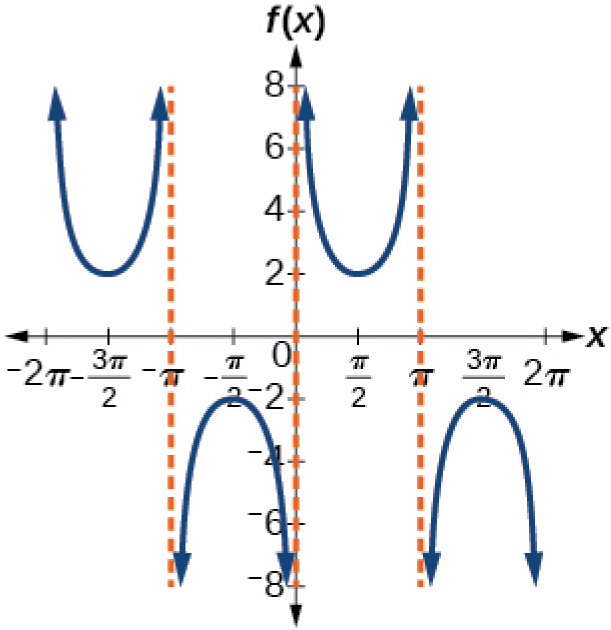

f(x)
8
6
4
2
x
−1
−0.75
−0.5
−0.25
0
0.25
0.5
0.75
1
−2
−4
−6
−8



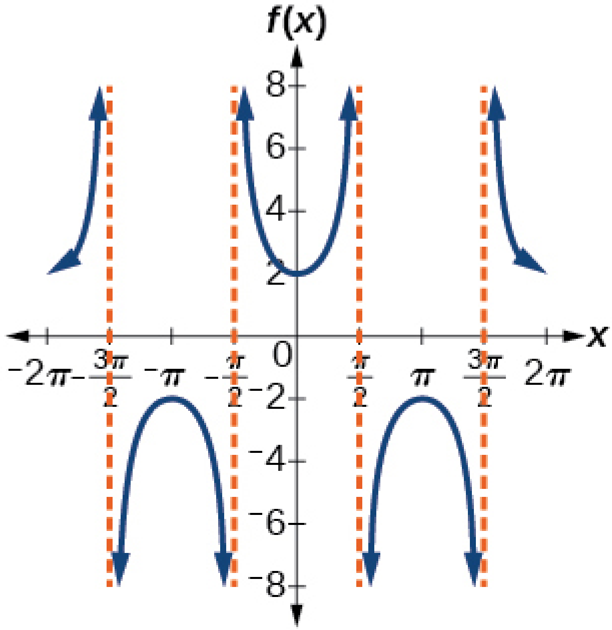
f(x)
8
7
6
5
4
3
2
1
0
x
$-\frac{\pi}{2}$
$-\frac{3\pi}{8}$
$-\frac{\pi}{4}$
$-\frac{\pi}{8}$
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
−1
−2
−3
−4
−5
−6
−7
−8

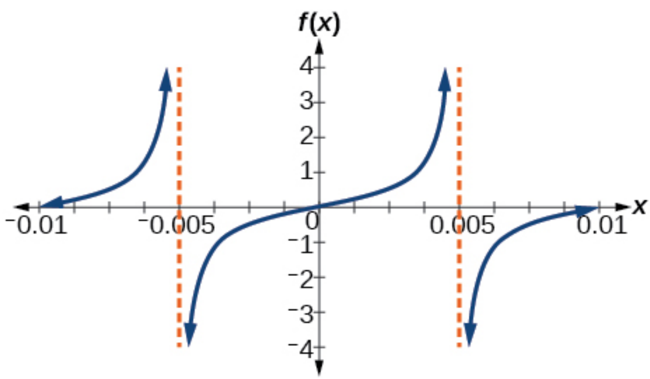
$f(x) = \csc(4x)$


f(x)
x
8
6
4
2
0
−2
−4
−6
−8
−5π/8
−π/2
−3π/8
−π/4
−π/8
π/8
π/4
3π/8



$f(x) = 2\csc x$





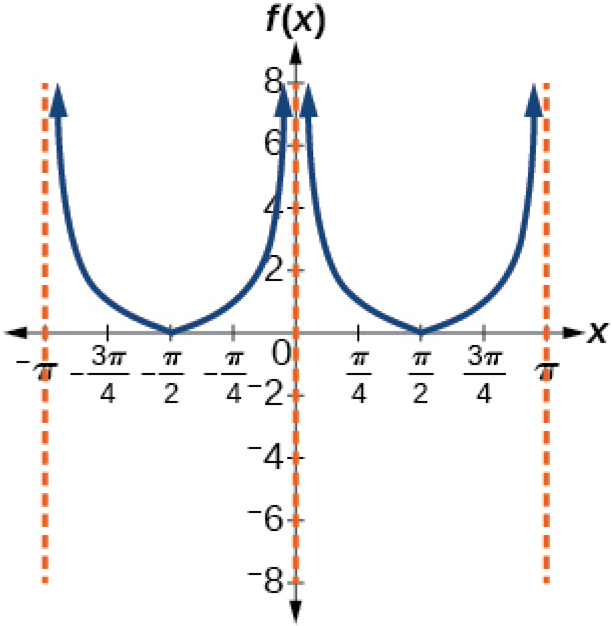
$f(x) = \frac{1}{2}\tan(100\pi x)$

Technology

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input cscx as 1 sinx .

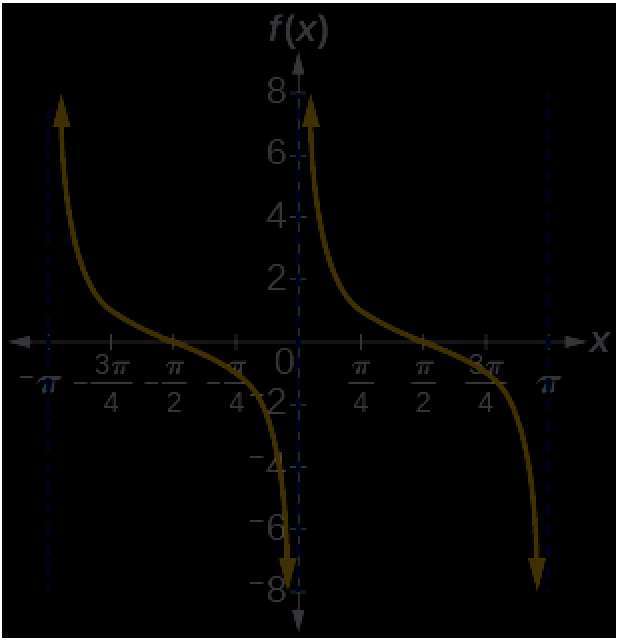
$f(x) = |\csc(x)|$

$f(x) = |\cot(x)|$



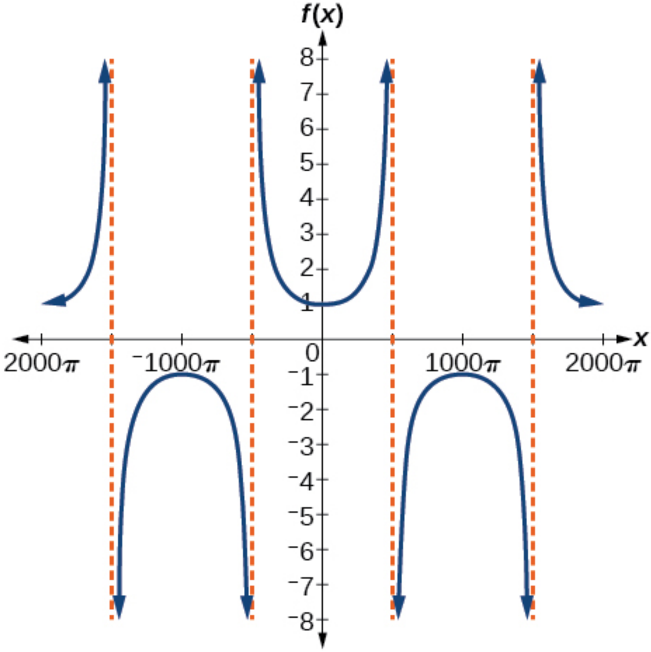
$f(x) = 2\csc(x)$

$f(x) = \csc(x)\sec(x)$



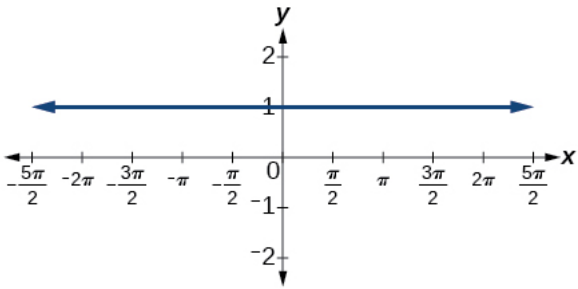
Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

$f(x) = \sec(0.001x)$



$f(x) = \cot(100\pi x)$

$f(x) = \sin 2x + \cos 2x$



Real-World Applications

The function $f(x)=20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance f(x), in feet.

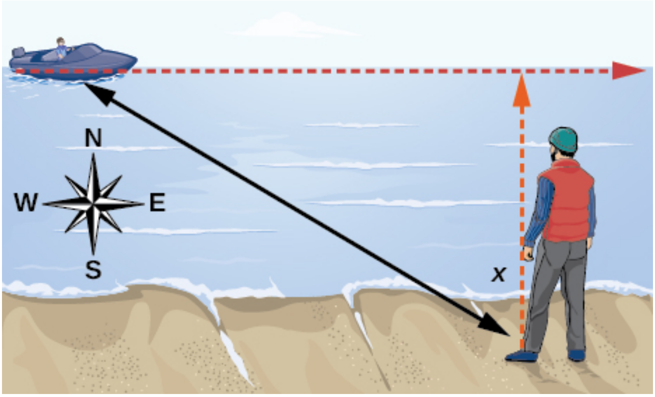
1. Graph on the interval $[0,5]$.
2. Find and interpret the stretching factor, period, and asymptote.
3. Evaluate $f(1)$ and $f(2.5)$ and discuss the function's values at those inputs.

Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See [link].) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x)=1.5\sec(x)$.

1. What is a reasonable domain for $d(x)$?
2. Graph $d(x)$ on this domain.
3. Find and discuss the meaning of any

vertical asymptotes on the graph of $d(x)$.

4. Calculate and interpret $d(-\frac{\pi}{3})$. Round to the second decimal place.
5. Calculate and interpret $d(\frac{\pi}{6})$. Round to the second decimal place.
6. What is the minimum distance between the fisherman and the boat? When does this occur?



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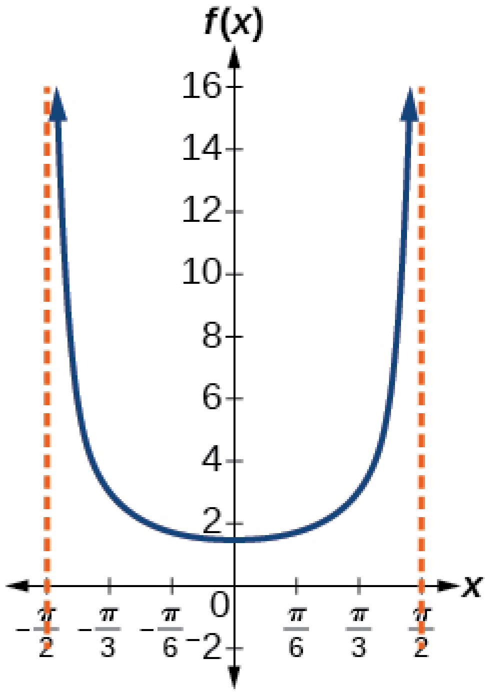
1. $(-\frac{\pi}{2}, \frac{\pi}{2})$;
2.



3. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$ — i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;
4. 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away;
5. 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73 km away;
6. 1.5 km; when $x = 0$

A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in

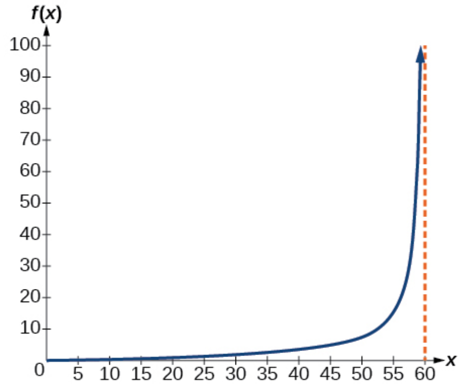
the interval 0 to 30 days, is given by $g(x) = 250{,}000\csc(\frac{\pi}{30}x)$.

1. Graph $g(x)$ on the interval $[0,30]$.
2. Evaluate $g(5)$ and interpret the information.
3. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
4. Find and discuss the meaning of any vertical asymptotes.

A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

1. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
2. Graph $h(x)$ on the interval $(0,60)$.
3. Evaluate and interpret the values $h(0)$ and $h(30)$.
4. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

1. $h(x) = 2\tan(\frac{\pi}{120}x)$;
2.



3. $h(0) = 0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$: after 30 seconds, the rockets is 2 mi high;
4. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.

Inverse Trigonometric Functions

In this section, you will:

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.
(a) Sine function on a restricted domain of $[-\frac{\pi}{2}, \frac{\pi}{2}]$; (b) Cosine function on a restricted domain of $[0,\pi]$ Tangent function on a restricted domain of $(-\frac{\pi}{2}, \frac{\pi}{2})$ The sine function and inverse sine (or arcsine) function The cosine function and inverse cosine (or arccosine) function The tangent function and inverse tangent (or arctangent) function

Understanding and Using the Inverse

Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in [link].

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

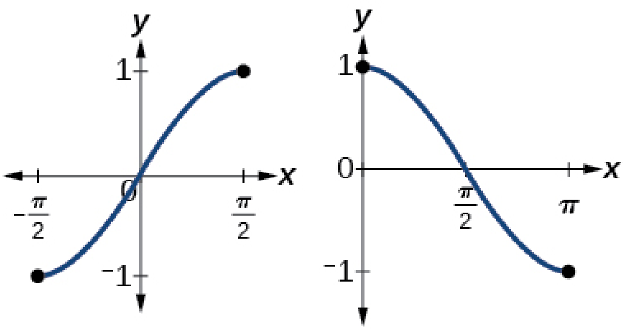
For example, if $f(x)=\sin x$, then we would write $f^{-1}(x)=\sin^{-1}x$. Be aware that $\sin^{-1}x$ does not mean $\frac{1}{\sin x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$, then $\frac{\pi}{6}=\sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi)=-1$, then $\pi=\cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right)=1$, then $\frac{\pi}{4}=\tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a)=b$, then an inverse function would satisfy $f^{-1}(b)=a$.

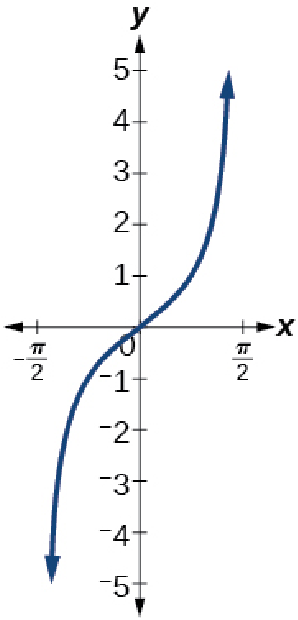
Bear in mind that the sine, cosine, and tangent

functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the domain of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. [link] shows the graph of the sine function limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.



(a) (b)

[link] shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one vertical asymptote to the next instead of being divided into two parts by an asymptote.

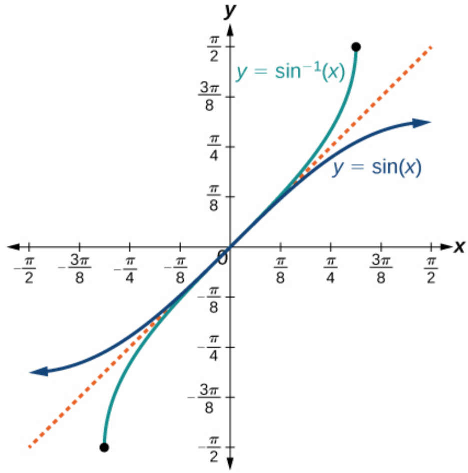
On these restricted domains, we can define the inverse trigonometric functions.

- The **inverse sine function** $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the **arcsine** function, and notated

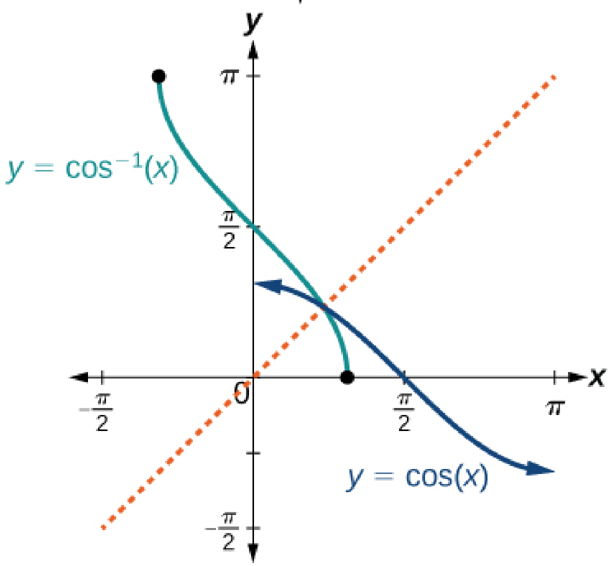
arcsinx.
$y=\sin^{-1} x$ has domain $[-1,1]$ and range $[-\frac{\pi}{2},\frac{\pi}{2}]$

- The **inverse cosine function** $y=\cos^{-1} x$ means $x=\cos y$. The inverse cosine function is sometimes called the **arccosine** function, and notated arccosx.
  $y=\cos^{-1} x$ has domain $[-1,1]$ and range $[0,\pi]$
- The **inverse tangent function** $y=\tan^{-1} x$ means $x=\tan y$. The inverse tangent function is sometimes called the **arctangent** function, and notated arctanx.
  $y=\tan^{-1} x$ has domain $(-\infty,\infty)$ and range $(-\frac{\pi}{2},\frac{\pi}{2})$

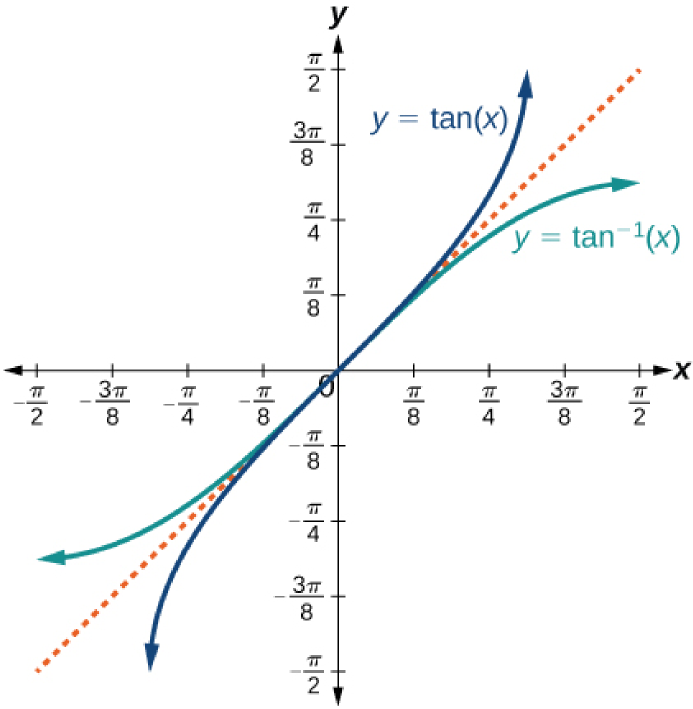
The graphs of the inverse functions are shown in [link], [link], and [link]. Notice that the output of each of these inverse functions is a *number,* an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1,1]$ and range $[-\frac{\pi}{2},\frac{\pi}{2}]$, $\cos^{-1} x$ has domain $[-1,1]$ and range $[0,\pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $(-\frac{\pi}{2},\frac{\pi}{2})$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y=x$.



y
x
$y = \sin^{-1}(x)$
$y = \sin(x)$
$\frac{\pi}{2}$
$\frac{3\pi}{8}$
$\frac{\pi}{4}$
$\frac{\pi}{8}$
0
$-\frac{\pi}{8}$
$-\frac{\pi}{4}$
$-\frac{3\pi}{8}$
$-\frac{\pi}{2}$



y
x
$y = \cos^{-1}(x)$
$y = \cos(x)$
π
$\frac{\pi}{2}$
0
$-\frac{\pi}{2}$



Relations for Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, if $\tan y = x$, then $\tan^{-1} x = y$.

Writing a Relation for an Inverse Function

Given $\sin(\frac{5\pi}{12}) \approx 0.96593$, write a relation involving the inverse sine.

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their

domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically π 6 (30°), π 4 (45°), and π 3 (60°), and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

1. $\sin\ -1\ (\ 1\ 2\ )$
2. $\sin\ -1\ (\ -\ 2\ 2\ )$
3. $\cos\ -1\ (\ -\ 3\ 2\ )$
4. $\tan\ -1\ (\ 1\ )$

---

1. Evaluating $\sin\ -1\ (\ 1\ 2\ )$ is the same as determining the angle that would have a sine value of $1\ 2$. In other words, what angle x would satisfy $\sin(x)=\ 1\ 2$? There are multiple values that would satisfy this relationship, such as $\pi\ 6$ and $5\pi\ 6$, but we know we need the angle in the interval $[\ -\ \pi\ 2\ ,\ \pi\ 2\ ]$, so the answer will be $\sin\ -1\ (\ 1\ 2\ )=\ \pi\ 6$. Remember that the inverse is a function, so for each input, we will get exactly one output.
2. To evaluate $\sin\ -1\ (\ -\ 2\ 2\ )$, we know that $5\pi\ 4$ and $7\pi\ 4$ both have a sine value of $-\ 2\ 2$, but neither is in the interval $[\ -\ \pi\ 2\ ,\ \pi\ 2\ ]$. For that, we need the negative angle coterminal with $7\pi\ 4$: $\sin\ -1\ (-\ 2\ 2\ )=-\ \pi\ 4$.
3. To evaluate $\cos\ -1\ (\ -\ 3\ 2\ )$, we are looking for an angle in the interval $[\ 0,\pi\ ]$ with a cosine value of $-\ 3\ 2$. The angle that satisfies this is $\cos\ -1\ (\ -\ 3\ 2\ )=\ 5\pi\ 6$.
4. Evaluating $\tan\ -1\ (\ 1\ )$, we are looking for an angle in the interval $(\ -\ \pi\ 2\ ,\ \pi\ 2\ )$ with a tangent value of 1. The correct

angle is $\tan^{-1}\left(1\right)=\frac{\pi}{4}$.

Evaluate each of the following.

1. $\sin^{-1}(-1)$
2. $\tan^{-1}\left(-1\right)$
3. $\cos^{-1}\left(-1\right)$
4. $\cos^{-1}\left(\frac{1}{2}\right)$

a. $-\frac{\pi}{2}$; b. $-\frac{\pi}{4}$; c. π; d. $\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent

functions. These may be labeled, for example, **SIN -1**, **ARCSIN**, or **ASIN**.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

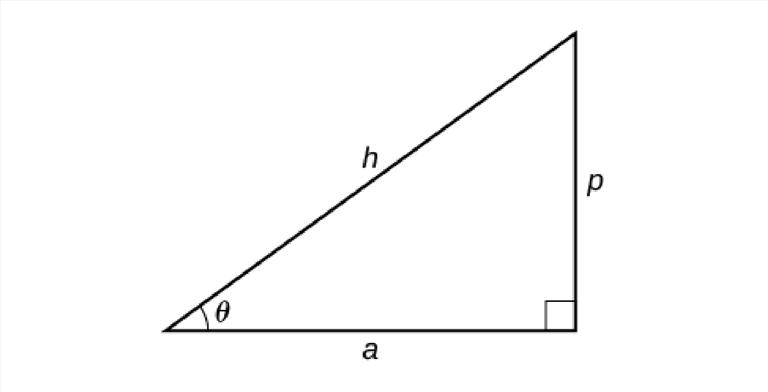
Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

1.9823 or 113.578°

Given two sides of a right triangle like the one shown in [link], find an angle.



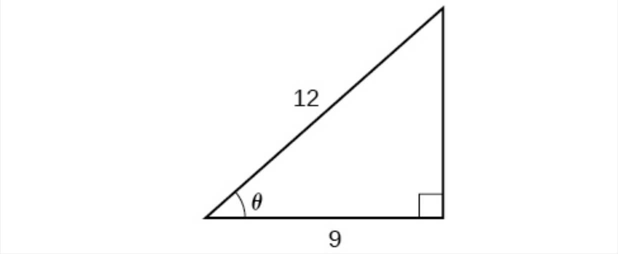
1. If one given side is the hypotenuse of length h and the side of length a adjacent to the

desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.

2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Applying the Inverse Cosine to a Right Triangle

Solve the triangle in [link] for the angle θ.



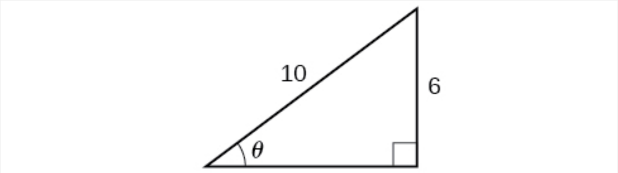
Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12} \quad \theta = \cos^{-1}\left(\frac{9}{12}\right)$$

Apply definition of the inverse.

$\theta \approx 0.7227$ or about $41.4096°$ Evaluate.

Solve the triangle in [link] for the angle θ.



$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$ radians

Right triangle illustrating the cofunction relationships

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let f(x) and g(x) be two different trigonometric functions belonging to the set {

$\sin(x), \cos(x), \tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

> Compositions of a trigonometric function and its inverse
>
> $\sin(\sin^{-1} x) = x \text{ for } -1 \le x \le 1$ $\cos(\cos^{-1} x) = x \text{ for } -1 \le x \le 1$ $\tan(\tan^{-1} x) = x \text{ for } -\infty < x < \infty$
>
> $\sin^{-1}(\sin x) = x \text{ only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $\cos^{-1}(\cos x) = x \text{ only for } 0 \le x \le \pi$ $\tan^{-1}(\tan x) = x \text{ only for } -\frac{\pi}{2} < x < \frac{\pi}{2}$

> **Is it correct that $\sin^{-1}(\sin x) = x$?**
>
> *No. This equation is correct if x belongs to the restricted domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, but sine is defined*

for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta, \cos\theta,$ or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Using Inverse Trigonometric Functions

Evaluate the following:

1. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
2. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
3. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
4. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

1. $\frac{\pi}{3}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\sin^{-1}(\sin($

$\frac{\pi}{3}))=\frac{\pi}{3}$.

2. $\frac{2\pi}{3}$ is not in $[-\frac{\pi}{2},\frac{\pi}{2}]$, but $\sin(\frac{2\pi}{3})=\sin(\frac{\pi}{3})$, so $\sin^{-1}(\sin(\frac{2\pi}{3}))=\frac{\pi}{3}$.
3. $\frac{2\pi}{3}$ is in $[0,\pi]$, so $\cos^{-1}(\cos(\frac{2\pi}{3}))=\frac{2\pi}{3}$.
4. $-\frac{\pi}{3}$ is not in $[0,\pi]$, but $\cos(-\frac{\pi}{3})=\cos(\frac{\pi}{3})$ because cosine is an even function.
5. $\frac{\pi}{3}$ is in $[0,\pi]$, so $\cos^{-1}(\cos(-\frac{\pi}{3}))=\frac{\pi}{3}$.

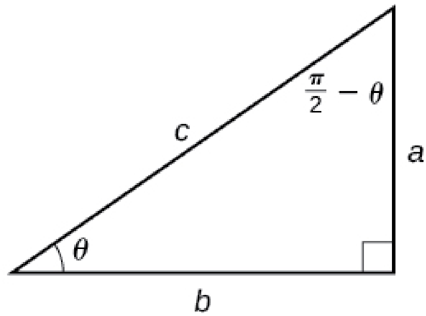
Evaluate $\tan^{-1}(\tan(\frac{\pi}{8}))$ and $\tan^{-1}(\tan(\frac{11\pi}{9}))$.

$\frac{\pi}{8}$; $\frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will

begin with compositions of the form $f^{-1}(g(x))$. For special values of x, we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2}-\theta$. Consider the sine and cosine of each angle of the right triangle in [link].



Because $\cos\theta=\frac{b}{c}=\sin(\frac{\pi}{2}-\theta)$, we have $\sin^{-1}(\cos\theta)=\frac{\pi}{2}-\theta$ if $0\le\theta\le\pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$. Similarly, $\sin\theta=\frac{a}{c}=\cos(\frac{\pi}{2}-\theta)$, so $\cos^{-1}(\sin\theta)=\frac{\pi}{2}-\theta$ if $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.
   $\sin^{-1}(\cos x) = \frac{\pi}{2} - y$
3. If x is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then find another angle y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin y = \sin x$.
   $\cos^{-1}(\sin x) = \frac{\pi}{2} - y$

Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

1. by direct evaluation.
2. by the method described previously.

---

1. Here, we can directly evaluate the inside of the composition.
   $$\cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6} + 2\pi\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

   Now, we can evaluate the inverse function as we did earlier.
   $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$
2. We have $x = \frac{13\pi}{6}, y = \frac{\pi}{6}$, and

$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin(\cos^{-1}(\frac{4}{5}))$.

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}(\frac{4}{5})$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

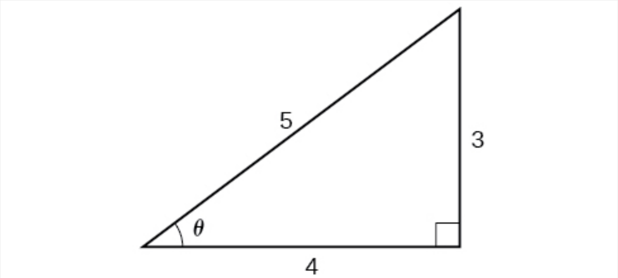
$$\sin^2\theta + \cos^2\theta = 1$$

Use our known value for cosine. $\sin^2\theta + (\frac{4}{5})^2 = 1$ Solve for sine. $\sin^2\theta = 1 - \frac{16}{25}$

$$\sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Since $\theta = \cos^{-1}(\frac{4}{5})$ is in quadrant I, $\sin\theta$ must be positive, so the solution is $\frac{3}{5}$. See [link].

Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$



We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive;

therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)=\sin\theta=\frac{3}{5}$.

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

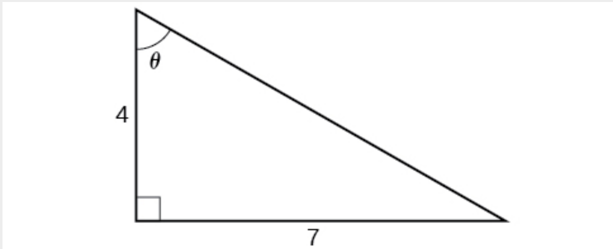
$\frac{12}{13}$

Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

While we could use a similar technique as in [link], we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta=\frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in [link].

A right triangle with two sides known



Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

4 2 + 7 2 = hypotenuse 2 hypotenuse = 65

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

sinθ = 7 65

This gives us our desired composition.

sin(tan −1 (7 4)) = sinθ = 7 65 = 7 65 65

Evaluate cos(sin −1 (7 9)).

4 2 9

Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos(\sin -1 (x\ 3))$ for $-3 \le x \le 3$.

We know there is an angle θ such that $\sin\theta = x\ 3$.

$\sin 2\ \theta + \cos 2\ \theta = 1$

Use the Pythagorean Theorem. $(x\ 3)\ 2 + \cos 2\ \theta = 1$ Solve for cosine. $\cos 2\ \theta = 1 - x\ 2\ 9$

$\cos\theta = \pm\ 9 - x\ 2\ 9 = \pm\ 9 - x\ 2\ 3$

Because we know that the inverse sine must give an angle on the interval $[-\pi\ 2 , \pi\ 2]$, we can deduce that the cosine of that angle must be positive.

$\cos(\sin -1 (x\ 3)) = 9 - x\ 2\ 3$

Find a simplified expression for $\sin(\tan -1 (4x))$ for $-\ 1\ 4 \le x \le 1\ 4$.

$4x\ 16\ x\ 2\ +1$

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x)=y$. However, $f(x)=y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See [link].
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6}$

$= \sin^{-1}\left(\frac{1}{2}\right)$. See [link].

- A calculator will return an angle within the restricted domain of the original trigonometric function. See [link].
- Inverse functions allow us to find an angle when given two sides of a right triangle. See [link].
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example, $\sin\left(\cos^{-1}(x)\right) = \sqrt{1-x^2}$. See [link].
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \le x \le \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. See [link] and [link].
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See [link].
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See [link].

Section Exercises

Verbal

Why do the functions $f(x)=\sin^{-1}x$ and $g(x)=\cos^{-1}x$ have different ranges?

The function $y=\sin x$ is one-to-one on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y=\sin x$, $f(x)=\sin^{-1}x$. The function $y=\cos x$ is one-to-one on $[0,\pi]$; thus, this interval is the range of the inverse function of $y=\cos x, f(x)=\cos^{-1}x$.

Since the functions $y=\cos x$ and $y=\cos^{-1}x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

Explain the meaning of $\frac{\pi}{6}=\arcsin(0.5)$.

$\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

Why must the domain of the sine function, $\sin x$, be restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for the inverse sine function to exist?

In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so that it is one-to-one and possesses an inverse.

Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$.

True . The angle, θ_1 that equals $\arccos(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2 , where θ_2 equals $\arccos x$, $x > 0$. Since θ_2 is the reference angle for θ_1 , $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x$ -

Algebraic

For the following exercises, evaluate the expressions.

$\sin^{-1} (2 \ 2)$

$\sin^{-1} (- \ 1 \ 2)$

$- \ \pi \ 6$

$\cos^{-1} (1 \ 2)$

$\cos^{-1} (- \ 2 \ 2)$

$3\pi \ 4$

$\tan^{-1} (1)$

$\tan^{-1} (- \ 3)$

$- \ \pi \ 3$

$\tan^{-1} (-1)$

$\tan^{-1}(\sqrt{3})$

$\frac{\pi}{3}$

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

$\cos^{-1}(-0.4)$

1.98

$\arcsin(0.23)$

$\arccos\left(\frac{3}{5}\right)$

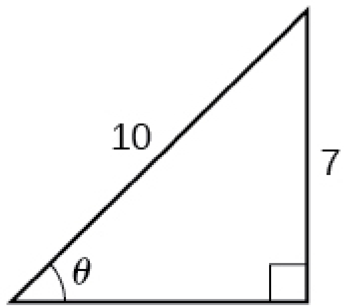
0.93

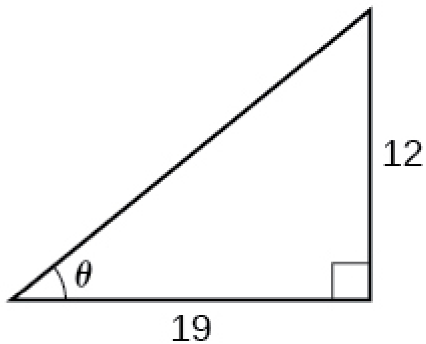
$\cos^{-1}(0.8)$

$\tan^{-1}(6)$

1.41

For the following exercises, find the angle θ in the given right triangle. Round answers to the nearest hundredth.





0.56 radians

For the following exercises, find the exact value, if

possible, without a calculator. If it is not possible, explain why.

$\sin^{-1}(\cos(\pi))$

$\tan^{-1}(\sin(\pi))$

0

$\cos^{-1}(\sin(\frac{\pi}{3}))$

$\tan^{-1}(\sin(\frac{\pi}{3}))$

0.71

$\sin^{-1}(\cos(\frac{-\pi}{2}))$

$\tan^{-1}(\sin(\frac{4\pi}{3}))$

-0.71

$\sin^{-1}(\sin(\frac{5\pi}{6}))$

$\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$

$-\frac{\pi}{4}$

$\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

$\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

0.8

$\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

$\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

$\frac{5}{13}$

$\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the following exercises, find the exact value of the expression in terms of x with the help of a reference triangle.

$\tan\left(\sin^{-1}\left(x-1\right)\right)$

$x-1 \; - \; x\,2 \; +2x$

$\sin(\cos \; -1 \; (1-x))$

$\cos(\sin \; -1 \; (1 \; x))$

$x\,2 \; -1 \; x$

$\cos(\tan \; -1 \; (3x-1))$

$\tan(\sin \; -1 \; (x+ \; 1 \; 2))$

$x+0.5 \; - \; x\,2 \; -x+ \; 3\;4$

Extensions

For the following exercises, evaluate the expression without using a calculator. Give the exact value.

$\sin \; -1 \; (1\;2)- \; \cos \; -1 \; (2\;2)+ \; \sin \; -1 \; (3\;2$
$)- \; \cos \; -1 \; (1) \; \cos \; -1 \; (3\;2)- \; \sin \; -1 \; (2\;2$
$)+ \; \cos \; -1 \; (1\;2)- \; \sin \; -1 \; (0)$

For the following exercises, find the function if $\sin t= \; x \; x+1$.

cost

2x+1 x+1

sect

cott

2x+1 x

cos(sin −1 (x x+1))

tan −1 (x 2x+1)

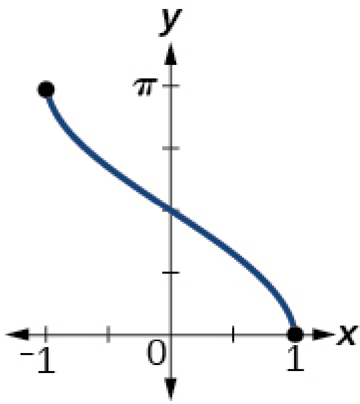
---

t

Graphical

Graph y= sin −1 x and state the domain and range of the function.

Graph y=arccosx and state the domain and range of the function.



domain $[-1,1]$; range $[0,\pi]$

Graph one cycle of $y=\tan^{-1}x$ and state the domain and range of the function.

For what value of x does $\sin x=\sin^{-1}x$? Use a graphing calculator to approximate the answer.

---

approximately $x=0.00$

For what value of x does $\cos x=\cos^{-1}x$? Use a graphing calculator to approximate the answer.

Real-World Applications

Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

0.395 radians

Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

1.11 radians

Without using a calculator, approximate the value of arctan(10,000). Explain why your answer is reasonable.

A truss for the roof of a house is constructed from two identical right triangles. Each has a

base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

1.25 radians

The line $y = 3\ 5\ x$ passes through the origin in the *x*,*y*-plane. What is the measure of the angle that the line makes with the positive *x*-axis?

The line $y = -3\ 7\ x$ passes through the origin in the *x*,*y*-plane. What is the measure of the angle that the line makes with the negative *x*-axis?

0.405 radians

What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

A 20-foot ladder leans up against the side of a

building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

No. The angle the ladder makes with the horizontal is 60 degrees.

Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

Chapter Review Exercises

Graphs of the Sine and Cosine Functions

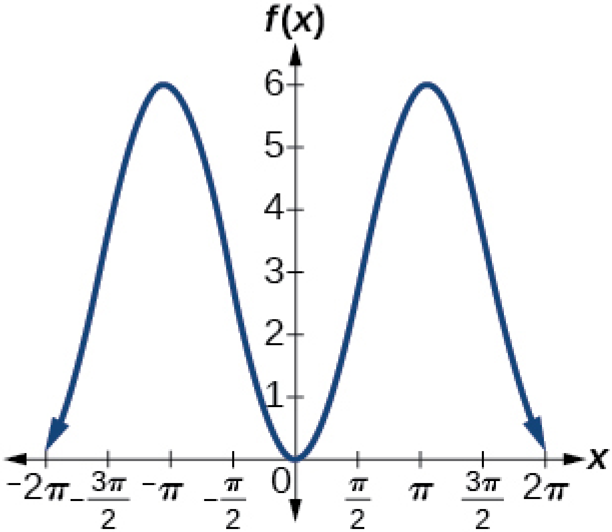
For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

$f(x)=-3\cos x+3$

---

amplitude: 3; period: 2π; midline: $y=3$; no

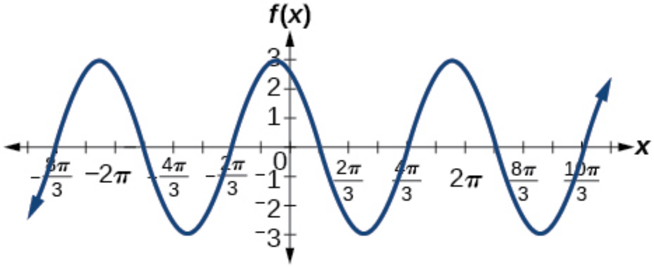
asymptotes



$f(x) = \frac{1}{4}\sin x$

$f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

amplitude: 3; period: 2π; midline: $y = 0$; no asymptotes

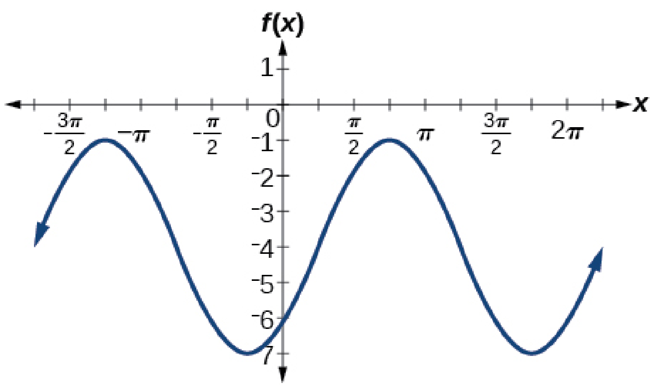


$f(x) = -2\sin\left(x - \frac{2\pi}{3}\right)$

$f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

---

amplitude: 3; period: 2π; midline: $y = -4$; no asymptotes

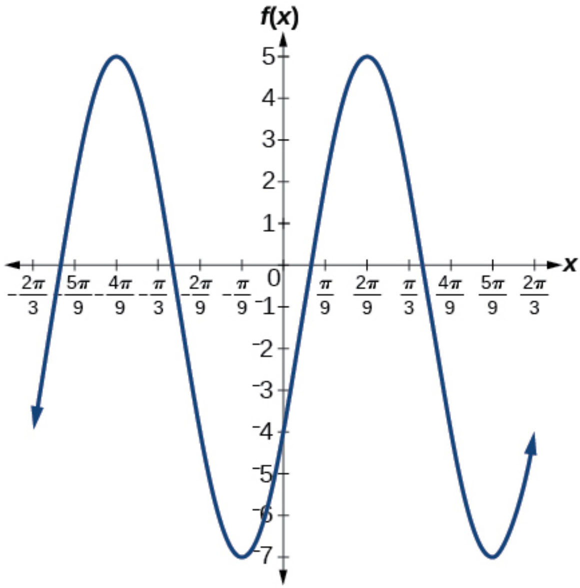


$f(x) = 2\left(\cos\left(x - \frac{4\pi}{3}\right) + 1\right)$

$f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

---

amplitude: 6; period: $\frac{2\pi}{3}$; midline: $y = -1$; no asymptotes



$f(x) = -100\sin(50x - 20)$

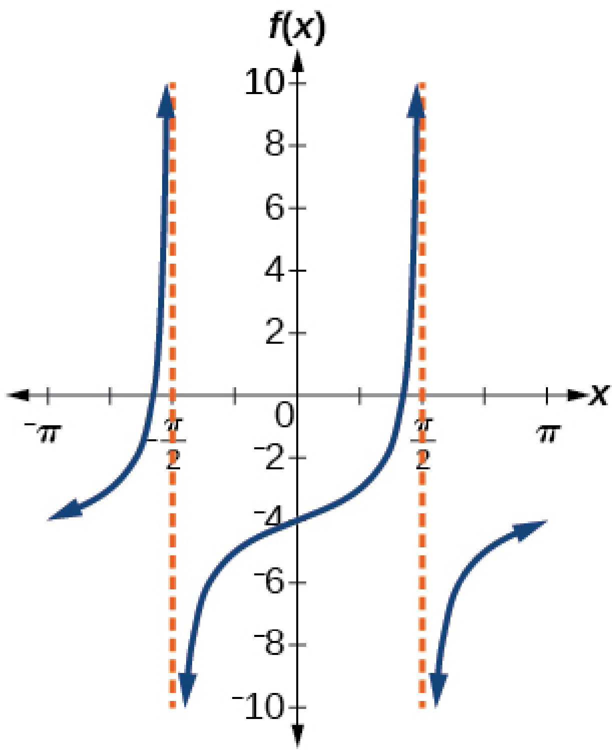
Graphs of the Other Trigonometric Functions

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

$f(x) = \tan x - 4$

---

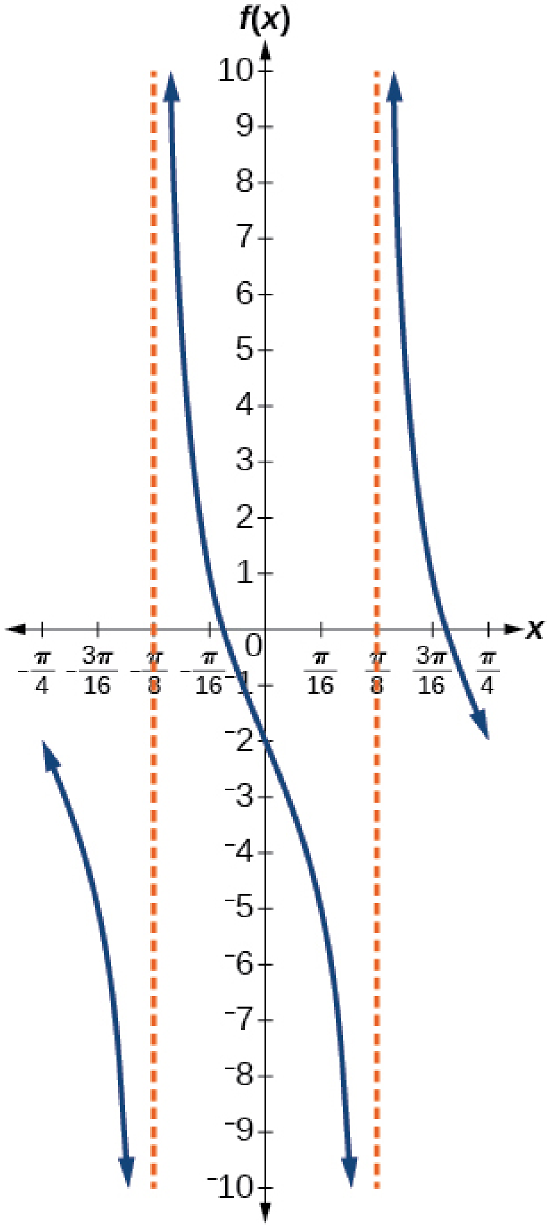
stretching factor: none; period: π; midline: $y = -4$; asymptotes: $x = \frac{\pi}{2} + \pi k$, where k is an integer



$f(x) = 2\tan\left(x - \frac{\pi}{6}\right)$

$f(x) = -3\tan(4x) - 2$

stretching factor: 3; period: $\frac{\pi}{4}$; midline: $y = -2$; asymptotes: $x = \frac{\pi}{8} + \frac{\pi}{4}k$, where k is an integer



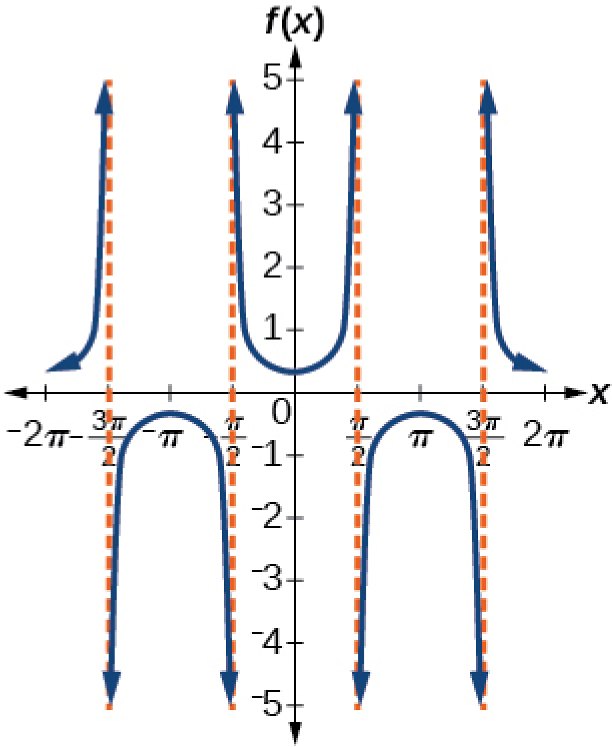
$f(x) = 0.2\cos(0.1x) + 0.3$

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

$f(x) = \frac{1}{3}\sec x$

---

amplitude: none; period: 2π; no phase shift; asymptotes: $x = \frac{\pi}{2}k$, where k is an odd integer

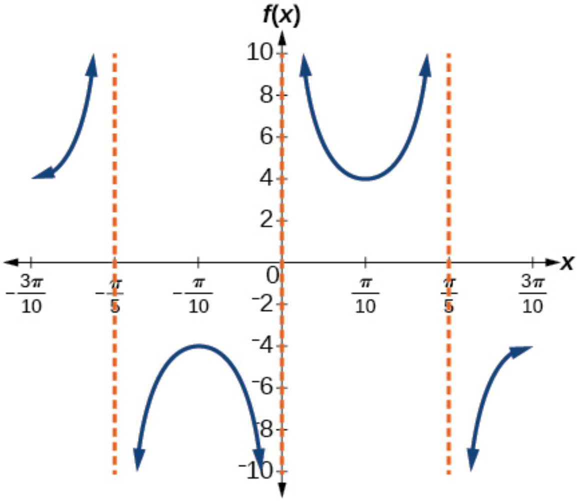


$f(x) = 3\cot x$

$f(x) = 4\csc(5x)$

---

amplitude: none; period: $\frac{2\pi}{5}$; no phase shift; asymptotes: $x = \frac{\pi}{5}k$, where k is an integer

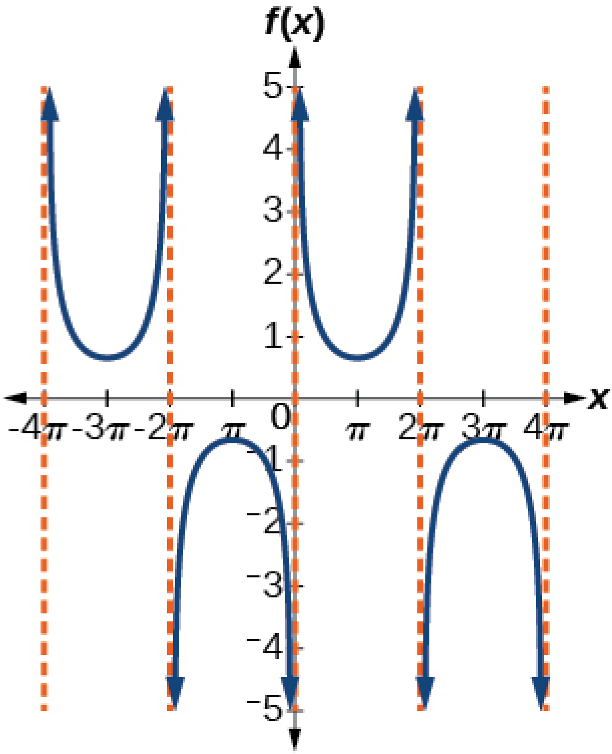


$f(x) = 8\sec\left(\frac{1}{4}x\right)$

$f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$

---

amplitude: none; period: 4π; no phase shift; asymptotes: $x = 2\pi k$, where k is an integer



$f(x) = -\csc(2x + \pi)$

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

What is the largest and smallest population the city may have?

---

largest: 20,000; smallest: 4,000

Graph the function on the domain of $[0,40]$.

What are the amplitude, period, and phase shift for the function?

---

amplitude: 8,000; period: 10; phase shift: 0

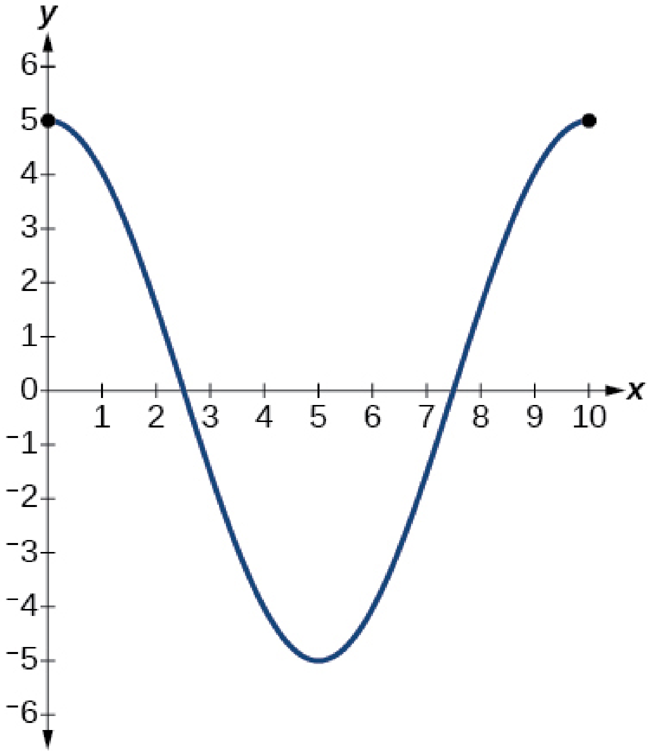
Over this domain, when does the population reach 18,000? 13,000?

What is the predicted population in 2007? 2010?

In 2007, the predicted population is 4,413. In 2010, the population will be 11,924.

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

Suppose the graph of the displacement function is shown in [link], where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches. Give the equation that models the vertical displacement of the weight on the spring.



At time $= 0$, what is the displacement of the weight?

---

5 in.

At what time does the displacement from the equilibrium point equal zero?

What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

---

10 seconds

Inverse Trigonometric Functions

For the following exercises, find the exact value without the aid of a calculator.

$\sin -1 (1)$

$\cos -1 (3\ 2)$

$\pi\ 6$

$\tan -1 (-1)$

$\cos -1 (1\ 2)$

$\pi\ 4$

$\sin^{-1}(-\ 3\ 2)$

$\sin^{-1}(\cos(\pi\ 6))$

$\pi\ 3$

$\cos^{-1}(\tan(3\pi\ 4))$

$\sin(\sec^{-1}(3\ 5))$

No solution

$\cot(\sin^{-1}(3\ 5))$

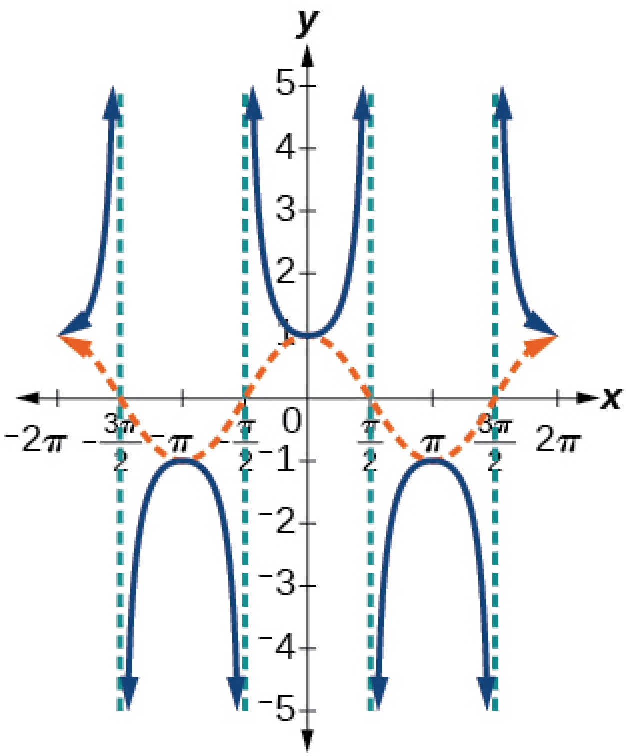
$\tan(\cos^{-1}(5\ 13))$

$12\ 5$

$\sin(\cos^{-1}(x\ x+1))$

Graph $f(x)=\cos x$ and $f(x)=\sec x$ on the interval $[0, 2\pi)$ and explain any observations.

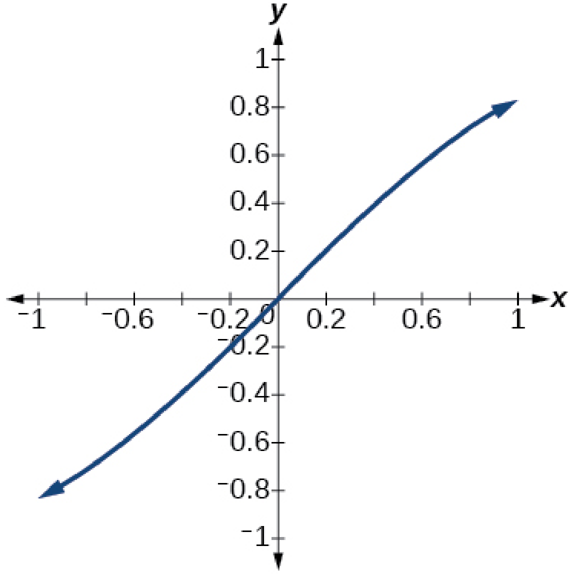
The graphs are not symmetrical with respect to the line $y=x$. They are symmetrical with respect to the y-axis.

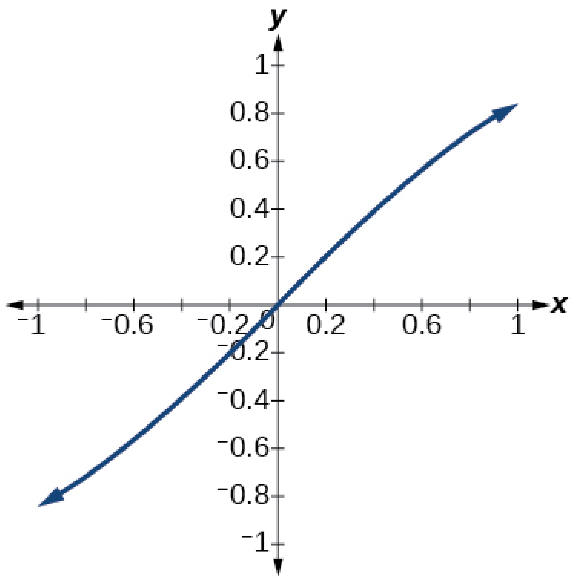


Graph $f(x)=\sin x$ and $f(x)=\csc x$ and explain any observations.

Graph the function $f(x)=\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$ on the interval $[-1,1]$ and compare the graph to the graph of $f(x)=\sin x$ on the same interval. Describe any observations.

The graphs appear to be identical.


y
x
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.6
−0.2
0
0.2
0.6
1


y
x
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.6
−0.2
0
0.2
0.6
1

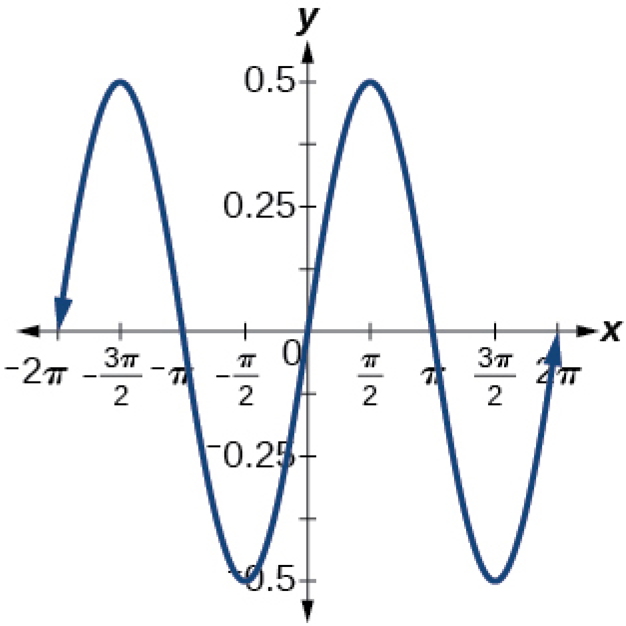
Chapter Practice Test

For the following exercises, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

$f(x) = 0.5\sin x$

---

amplitude: 0.5; period: 2π; midline $y = 0$

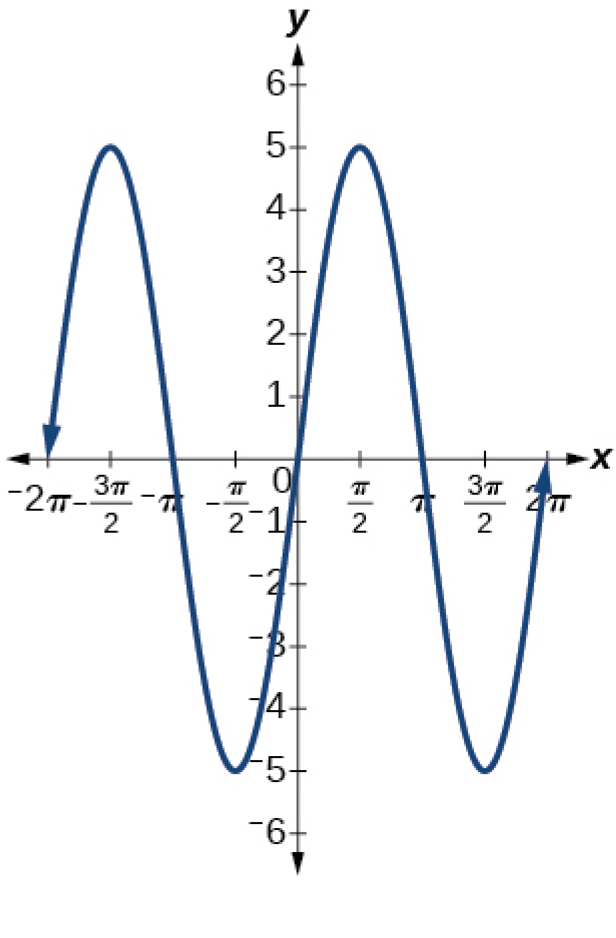


$f(x) = 5\cos x$

$f(x) = 5\sin x$

---

amplitude: 5; period: 2π; midline: $y = 0$

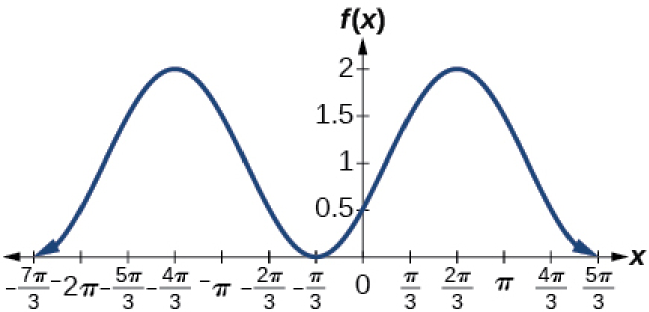


$f(x) = \sin(3x)$

$f(x) = -\cos\left(x + \frac{\pi}{3}\right) + 1$

---

amplitude: 1; period: 2π; midline: $y = 1$

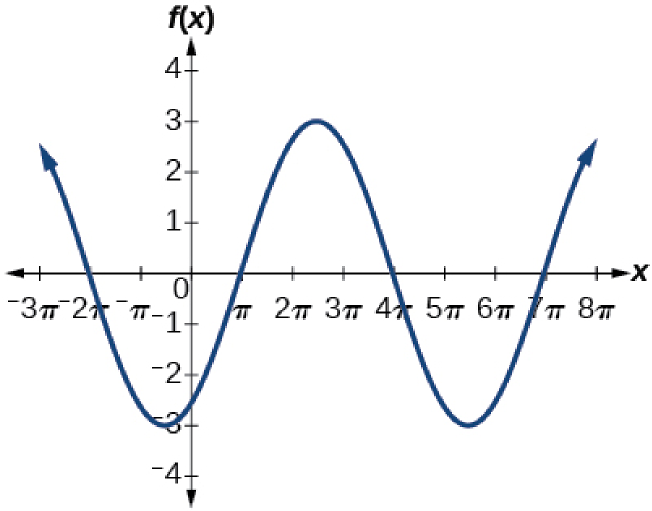


$f(x) = 5\sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 4$

$f(x) = 3\cos\left(\frac{1}{3}x - \frac{5\pi}{6}\right)$

---

amplitude: 3; period: 6π; midline: $y = 0$

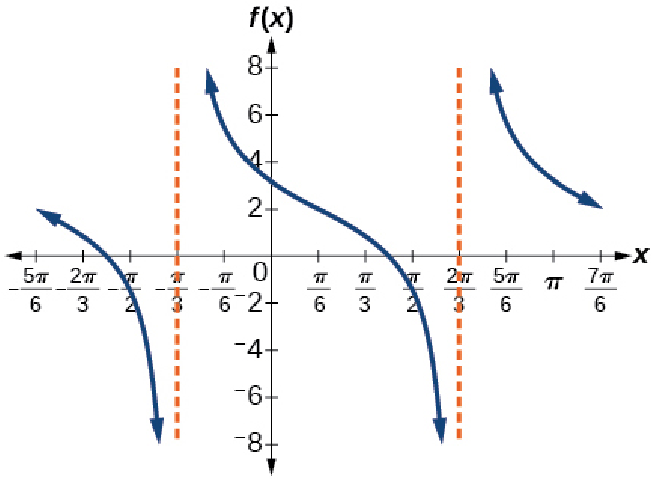


$f(x) = \tan(4x)$

$f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

---

amplitude: none; period: π; midline: $y = 0$, asymptotes: $x = \frac{2\pi}{3} + \pi k$, where k is an integer

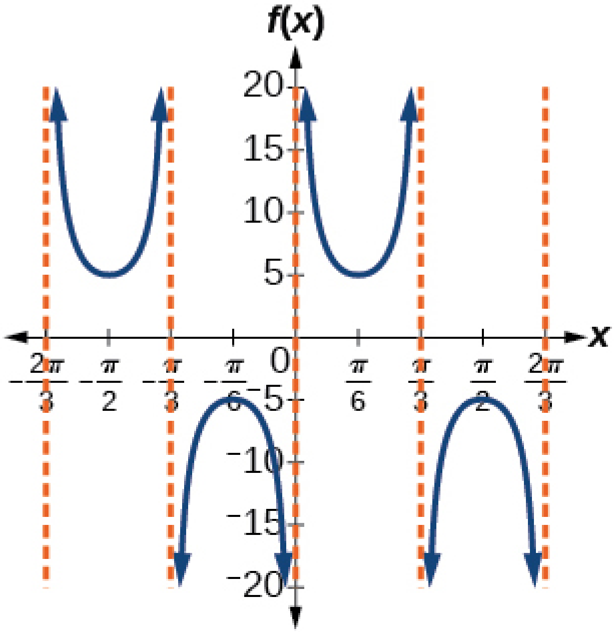


$f(x)=\pi\cos(3x+\pi)$

$f(x)=5\csc(3x)$

---

amplitude: none; period: $\frac{2\pi}{3}$; midline: $y=0$, asymptotes: $x=\frac{\pi}{3}k$, where k is an integer

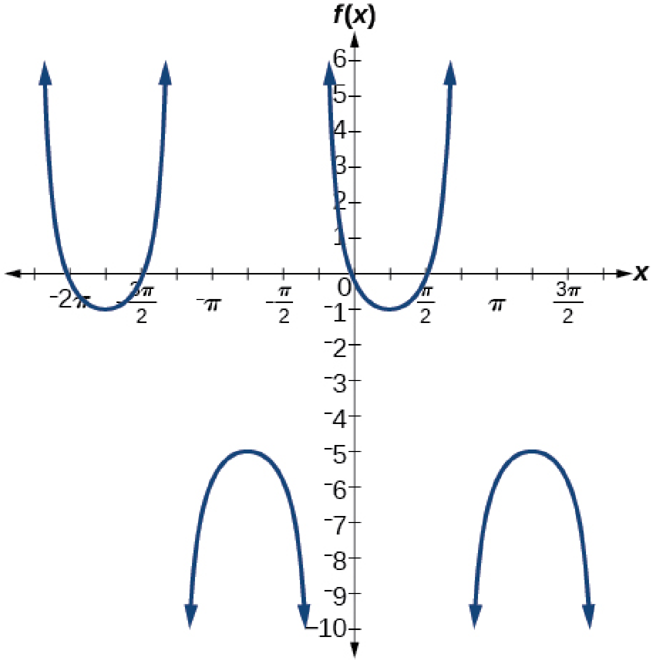


$f(x)=\pi\sec\left(\frac{\pi}{2}x\right)$

$f(x)=2\csc\left(x+\frac{\pi}{4}\right)-3$

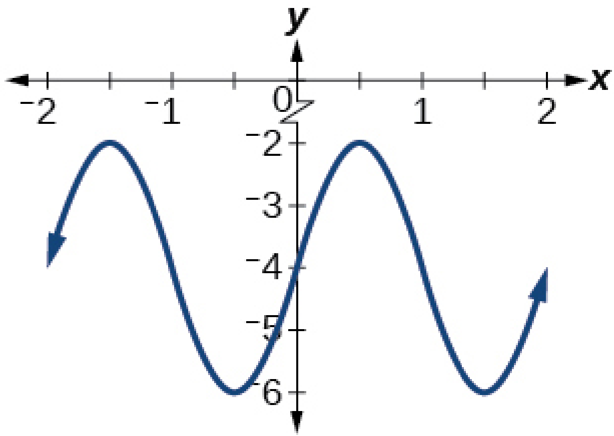
---

amplitude: none; period: 2π; midline: $y=-3$

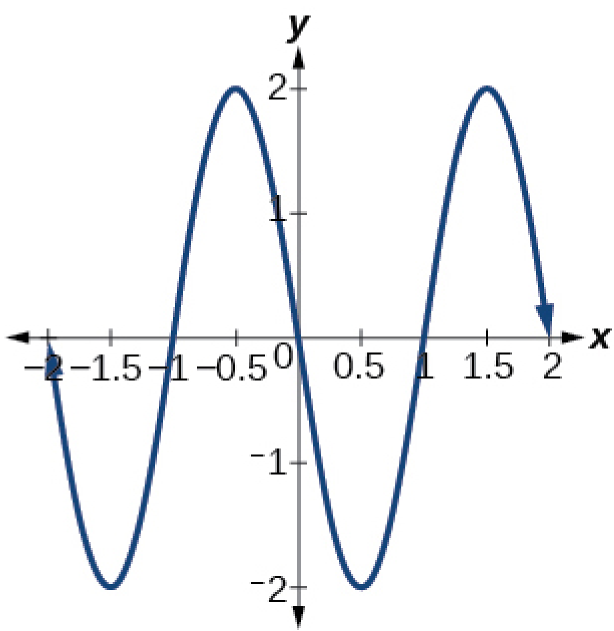


For the following exercises, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

Give in terms of a sine function.

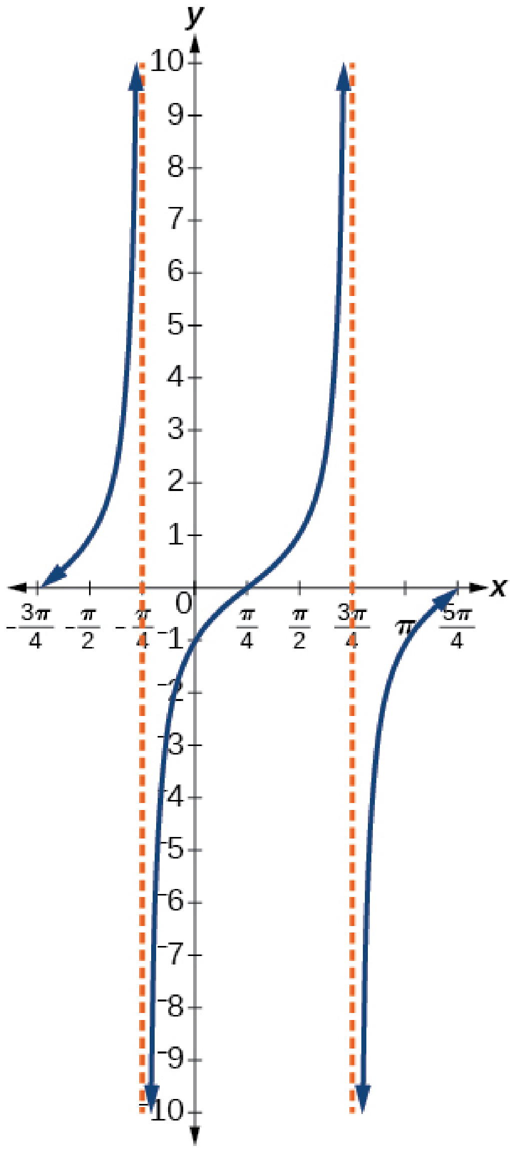


Give in terms of a sine function.



amplitude: 2; period: 2; midline: $y=0$; $f(x)=2\sin(\pi(x-1))$

Give in terms of a tangent function.



For the following exercises, find the amplitude, period, phase shift, and midline.

$y=\sin(\frac{\pi}{6}x+\pi)-3$

---

amplitude: 1; period: 12; phase shift: -6; midline $y=-3$

$y=8\sin(\frac{7\pi}{6}x+\frac{7\pi}{2})+6$

The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

$D(t)=68-12\sin(\frac{\pi}{12}x)$

Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the following exercises, find the period and

horizontal shift of each function.

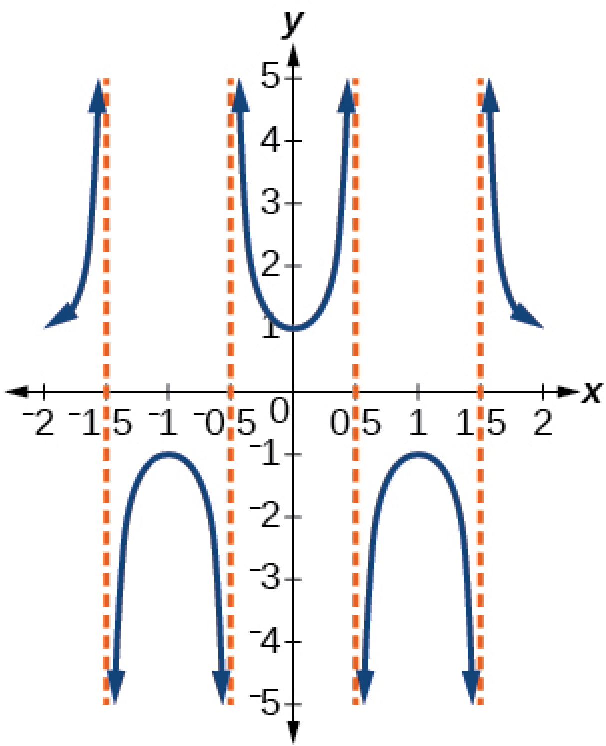
$g(x)=3\tan(6x+42)$

---

period: $\frac{\pi}{6}$; horizontal shift: -7

$n(x)=4\csc\left(\frac{5\pi}{3}x-\frac{20\pi}{3}\right)$

Write the equation for the graph in [link] in terms of the secant function and give the period and phase shift.



$f(x) = \sec(\pi x)$; period: 2; phase shift: 0

If $\tan x = 3$, find $\tan(-x)$.

If $\sec x = 4$, find $\sec(-x)$.

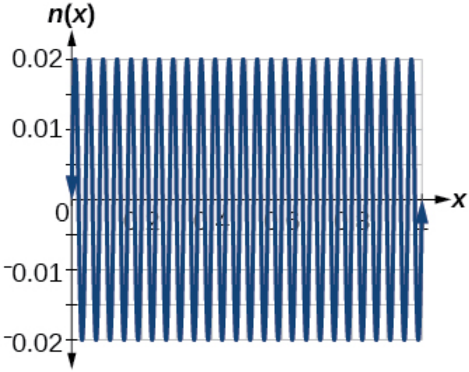
4

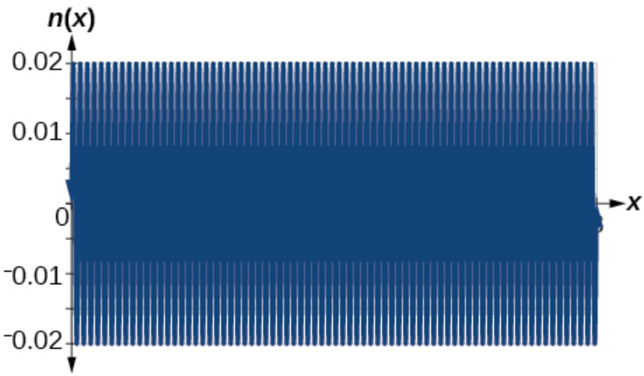
For the following exercises, graph the functions on the specified window and answer the questions.

Graph $m(x) = \sin(2x) + \cos(3x)$ on the viewing window $[-10, 10]$ by $[-3, 3]$. Approximate the graph's period.

Graph $n(x) = 0.02\sin(50\pi x)$ on the following domains in x: $[0, 1]$ and $[0, 3]$. Suppose this function models sound waves. Why would these views look so different?

The views are different because the period of the wave is $\frac{1}{25}$. Over a bigger domain, there will be more cycles of the graph.





Graph $f(x) = \text{sinx x}$ on $[-0.5, 0.5]$ and explain any observations.

For the following exercises, let $f(x) = 3\ 5 \cos(6x)$.

What is the largest possible value for $f(x)$?

---

3 5

What is the smallest possible value for $f(x)$?

Where is the function increasing on the interval $[0, 2\pi]$?

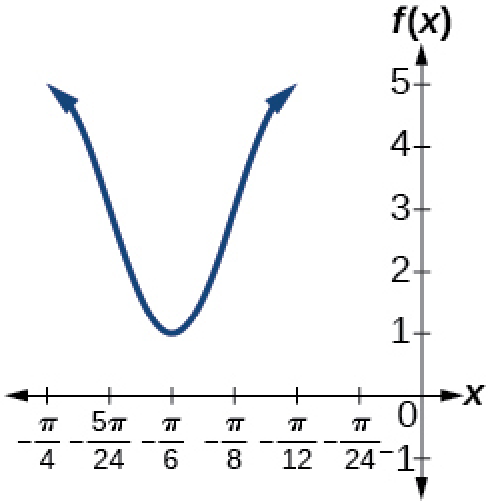
On the approximate intervals $(0.5, 1), (1.6, 2.1), (2.6, 3.1), (3.7, 4.2), (4.7, 5.2), (5.6, 6.28)$

For the following exercises, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

Sine curve with amplitude 3, period $\frac{\pi}{3}$, and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

Cosine curve with amplitude 2, period $\frac{\pi}{6}$, and phase shift $(h, k) = \left(-\frac{\pi}{4}, 3\right)$

$f(x) = 2\cos\left(12\left(x + \frac{\pi}{4}\right)\right) + 3$

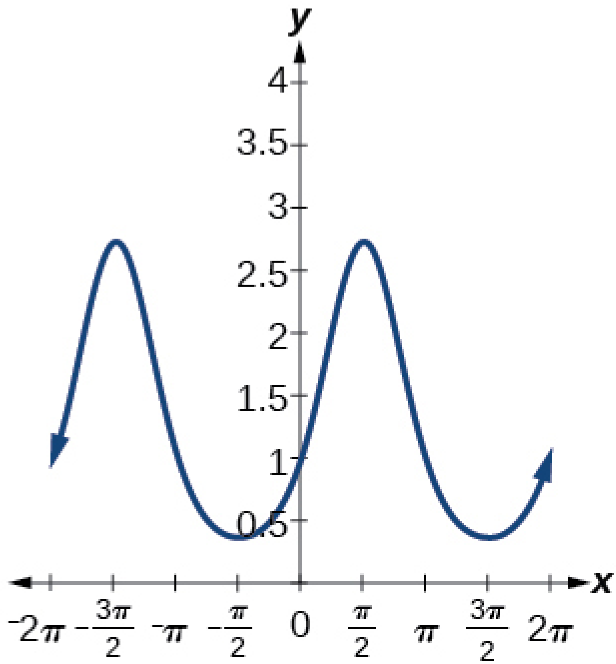


For the following exercises, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

$f(x) = 5\cos(3x) + 4\sin(2x)$

$f(x) = e^{\sin t}$

This graph is periodic with a period of 2π.



For the following exercises, find the exact value.

sin −1 (3 2)

tan −1 (3)

π 3

cos −1 (− 3 2)

cos −1 (sin(π))

$\frac{\pi}{2}$

$\cos^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

$\cos\left(\sin^{-1}\left(1-2x\right)\right)$

$1-\left(1-2x\right)^2$

$\cos^{-1}\left(-0.4\right)$

$\cos\left(\tan^{-1}\left(x^2\right)\right)$

$\frac{1}{1+x^4}$

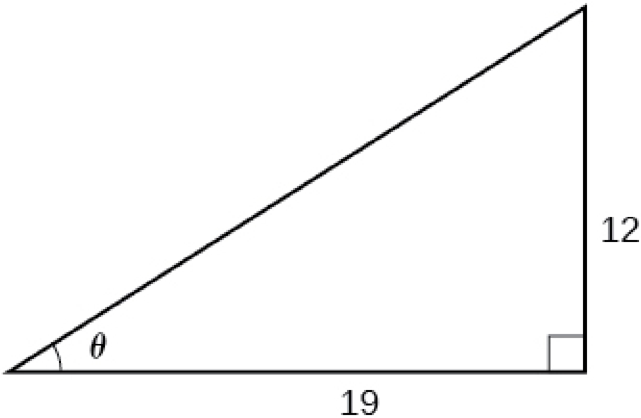
For the following exercises, suppose $\sin t=\frac{x}{x+1}$. Evaluate the following expressions.

$\tan t$

$\csc t$

$\frac{x+1}{x}$

Given [link], find the measure of angle θ to three decimal places. Answer in radians.



For the following exercises, determine whether the equation is true or false.

$\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

False

$\arccos\left(\cos\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.

---

approximately 0.07 radians

Glossary

arccosine
: another name for the inverse cosine; $\arccos x = \cos^{-1} x$

arcsine
: another name for the inverse sine; $\arcsin x = \sin^{-1} x$

arctangent
: another name for the inverse tangent; $\arctan x = \tan^{-1} x$

inverse cosine function
: the function $\cos^{-1} x$, which is the inverse of the cosine function and the angle that has a cosine equal to a given number

inverse sine function
: the function $\sin^{-1} x$, which is the inverse of the sine function and the angle that has a sine equal to a given number

inverse tangent function
: the function $\tan^{-1} x$, which is the inverse of the tangent function and the angle that has a tangent equal to a given number

Solving Trigonometric Equations with Identities

In this section, you will:

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

International passports and travel documents

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to

represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Graph of $y = \sin\theta$ Graph of $y = \cos\theta$

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the Pythagorean Identities, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean Identities** (see [link]), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Pythagorean Identities		
$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+ \cot 2 \theta= \csc 2 \theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+ \cot 2 \theta= \csc 2 \theta$
$1+ \cot 2 \theta=(1+ \cos 2 \theta \sin 2 \theta)$
Rewrite the left side. $=(\sin 2 \theta \sin 2 \theta)+(\cos 2 \theta \sin 2 \theta)$
Write both terms with the common denominator. $= \sin 2 \theta+ \cos 2 \theta \sin 2 \theta = 1 \sin 2 \theta = \csc 2 \theta$

Similarly, $1+ \tan 2 \theta= \sec 2 \theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives
$1+ \tan 2 \theta=1+ (\sin\theta \cos\theta) 2$ Rewrite left side. $= (\cos\theta \cos\theta) 2 + (\sin\theta \cos\theta) 2$
Write both terms with the common denominator. $= \cos 2 \theta+ \sin 2 \theta \cos 2 \theta = 1 \cos 2 \theta = \sec 2 \theta$

The next set of fundamental identities is the set of **even-odd identities.** The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle and determine whether the identity is odd or even. (See [link]).

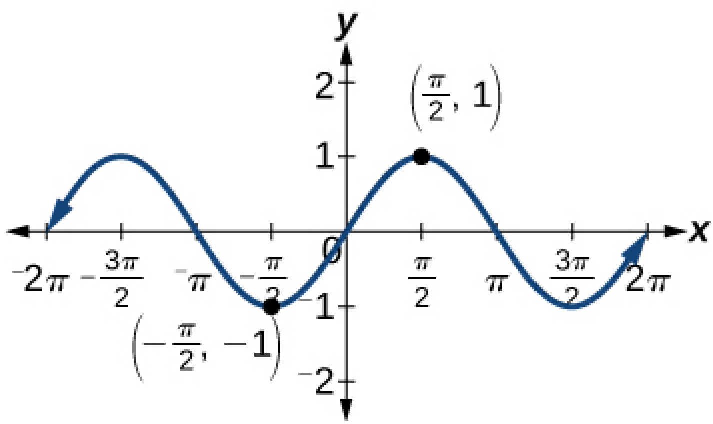
Even-Odd Identities

$\tan(-\theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\cot(-\theta) = -\cot\theta$	$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$

Recall that an odd function is one in which $f(-x) = -f(x)$ for all x in the domain of f. The sine function is an odd function because $\sin(-\theta) = -\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\sin\left(\frac{\pi}{2}\right) = 1 \text{ and } \sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

This is shown in [link].



Recall that an even function is one in which

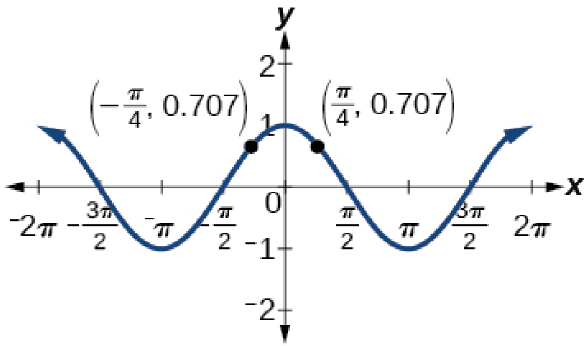
$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function

because $\cos(-\theta)=\cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right) \approx 0.707$$

See [link].



For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta)=-\sin\theta$, sine is an odd function.
- Since, $\cos(-\theta)=\cos\theta$, cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity, $\tan(-\theta)=-\tan\theta$. We can interpret the tangent of a negative angle as $\tan(-\theta)=\frac{\sin(-\theta)}{\cos(-\theta)}=\frac{-\sin\theta}{\cos\theta}=-\tan\theta$. Tangent is therefore an odd function, which means that $\tan(-\theta)=-\tan(\theta)$ for all θ in the domain of the tangent function.

The cotangent identity, $\cot(-\theta)=-\cot\theta$, also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as $\cot(-\theta)=\frac{\cos(-\theta)}{\sin(-\theta)}=\frac{\cos\theta}{-\sin\theta}=-\cot\theta$. Cotangent is therefore an odd function, which means that $\cot(-\theta)=-\cot(\theta)$ for all θ in the domain of the cotangent function.

The cosecant function is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as $\csc(-\theta)=\frac{1}{\sin(-\theta)}=\frac{1}{-\sin\theta}=-\csc\theta$. The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as $\sec(-\theta)=\frac{1}{\cos(-\theta)}=\frac{1}{\cos\theta}=\sec\theta$. The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other. See [link].

Reciprocal Identities	
$\sin\theta = 1 \csc\theta$	$\csc\theta = 1 \sin\theta$
$\cos\theta = 1 \sec\theta$	$\sec\theta = 1 \cos\theta$
$\tan\theta = 1 \cot\theta$	$\cot\theta = 1 \tan\theta$

The final set of identities is the set of quotient identities, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities. See [link].

Quotient Identities	
$\tan\theta = \sin\theta \cos\theta$	$\cot\theta = \cos\theta \sin\theta$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

Summarizing Trigonometric Identities

The **Pythagorean Identities** are based on the properties of a right triangle.

$$\cos 2 \theta + \sin 2 \theta = 1$$

$$1 + \cot 2 \theta = \csc 2 \theta$$

$$1 + \tan 2 \theta = \sec 2 \theta$$

The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$\tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$

$\sin(-\theta) = -\sin\theta$

$\csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$

$\sec(-\theta) = \sec\theta$

The **reciprocal identities** define reciprocals of the trigonometric functions.

$\sin\theta = \ 1 \ \csc\theta$

$\cos\theta = \ 1 \ \sec\theta$

$\tan\theta = \ 1 \ \cot\theta$

$\csc\theta = \ 1 \ \sin\theta$

$\sec\theta = \ 1 \ \cos\theta$

$\cot\theta = \ 1 \ \tan\theta$

The **quotient identities** define the relationship among the trigonometric functions.

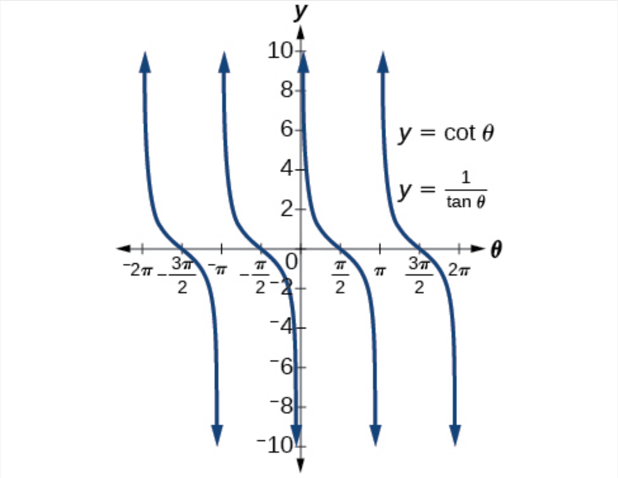
$\tan\theta = \ \sin\theta \ \cos\theta$

$\cot\theta = \ \cos\theta \ \sin\theta$

Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \ 1 \ \tan\theta$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \ 1 \ \tan\theta$.

See [link].



Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to confirm an identity verified with analytical means. If both expressions give the same graph, then they are most likely identities.

Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions,

square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

We will start on the left side, as it is the more complicated side:

$$\tan\theta\cos\theta = \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta = \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta = \sin\theta$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\end{aligned}$$

$$= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$= 1$$

Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:
$(1+\sin x)[1+\sin(-x)] = \cos^2 x$

Working on the left side of the equation, we have
$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$ Since $\sin(-x) = -\sin x$
$= 1 - \sin^2 x$ Difference of squares
$= \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$

Verifying a Trigonometric Identity Involving $sec^2\theta$

Verify the identity $\frac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

As the left side is more complicated, let's begin there.

$\sec 2\,\theta - 1 \sec 2\,\theta = (\tan 2\,\theta + 1) - 1 \sec 2\,\theta$
$\sec 2\,\theta = \tan 2\,\theta + 1 \qquad = \tan 2\,\theta \sec 2$
$\theta \qquad = \tan 2\,\theta(1 \sec 2\,\theta)$
$\qquad = \tan 2\,\theta(\cos 2\,\theta) \cos 2\,\theta = 1 \sec 2$
$\theta \qquad = (\sin 2\,\theta \cos 2\,\theta)(\cos 2\,\theta) \tan 2$
$\theta = \sin 2\,\theta \cos 2\,\theta \qquad = (\sin 2\,\theta \cos 2$
$\theta)(\cos 2\,\theta) \qquad = \sin 2\,\theta$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$\sec 2\,\theta - 1 \sec 2\,\theta = \sec 2\,\theta \sec 2\,\theta - 1 \sec 2$
$\theta \qquad = 1 - \cos 2\,\theta \qquad = \sin 2$
θ

Analysis

In the first method, we used the identity $\sec 2\,\theta = \tan 2\,\theta + 1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Show that $\cot\theta \csc\theta = \cos\theta$.

$\cot\theta \csc\theta = \cos\theta \sin\theta \ 1 \sin\theta = \cos\theta \sin\theta \cdot \sin\theta \ 1 = \cos\theta$

Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:
$2\tan\theta\sec\theta = 2(\sin\theta \cos\theta)(1 \cos\theta) = 2\sin\theta \cos 2 \theta = 2\sin\theta \ 1 - \sin 2 \theta$ Substitute $1 - \sin 2 \theta$ for $\cos 2 \theta$

Thus,
$2\tan\theta\sec\theta = 2\sin\theta \ 1 - \sin 2 \theta$

Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$\sin 2(-\theta)-\cos 2(-\theta)\sin(-\theta)-\cos(-\theta)=\cos\theta-\sin\theta$

Let’s start with the left side and simplify:

$\sin 2(-\theta)-\cos 2(-\theta)\sin(-\theta)-\cos(-\theta)=[\sin(-\theta)]2-[\cos(-\theta)]2\sin(-\theta)-\cos(-\theta)$

$=(-\sin\theta)2-(\cos\theta)2-\sin\theta-\cos\theta$

$\sin(-x)=-\sin x \text{and} \cos(-x)=\cos x$

$=(\sin\theta)2-(\cos\theta)2-\sin\theta-\cos\theta$ Difference of squares

$=(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)-(\sin\theta+\cos\theta)$

$=(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)-(\sin\theta+\cos\theta)$

$=\cos\theta-\sin\theta$

Verify the identity $\sin 2\theta-1\tan\theta\sin\theta-\tan\theta=\sin\theta+1\tan\theta$.

$\sin 2\theta-1\tan\theta\sin\theta-\tan\theta=(\sin\theta+1)(\sin\theta-1)\tan\theta(\sin\theta-1)=\sin\theta+1\tan\theta$

Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1- \cos 2 x)(1+ \cot 2 x) = 1$.

We will work on the left side of the equation.

$(1- \cos 2 x)(1+ \cot 2 x) = (1- \cos 2 x)(1 + \cos 2 x \sin 2 x)$

$= (1- \cos 2 x)(\sin 2 x \sin 2 x + \cos 2 x \sin 2 x)$

Find the common denominator.

$= (1- \cos 2 x)(\sin 2 x + \cos 2 x \sin 2 x)$

$= (\sin 2 x)(1 \sin 2 x)$

$= 1$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the

difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x+1)(\sin x-1)=0$ resembles the equation $(x+1)(x-1)=0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2 - b^2 = (a-b)(a+b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta+\cos\theta-1$.

Notice that the pattern displayed has the same form as a standard quadratic expression, ax^2+bx+c. Letting $\cos\theta=x$, we can rewrite the expression as follows:

$$2x^2+x-1$$

This expression can be factored as $(2x-1)(x+1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression: $4\cos^2\theta-1$.

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares. Thus,

$$\begin{aligned}4\cos^2\theta-1&=(2\cos\theta)^2-1\\&=(2\cos\theta-1)(2\cos\theta+1)\end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta = x$, rewrite the expression as $4x^2 - 1$, and factor $(2x - 1)(2x + 1)$. Then replace x with $\cos\theta$ and solve for the angle.

Rewrite the trigonometric expression: $25 - 9\sin^2\theta$.

This is a difference of squares formula: $25 - 9\sin^2\theta = (5 - 3\sin\theta)(5 + 3\sin\theta)$.

Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$\csc^2\theta - \cot^2\theta$

We can start with the Pythagorean identity.

$1 + \cot^2\theta = \csc^2\theta$

Now we can simplify by substituting $1+\cot^2\theta$ for $\csc^2\theta$. We have

$$\begin{aligned}\csc^2\theta-\cot^2\theta &= 1+\cot^2\theta-\cot^2\theta\\ &= 1\end{aligned}$$

Use algebraic techniques to verify the identity: $\frac{\cos\theta}{1+\sin\theta}=\frac{1-\sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1-\sin\theta$.)

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)=\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} &= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}\\ &= \frac{1-\sin\theta}{\cos\theta}\end{aligned}$$

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean Identities	$\sin^2\theta + \cos^2\theta = 1$ $1 + \cot^2\theta = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$
Even-odd identities	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$ $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$
Reciprocal identities	$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Quotient identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the

identities illustrates how expressions can be rewritten to simplify a problem.

- Graphing both sides of an identity will verify it. See [link].
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See [link] and [link].
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See [link].
- We can create an identity by simplifying an expression and then verifying it. See [link].
- Verifying an identity may involve algebra with the fundamental identities. See [link] and [link].
- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See [link], [link], and [link].

Section Exercises

Verbal

We know $g(x)=\cos x$ is an even function, and

$f(x) = \sin x$ and $h(x) = \tan x$ are odd functions. What about $G(x) = \cos 2 x, F(x) = \sin 2 x$, and $H(x) = \tan 2 x$? Are they even, odd, or neither? Why?

All three functions, F, G, and H, are even.

This is because $F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin 2 x = F(x), G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = \cos 2 x = G(x)$ and $H(-x) = \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan 2 x = H(x)$.

Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

When $\cos t = 0$, then $\sec t = 1\ 0$, which is undefined.

All of the Pythagorean Identities are related. Describe how to manipulate the equations to

get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

Algebraic

For the following exercises, use the fundamental identities to fully simplify the expression.

$\sin x \cos x \sec x$

$\sin x$

$\sin(-x)\cos(-x)\csc(-x)$

$\tan x \sin x + \sec x \cos^2 x$

$\sec x$

$\csc x + \cos x \cot(-x)$

$\cot t + \tan t \sec(-t)$

$\csc t$

$3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

$-\tan(-x)\cot(-x)$

-1

$-\sin(-x)\cos x\sec x\csc x\tan x \cot x$

$1+ \tan 2 \theta \csc 2 \theta + \sin 2 \theta + 1 \sec 2 \theta$

$\sec 2 x$

$(\tan x \csc 2 x + \tan x \sec 2 x)(1+\tan x 1+\cot x$
$)- 1 \cos 2 x$

$1- \cos 2 x \tan 2 x +2 \sin 2 x$

$\sin 2 x+1$

For the following exercises, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

$\tan x+\cot x \csc x ;\cos x$

secx + cscx 1 + tanx ;sinx

---

1 sinx

cosx 1 + sinx + tanx;cosx

1 sinxcosx − cotx;cotx

---

1 cotx

1 1 − cosx − cosx 1 + cosx ;cscx

(secx + cscx)(sinx + cosx) − 2 − cotx;tanx

---

tanx

1 cscx − sinx ;secx and tanx

1 − sinx 1 + sinx − 1 + sinx 1 − sinx
;secx and tanx

− 4secxtanx

$\tan x; \sec x$

$\sec x; \cot x$

$\pm\frac{1}{\sqrt{\cot^2 x+1}}$

$\sec x; \sin x$

$\cot x; \sin x$

$\pm\frac{\sqrt{1-\sin^2 x}}{\sin x}$

$\cot x; \csc x$

For the following exercises, verify the identity.

$\cos x-\cos^3 x=\cos x\sin^2 x$

Answers will vary. Sample proof:

$\cos x-\cos^3 x=\cos x(1-\cos^2 x)$
$=\cos x\sin^2 x$

$\cos x(\tan x-\sec(-x))=\sin x-1$

$$\frac{1+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1+2\tan^2 x$$

Answers will vary. Sample proof:
$$\frac{1+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x + \tan^2 x = \tan^2 x + 1 + \tan^2 x = 1+2\tan^2 x$$

$$(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$$

$$\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$$

Answers will vary. Sample proof:
$$\cos^2 x - \tan^2 x = 1 - \sin^2 x - (\sec^2 x - 1) = 1 - \sin^2 x - \sec^2 x + 1 = 2 - \sin^2 x - \sec^2 x$$

Extensions

For the following exercises, prove or disprove the identity.

$$\frac{1}{1+\cos x} - \frac{1}{1-\cos(-x)} = -2\cot x\csc x$$

$$\csc^2 x(1+\sin^2 x) = \cot^2 x$$

False

$(\sec^2(-x) - \tan^2 x \tan x)\left(\frac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x = \cos 2x$

$\frac{\tan x \sec x}{\sin(-x)} = \cos^2 x$

False

$\frac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

$\frac{1+\sin x}{\cos x} = \frac{\cos x}{1+\sin(-x)}$

Proved with negative and Pythagorean Identities

For the following exercises, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

$\frac{\cos^2\theta - \sin^2\theta}{1-\tan^2\theta} = \sin^2\theta$

$3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

True $3\sin^2\theta + 4\cos^2\theta = 3\sin^2\theta + 3\cos^2\theta + \cos^2\theta = 3(\sin^2\theta + \cos^2\theta) + \cos^2\theta = 3 + \cos^2\theta$

$\sec\theta + \tan\theta \cot\theta + \cos\theta = \sec^2\theta$

Glossary

even-odd identities
: set of equations involving trigonometric functions such that if $f(-x) = -f(x)$, the identity is odd, and if $f(-x) = f(x)$, the identity is even

Pythagorean identities
: set of equations involving trigonometric functions based on the right triangle properties

quotient identities
: pair of identities based on the fact that tangent is the ratio of sine and cosine, and cotangent is the ratio of cosine and sine

reciprocal identities
: set of equations involving the reciprocals of basic trigonometric definitions

Sum and Difference Identities

In this section, you will:

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like

many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

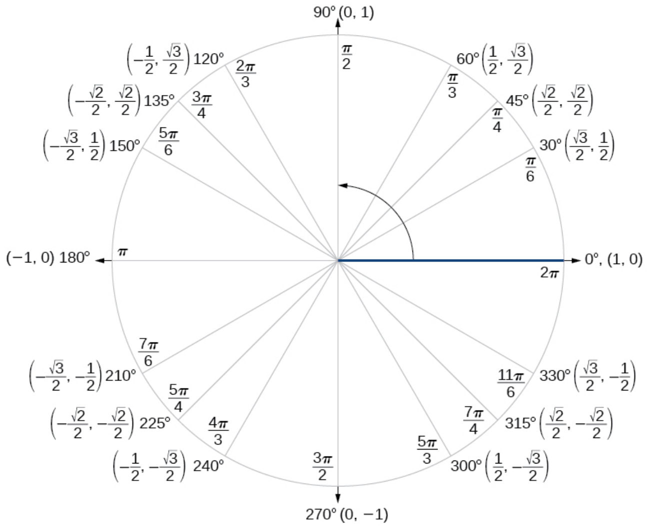
The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.
The Unit Circle

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special

angles, which we can review in the unit circle shown in [link].



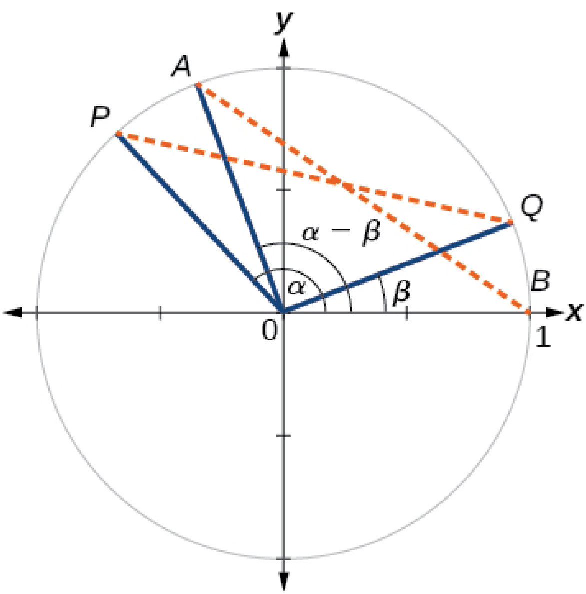
We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles. See [link].

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let’s consider two points on the unit circle.

See [link]. Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$. Note the measure of angle POQ is $\alpha - \beta$.

Label two more points: A at an angle of $(\alpha - \beta)$ from the positive x-axis with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.



We can find the distance from P to Q using the distance formula.

$$d_{PQ} = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$$

Then we apply the Pythagorean Identity and simplify.

$= (\cos 2 \alpha + \sin 2 \alpha) + (\cos 2 \beta + \sin 2 \beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$

Similarly, using the distance formula we can find the distance from A to B.

$d \ AB = (\cos(\alpha - \beta) - 1) \ 2 + (\sin(\alpha - \beta) - 0) \ 2 = \cos 2 (\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin 2 (\alpha - \beta)$

Applying the Pythagorean Identity and simplifying we get:

$= (\cos 2 (\alpha - \beta) + \sin 2 (\alpha - \beta)) - 2\cos(\alpha - \beta) + 1 = 1 - 2\cos(\alpha - \beta) + 1 = 2 - 2\cos(\alpha - \beta)$

Because the two distances are the same, we set them equal to each other and simplify.

$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$

$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$

Finally we subtract 2 from both sides and divide both sides by -2.

$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

Sum and Difference Formulas for Cosine
These formulas can be used to calculate the cosine of sums and differences of angles.
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Given two angles, find the cosine of the difference between the angles.

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)$.

Use the formula for the cosine of the difference of two angles. We have

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)=\cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\pi\right.$

6) =(− 2 2)(3 2)−(2 2)(1 2) = − 6 4 − 2 4 = − 6 − 2 4

Find the exact value of cos(π 3 − π 4).

2 + 6 4

Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of cos(75 ∘).

As 75 ∘ = 45 ∘ + 30 ∘ , we can evaluate cos(75 ∘) as cos(45 ∘ + 30 ∘). Thus,

cos(45 ∘ + 30 ∘)=cos(45 ∘)cos(30 ∘)−sin(45 ∘)sin(30 ∘) = 2 2 (3 2)− 2 2 (1 2) = 6 4 − 2 4 = 6 − 2 4

Find the exact value of $\cos(105^\circ)$.

2 − 6 4

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

Sum and Difference Formulas for Sine

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Given two angles, find the sine of the difference between the angles.

1. Write the difference formula for sine.

2. Substitute the given angles into the formula.
3. Simplify.

Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

1. $\sin(45^\circ - 30^\circ)$
2. $\sin(135^\circ - 120^\circ)$

---

1. Let's begin by writing the formula and substitute the given angles.
   $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\sin(45^\circ - 30^\circ) = \sin(45^\circ)\cos(30^\circ) - \cos(45^\circ)\sin(30^\circ)$

   Next, we need to find the values of the trigonometric expressions.
   $\sin(45^\circ) = 2\ 2, \cos(30^\circ) = 3\ 2, \cos(45^\circ) = 2\ 2, \sin(30^\circ) = 1\ 2$

   Now we can substitute these values into the equation and simplify.
   $\sin(45^\circ - 30^\circ) = 2\ 2\ (3\ 2) - 2\ 2\ (1\ 2) = 6 - 2\ 4$

2. Again, we write the formula and substitute the given angles.

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

sin(135 ∘ − 120 ∘)=sin(135 ∘)cos(120 ∘)−cos(135 ∘)sin(120 ∘)

Next, we find the values of the trigonometric expressions.

sin(135 ∘)= 2 2 ,cos(120 ∘)= − 1 2 ,cos(135 ∘)= − 2 2 ,sin(120 ∘)= 3 2

Now we can substitute these values into the equation and simplify.

sin(135 ∘ − 120 ∘)= 2 2 (− 1 2)−(− 2 2)(3 2) = − 2 + 6 4 = 6 − 2 4 sin(135 ∘ − 120 ∘)= 2 2 (− 1 2)−(− 2 2)(3 2) = − 2 + 6 4 = 6 − 2 4

Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of sin(cos −1 1 2 + sin −1 3 5).

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\cos\alpha=\frac{1}{2}, 0\le\alpha\le\pi \quad \sin\beta=\frac{3}{5}, -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$$

We will use the Pythagorean Identities to find $\sin\alpha$ and $\cos\beta$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \quad \cos\beta=\sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Using the sum formula for sine,

$$\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)=\sin(\alpha+\beta) =\sin\alpha\cos\beta+\cos\alpha\sin\beta = \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5} = \frac{4\sqrt{3}+3}{10}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for

sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x}, \cos x \neq 0$.

Let’s derive the sum formula for tangent.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}
\end{aligned}$$

Divide the numerator and denominator by $\cos\alpha\cos\beta$

$$\begin{aligned}
&= \frac{\frac{\sin\alpha \cos\beta}{\cos\alpha \cos\beta} + \frac{\cos\alpha \sin\beta}{\cos\alpha \cos\beta}}{\frac{\cos\alpha \cos\beta}{\cos\alpha \cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

Sum and Difference Formulas for Tangent

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Given two angles, find the tangent of the sum of the angles.

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.

3. Simplify.

Finding the Exact Value of an Expression Involving Tangent

Find the exact value of tan(π 6 + π 4).

Let’s first write the sum formula for tangent and substitute the given angles into the formula.

$\tan(\alpha+\beta)=\ \tan\alpha+\tan\beta\ 1-\tan\alpha\tan\beta\ \tan(\pi\ 6\ +\ \pi\ 4)=\ \tan(\pi\ 6)+\tan(\pi\ 4)\ 1-(\tan(\pi\ 6))(\tan(\pi\ 4))$

Next, we determine the individual tangents within the formula:

$\tan(\pi\ 6)=\ 1\ 3\ ,\tan(\pi\ 4)=1$

So we have

$\tan(\pi\ 6\ +\ \pi\ 4)=\ 1\ 3\ +1\ 1-(\ 1\ 3\ )(1)$ $=\ 1+\ 3\ 3\ 3\ -1\ 3$ $=\ 1+\ 3\ 3\ (\ 3\ 3\ -1\ )$ $=\ 3\ +1\ 3\ -1$

Find the exact value of tan(2π 3 + π 4).

$1-3\ 1+3$

Finding Multiple Sums and Differences of Angles

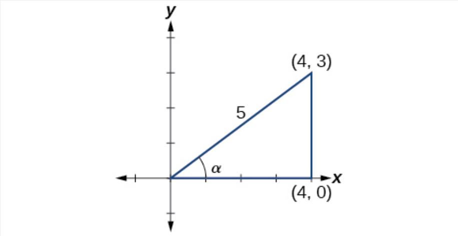
Given $\sin\alpha=\frac{3}{5}, 0<\alpha<\frac{\pi}{2}, \cos\beta=-\frac{5}{13}, \pi<\beta<\frac{3\pi}{2}$, find

1. $\sin(\alpha+\beta)$
2. $\cos(\alpha+\beta)$
3. $\tan(\alpha+\beta)$
4. $\tan(\alpha-\beta)$

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

1. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See [link]. Using the Pythagorean Theorem, we can find the length of side a:

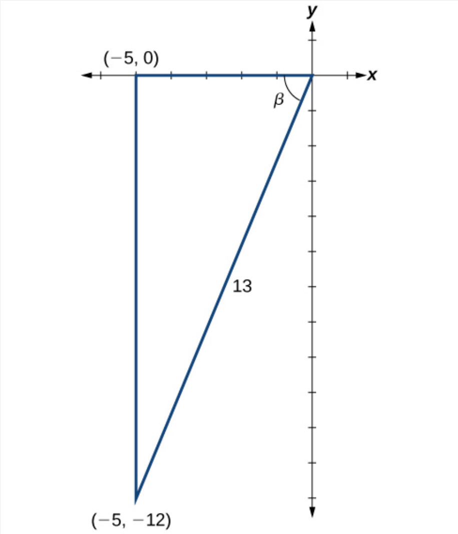
$a^2 + 3^2 = 5^2$ $a^2 = 16$
$a = 4$



Since $\cos\beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5, the hypotenuse is 13, and β is in the third quadrant. See [link]. Again, using the Pythagorean Theorem, we have

$$(-5)^2 + a^2 = 13^2 \quad 25 + a^2 = 169 \quad a^2 = 144 \quad a = \pm 12$$

Since β is in the third quadrant, $a = -12$.



The next step is finding the cosine of α and the sine of β. The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in [link]: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in [link], as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

2. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

   $$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

3. For $\tan(\alpha+\beta)$, if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

   $$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

   If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

   $$\tan\beta = \frac{-12}{13}{\Big/}\frac{-5}{13} = \frac{12}{5}$$

   Then,

   $$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\left(\frac{12}{5}\right)}\end{aligned}$$

$$= \frac{63}{20} - \frac{16}{20}$$
$$= -\frac{63}{16}$$

4. To find $\tan(\alpha - \beta)$, we have the values we need. We can substitute them in and evaluate.

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4}\left(\frac{12}{5}\right)}$$
$$= \frac{-\frac{33}{20}}{\frac{56}{20}} = -\frac{33}{56}$$

Analysis

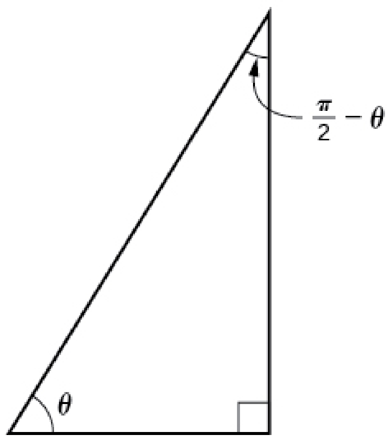
A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that
$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those

two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In [link], notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\left(\frac{\pi}{2}-\theta\right)$.

Notice also that $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$: opposite over hypotenuse. Thus, when two angles are complementary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.



From these relationships, the cofunction identities are formed.

Cofunction Identities

The cofunction identities are summarized in [link].

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Write $\sin \frac{\pi}{7}$ in terms of its cofunction.

$\cos\left(\frac{5\pi}{14}\right)$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules from Solving Trigonometric Equations with Identities may help simplify the process of verifying an identity.

Given an identity, verify using sum and difference formulas.

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one

side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \quad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta \; =2\sin\alpha\cos\beta$$

We see that the identity is verified.

Verifying an Identity Involving Tangent

Verify the following identity.

$\sin(\alpha-\beta)$ $\cos\alpha\cos\beta$ $=\tan\alpha-\tan\beta$

We can begin by rewriting the numerator on the left side of the equation.

$\sin(\ \alpha-\beta\ )$ $\cos\alpha\cos\beta$ $=$ $\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\cos\alpha\cos\beta$ $=$ $\sin\alpha\ \cos\beta$ $\cos\alpha\ \cos\beta$ $-$ $\cos\alpha\ \sin\beta$ $\cos\alpha\ \cos\beta$

Rewrite using a common denominator. $=$ $\sin\alpha$ $\cos\alpha$ $-$ $\sin\beta$ $\cos\beta$ Cancel. $=\tan\alpha-\tan\beta$

Rewrite in terms of tangent.

We see that the identity is verified. In many cases, verifying tangent identities can successfully be accomplished by writing the tangent in terms of sine and cosine.

Verify the identity: $\tan(\ \pi-\theta\ )=-\tan\theta$.

$\tan(\pi-\theta)=$ $\tan(\pi)-\tan\theta$ $1+\tan(\pi)\tan\theta$

$=$ $0-\tan\theta$ $1+0\cdot\tan\theta$ $=$

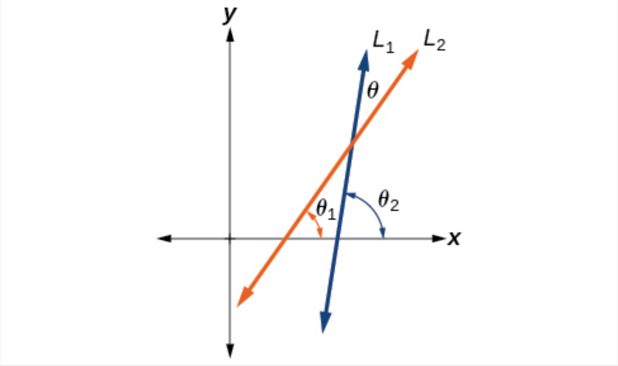
$-\tan\theta$

Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See [link]. Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.)

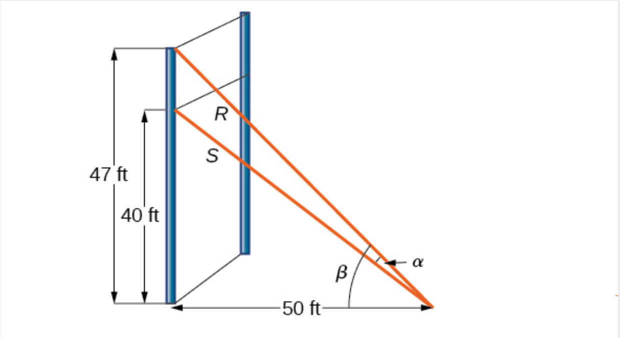


Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1) \\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} \\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See [link].



Let’s first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

Now, substituting the values we know into the formula, we have

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$

$\tan\alpha$ $4\left(1+\frac{47}{50}\tan\alpha\right)=5\left(\frac{47}{50}-\tan\alpha\right)$

Use the distributive property, and then simplify the functions.

$4(1)+4\left(\frac{47}{50}\right)\tan\alpha=5\left(\frac{47}{50}\right)-5\tan\alpha$

$4+3.76\tan\alpha=4.7-5\tan\alpha$ $5\tan\alpha$

$+3.76\tan\alpha=0.7$ $8.76\tan\alpha=0.7$

$\tan\alpha\approx 0.07991$ $\tan^{-1}(0.07991)\approx .079741$

Now we can calculate the angle in degrees.

$\alpha\approx 0.079741\left(\frac{180}{\pi}\right)\approx 4.57^\circ$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
Cofunction identities	$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See [link] and [link].
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See [link].
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See [link].
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles.

The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See [link].

- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See [link].
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See [link].
- Sum and difference formulas are useful in verifying identities. See [link] and [link].
- Application problems are often easier to solve by using sum and difference formulas. See [link] and [link].

Section Exercises

Verbal

Explain the basis for the cofunction identities and when they apply.

The cofunction identities apply to

complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures x, the second angle measures $\frac{\pi}{2} - x$. Then $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. The same holds for the other cofunction identities. The key is that the angles are complementary.

Is there only one way to evaluate $\cos\left(\frac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (Hint: $0 - x = -x$)

$\sin(-x) = -\sin x$, so $\sin x$ is odd. $\cos(-x) = \cos(0 - x) = \cos x$, so $\cos x$ is even.

Algebraic

For the following exercises, find the exact value.

$\cos\left(\frac{7\pi}{12}\right)$

$\cos(\pi\ 12)$

$2 + 6\ 4$

$\sin(5\pi\ 12)$

$\sin(11\pi\ 12)$

$6 - 2\ 4$

$\tan(- \pi\ 12)$

$\tan(19\pi\ 12)$

$-2- 3$

For the following exercises, rewrite in terms of sinx and cosx.

$\sin(x+ 11\pi\ 6)$

$\sin(x- 3\pi\ 4)$

$-\ 2\ 2\ \sin x-\ 2\ 2\ \cos x$

$\cos(\ x-\ 5\pi\ 6\ )$

$\cos(\ x+\ 2\pi\ 3\ )$

$-\ 1\ 2\ \cos x-\ 3\ 2\ \sin x$

For the following exercises, simplify the given expression.

$\csc(\ \pi\ 2\ -t\ )$

$\sec(\ \pi\ 2\ -\theta\ )$

$\csc\theta$

$\cot(\ \pi\ 2\ -x\ )$

$\tan(\ \pi\ 2\ -x\ )$

$\cot x$

$\sin(2x)\cos(5x)-\sin(5x)\cos(2x)$

$\tan(3\ 2\ x)-\tan(7\ 5\ x)\ 1+\tan(3\ 2\ x)\tan(7\ 5\ x)$

$\tan(x\ 10)$

For the following exercises, find the requested information.

Given that $\sin a=\ 2\ 3$ and $\cos b=-\ 1\ 4$, with a and b both in the interval $[\ \pi\ 2\ ,\pi\ )$, find $\sin(a+b)$ and $\cos(a-b)$.

Given that $\sin a=\ 4\ 5$, and $\cos b=\ 1\ 3$, with a and b both in the interval $[\ 0,\ \pi\ 2\ )$, find $\sin(a-b)$ and $\cos(a+b)$.

$\sin(a-b)=(\ 4\ 5\ )(\ 1\ 3\ )-(\ 3\ 5\ )(\ 2\ 2\ 3\ )=\ 4-6\ 2\ 15$
$\cos(a+b)=(\ 3\ 5\ )(\ 1\ 3\ )-(\ 4\ 5\ )(\ 2\ 2\ 3\ )=\ 3-8\ 2\ 15$

For the following exercises, find the exact value of each expression.

$\sin(\cos^{-1}(0) - \cos^{-1}(\frac{1}{2}))$

$\cos(\cos^{-1}(\frac{2}{2}) + \sin^{-1}(\frac{3}{2}))$

$\frac{2-6}{4}$

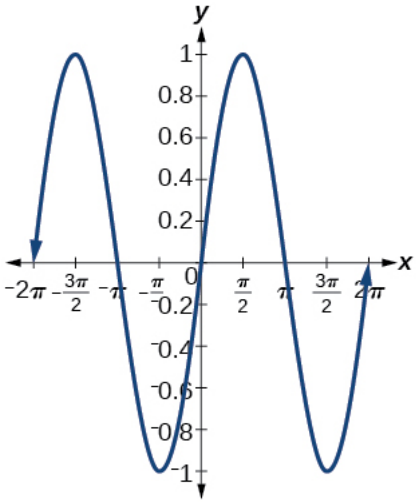
$\tan(\sin^{-1}(\frac{1}{2}) - \cos^{-1}(\frac{1}{2}))$

Graphical

For the following exercises, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

$\cos(\frac{\pi}{2} - x)$

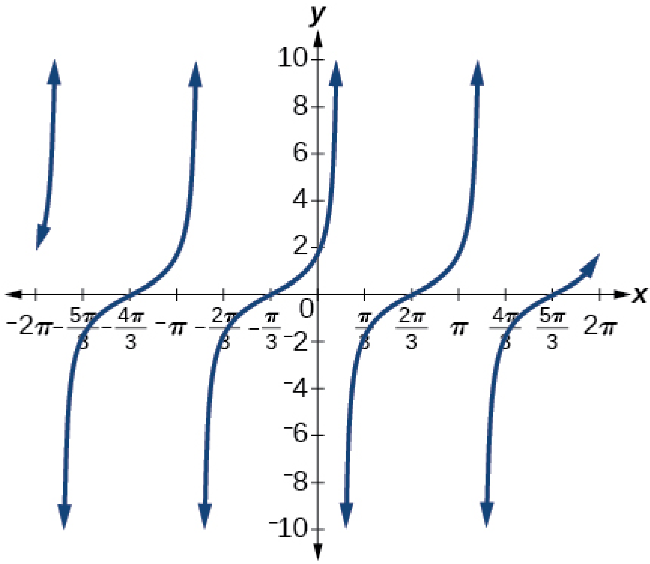
$\sin x$



$\sin(\pi - x)$

$\tan\left(\frac{\pi}{3} + x\right)$

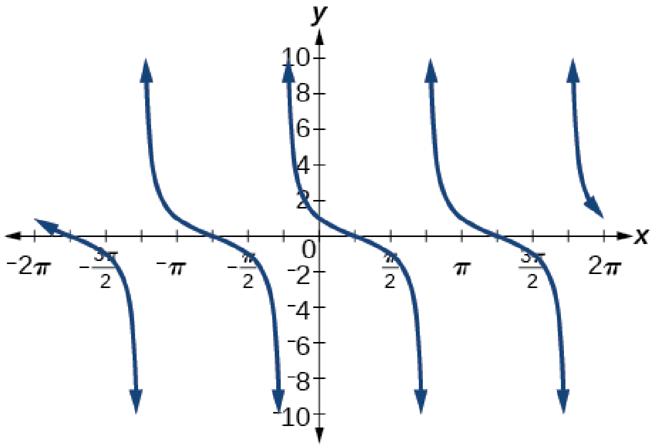
$\cot\left(\frac{\pi}{6} - x\right)$



$\sin\left(\frac{\pi}{3}+x\right)$

$\tan\left(\frac{\pi}{4}-x\right)$

$\cot\left(\frac{\pi}{4}+x\right)$



cos(7π 6 +x)

sin(π 4 +x)

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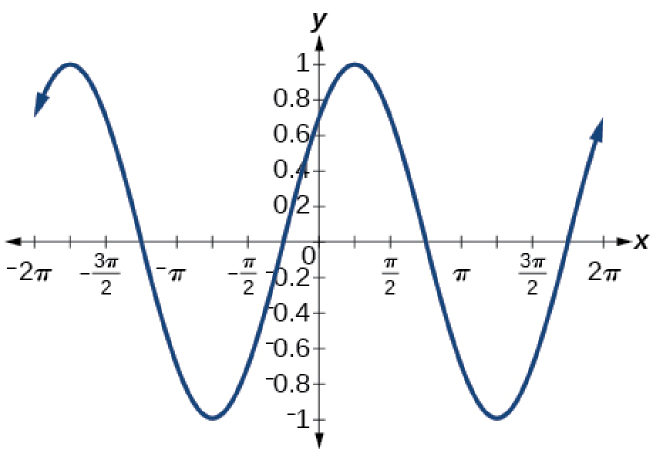
sinx 2 + cosx 2



$\cos\left(\frac{5\pi}{4}+x\right)$

For the following exercises, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x=x+x$.)

$f(x)=\sin(4x)-\sin(3x)\cos x, g(x)=\sin x\cos(3x)$

They are the same.

$f(x)=\cos(4x)+\sin x\sin(3x), g(x)=-\cos x\cos(3x)$

$f(x)=\sin(3x)\cos(6x), g(x)=-\sin(3x)\cos(6x)$

They are the different, try $g(x)=\sin(9x)-\cos(3x)\sin(6x)$.

$f(x)=\sin(4x), g(x)=\sin(5x)\cos x-\cos(5x)\sin x$

$f(x)=\sin(2x),g(x)=2\sin x\cos x$

They are the same.

$f(\theta)=\cos(2\theta),g(\theta)=\cos 2\theta-\sin 2\theta$

$f(\theta)=\tan(2\theta),g(\theta)=\tan\theta\ 1+\tan 2\theta$

They are the different, try $g(\theta)=2\tan\theta\ 1-\tan 2\theta$.

$f(x)=\sin(3x)\sin x,g(x)=\sin 2(2x)\cos 2x-\cos 2(2x)\sin 2x$

$f(x)=\tan(-x),g(x)=\tan x-\tan(2x)$
$1-\tan x\tan(2x)$

They are different, try $g(x)=\tan x-\tan(2x)$
$1+\tan x\tan(2x)$.

Technology

For the following exercises, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

$\sin(75^\circ)$

$\sin(195^\circ)$

$-\frac{\sqrt{3}-1}{2\sqrt{2}}$, or -0.2588

$\cos(165^\circ)$

$\cos(345^\circ)$

$\frac{1+\sqrt{3}}{2\sqrt{2}}$, or 0.9659

$\tan(-15^\circ)$

Extensions

For the following exercises, prove the identities provided.

$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+1}{1-\tan x}$

$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\left(\frac{\pi}{4}\right)}{1-\tan x\tan\left(\frac{\pi}{4}\right)}=\frac{\tan x+1}{1-\tan x(1)}=\frac{\tan x+1}{1-\tan x}$

$\frac{\tan(a+b)}{\tan(a-b)} = \frac{\sin a \cos a + \sin b \cos b}{\sin a \cos a - \sin b \cos b}$

$\frac{\cos(a+b)}{\cos a \cos b} = 1 - \tan a \tan b$

$\frac{\cos(a+b)}{\cos a \cos b} = \frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b} = 1 - \tan a \tan b$

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

$\frac{\cos(x+h) - \cos x}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}$

$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cosh - \sin x \sinh - \cos x}{h} = \frac{\cos x(\cosh - 1) - \sin x \sinh}{h} = \cos x \frac{\cosh - 1}{h} - \sin x \frac{\sinh}{h}$

For the following exercises, prove or disprove the statements.

$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$

$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

True

$\tan(x+y)\ 1+\tan x \tan x\ =\ \tan x+\tan y\ 1-\ \tan 2$
$x \tan 2\ y$

If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

True. Note that $\sin(\alpha+\beta)=\sin(\pi-\gamma)$ and expand the right hand side.

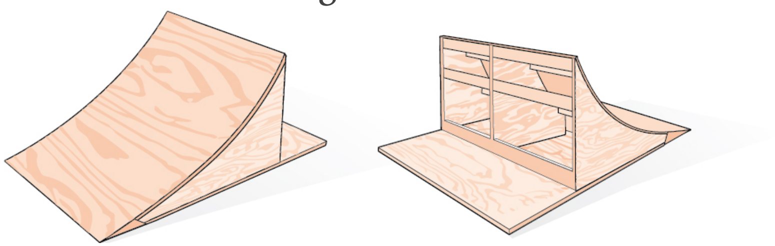
If α,β, and γ are angles in the same triangle, then prove or disprove $\tan\alpha+\tan\beta$
$+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

Double-Angle, Half-Angle, and Reduction Formulas

In this section, you will:

- Use double-angle formulas to find exact values.
- Use double-angle formulas to verify identities.
- Use reduction formulas to simplify an expression.
- Use half-angle formulas to find exact values.

Bicycle ramps for advanced riders have a steeper incline than those designed for novices.



Bicycle ramps made for competition (see [link]) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

Using Double-Angle Formulas to Find

Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The double-angle formulas are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

If we let $\alpha = \beta = \theta$, then we have
$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \quad \sin(2\theta) = 2\sin\theta\cos\theta$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and letting $\alpha = \beta = \theta$, we have
$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \quad \cos(2\theta) = \cos 2 \theta - \sin 2 \theta$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more interpretations. The first one is:
$\cos(2\theta) = \cos 2 \theta - \sin 2 \theta \quad = (1 - \sin 2 \theta) - \sin 2 \theta \quad = 1 - 2 \sin 2 \theta$

The second interpretation is:
$\cos(2\theta) = \cos 2 \theta - \sin 2 \theta \quad = \cos 2 \theta - (1 - \cos 2 \theta) \quad = 2 \cos 2 \theta - 1$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha=\beta=\theta$ in the sum formula gives

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$$
$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$$

Double-Angle Formulas

The **double-angle formulas** are summarized as follows:

$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta \qquad =1-2\sin^2\theta$$
$$=2\cos^2\theta-1$$
$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$$

Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value.

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Using a Double-Angle Formula to Find the Exact Value Involving Tangent

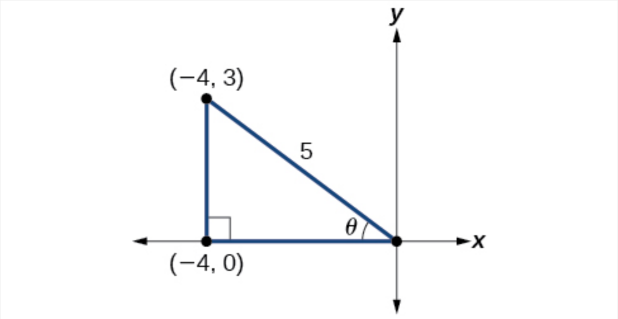
Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

1. $\sin(2\theta)$
2. $\cos(2\theta)$
3. $\tan(2\theta)$

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$, such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the Pythagorean Theorem to find the length of the hypotenuse:

$$(-4)^2 + (3)^2 = c^2 \quad 16+9 = c^2 \quad 25 = c^2$$
$$c = 5$$

Now we can draw a triangle similar to the one shown in [link].



1. Let's begin by writing the double-angle formula for sine.
   $\sin(2\theta)=2\sin\theta\cos\theta$

   We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on [link], we see that the hypotenuse equals 5, so $\sin\theta=\frac{3}{5}$, and $\cos\theta=-\frac{4}{5}$. Substitute these values into the equation, and simplify.

   Thus,
   $\sin(2\theta)=2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)=-\frac{24}{25}$
2. Write the double-angle formula for cosine.
   $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

   Again, substitute the values of the sine and cosine into the equation, and simplify.
   $$\begin{aligned}\cos(2\theta)&=\left(-\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2\\&=\frac{16}{25}-\frac{9}{25}\\&=\frac{7}{25}\end{aligned}$$
3. Write the double-angle formula for

tangent.

$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1 - \left(-\frac{3}{4}\right)^2} \\ &= \frac{-\frac{3}{2}}{1 - \frac{9}{16}} \\ &= -\frac{3}{2}\left(\frac{16}{7}\right) \\ &= -\frac{24}{7}\end{aligned}$$

Given $\sin\alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

$\cos(2\alpha) = \frac{7}{32}$

Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

$\cos(6x) = \cos(2(3x + 3x))$

$$= \cos 2(3x) - \sin 2(3x) \qquad = 2 \cos 2 (3x) - 1$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Using the Double-Angle Formulas to Establish an Identity

Establish the following identity using double-angle formulas:

$1+\sin(2\theta)=(\sin\theta+\cos\theta)^2$

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta \\ &=(\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta \\ &=1+2\sin\theta\cos\theta \\ &=1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

where $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Establish the identity: $\cos^4\theta-\sin^4\theta=\cos(2\theta)$.

$\cos^4\theta-\sin^4\theta=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)=\cos(2\theta)$

Verifying a Double-Angle Identity for Tangent

Verify the identity:
$\tan(2\theta) = 2\cot\theta - \tan\theta$

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.
$\tan(2\theta) = 2\tan\theta 1 - \tan 2 \theta$ Double-angle formula $= 2\tan\theta(1 \tan\theta)(1 - \tan 2 \theta)(1 \tan\theta)$
Multiply by a term that results in desired numerator.
$= 2 1 \tan\theta - \tan 2 \theta \tan\theta$ $= 2 \cot\theta - \tan\theta$ Use reciprocal identity for $1 \tan\theta$.

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show
$2\tan\theta 1 - \tan 2 \theta = 2 \cot\theta - \tan\theta$

Let's work on the right side.
$2 \cot\theta - \tan\theta = 2 1 \tan\theta - \tan\theta(\tan\theta \tan\theta)$
$= 2\tan\theta 1 \tan\theta(\tan\theta) - \tan\theta(\tan\theta)$
$= 2\tan\theta 1 - \tan 2 \theta$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

$\cos(2\theta)\cos\theta = (\cos^2\theta - \sin^2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are

especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let’s begin with $\cos(2\theta)=1-2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\cos(2\theta)=1-2\sin^2\theta \quad 2\sin^2\theta=1-\cos(2\theta) \quad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

Next, we use the formula $\cos(2\theta)=2\cos^2\theta-1$. Solve for $\cos^2\theta$:

$$\cos(2\theta)=2\cos^2\theta-1 \quad 1+\cos(2\theta)=2\cos^2\theta \quad \frac{1+\cos(2\theta)}{2}=\cos^2\theta$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\tan^2\theta=\frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} \quad \text{Substitute the reduction formulas.}$$

$$=\left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Reduction Formulas

The **reduction formulas** are summarized as follows:

$$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

$\tan 2\ \theta = \ 1 - \cos(\ 2\theta\ )\ 1 + \cos(\ 2\theta\ )$

Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos 4\ x$ that does not involve any powers of sine or cosine greater than 1.

We will apply the reduction formula for cosine twice.

$\cos 4\ x = \ (\ \cos 2\ x)\ 2 \qquad = \ (\ 1 + \cos(2x)\ 2\ )$

2 Substitute reduction formula for $\cos 2\ x$.

$= \ 1\ 4\ (\ 1 + 2\cos(2x) + \ \cos 2\ (2x)\ )$

$= \ 1\ 4\ +\ 1\ 2\ \cos(2x) + \ 1\ 4\ ($

$1 + \cos 2(2x)\ 2\ )$

Substitute reduction formula for $\cos 2\ x$.

$= \ 1\ 4\ +\ 1\ 2\ \cos(2x) + \ 1\ 8\ +\ 1\ 8$

$\cos(4x) \qquad = \ 3\ 8\ +\ 1\ 2\ \cos(2x) + \ 1\ 8$

$\cos(4x)$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove
$\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right]\left[1-\cos(4x)\right]$

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right]\end{aligned}$$

Substitute the power-reduction formula.

$$\begin{aligned} &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted
$\frac{1-\cos(4x)}{2}$

for $\sin^2(2x)$. The formula states
$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

$10 \cos^4 x = 10 \cos^4 x = 10 (\cos^2 x)^2$

$= 10 [1 + \cos(2x) \; 2]^2$

Substitute reduction formula for $\cos^2 x$.

$= 10 \; 4 \; [1 + 2\cos(2x) + \cos^2 (2x)]$

$= 10 \; 4 + 10 \; 2 \cos(2x) + 10 \; 4 \; ($

$1 + \cos2(2x) \; 2 \;)$

Substitute reduction formula for $\cos^2 x$.

$= 10 \; 4 + 10 \; 2 \cos(2x) + 10 \; 8 + 10 \; 8$

$\cos(4x) \quad = 30 \; 8 + 5\cos(2x) + 10 \; 8$

$\cos(4x) \quad = 15 \; 4 + 5\cos(2x) + 5 \; 4$

$\cos(4x)$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with α 2 , the half-angle formula for sine is found by simplifying the equation and solving for sin(α 2). Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which α 2 terminates.

The half-angle formula for sine is derived as follows:

$$\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$
$$\sin^2\left(\frac{\alpha}{2}\right)=\frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}=\frac{1-\cos\alpha}{2}$$
$$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

To derive the half-angle formula for cosine, we have

$$\cos^2\theta=\frac{1+\cos(2\theta)}{2}$$
$$\cos^2\left(\frac{\alpha}{2}\right)=\frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}=\frac{1+\cos\alpha}{2}$$
$$\cos\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

For the tangent identity, we have

$$\tan^2\theta=\frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$
$$\tan^2\left(\frac{\alpha}{2}\right)=\frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}=\frac{1-\cos\alpha}{1+\cos\alpha}$$
$$\tan\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Half-Angle Formulas

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$\cos\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\tan\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha}$$

Using a Half-Angle Formula to Find the Exact Value of a Sine Function

Find $\sin(15^\circ)$ using a half-angle formula.

Since $15 \circ = 30 \circ 2$, we use the half-angle formula for sine:

$$\sin 30 \circ 2 = 1 - \cos 30 \circ 2 \qquad = 1 - 3\ 2\ 2$$
$$= 2 - 3\ 2\ 2 \qquad = 2 - 3\ 4$$
$$= 2 - 3\ 2$$

Analysis

Notice that we used only the positive root because sin(15 o) is positive.

Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

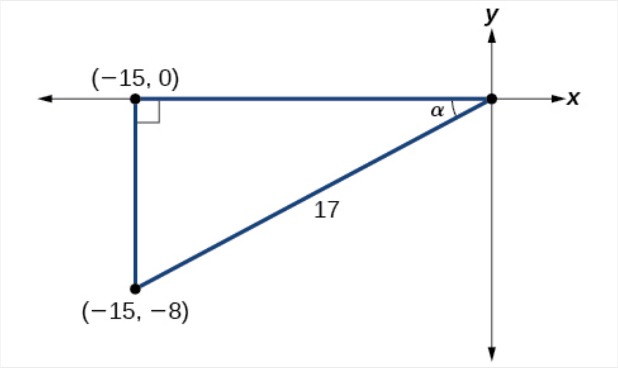
1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = 8\ 15$ and α lies in quadrant III, find the exact value of the following:

1. $\sin\left(\frac{\alpha}{2}\right)$
2. $\cos\left(\frac{\alpha}{2}\right)$
3. $\tan\left(\frac{\alpha}{2}\right)$

Using the given information, we can draw the triangle shown in [link]. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha=-\frac{8}{17}$ and $\cos\alpha=-\frac{15}{17}$.



1. Before we start, we must remember that, if α is in quadrant III, then $180°<\alpha<270°$, so $\frac{180°}{2}<\frac{\alpha}{2}<\frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90°<\frac{\alpha}{2}<135°$.

   To find $\sin\frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in [link] and

simplify.

$$\begin{aligned}\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}} \\ &= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\ &= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\ &= \pm\sqrt{\frac{16}{17}} \\ &= \pm\frac{4}{\sqrt{17}} \\ &= \frac{4\sqrt{17}}{17}\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

2. To find $\cos\frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in [link], and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} \\ &= \pm\sqrt{\frac{1}{17}} \\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

3. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in [link] and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} \\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\ &= \pm\sqrt{\frac{32}{2}} \\ &= -\sqrt{16} \\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and

tangent is negative in quadrant II.

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

$-\frac{2}{\sqrt{5}}$

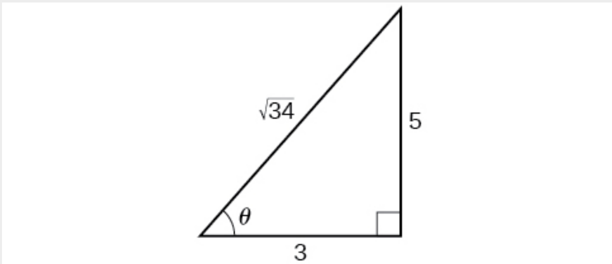
Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan\theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Since the angle for novice competition measures half the steepness of the angle for the high-level competition, and $\tan\theta = \frac{5}{3}$ for

high-competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See [link].

$$3^2 + 5^2 = 34 \qquad c = \sqrt{34}$$



We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$. Thus,

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}} = \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}} = \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}} \approx 0.57$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

Key Equations

Double-angle formulas	$\sin(2\theta)=2\sin\theta\cos\theta$ $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $=1-2\sin^2\theta$ $=2\cos^2\theta-1$ $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta=\dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta=\dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta=\dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ $\cos\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ $\tan\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$ $=\dfrac{\sin\alpha}{1+\cos\alpha}$ $=\dfrac{1-\cos\alpha}{\sin\alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See [link], [link], [link], and [link].
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See [link] and [link].
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See [link], [link], and [link].

Section Exercises

Verbal

Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

Use the Pythagorean identities and isolate the squared term.

Explain how to determine the double-angle formula for tan(2x) using the double-angle formulas for cos(2x) and sin(2x).

We can determine the half-angle formula for $\tan(x\ 2)=\ 1-\cos x\ 1+\cos x$ by dividing the formula for $\sin(x\ 2)$ by $\cos(x\ 2)$. Explain how to determine two formulas for $\tan(x\ 2)$ that do not involve any square roots.

$1-\cos x\ \sin x$, $\sin x\ 1+\cos x$, multiplying the top and bottom by $1-\cos x$ and $1+\cos x$, respectively.

For the half-angle formula given in the previous exercise for $\tan(x\ 2)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

Algebraic

For the following exercises, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

If $\sin x=\ 1\ 8$, and x is in quadrant I.

---

a) 3 7 32 b) 31 32 c) 3 7 31

If $\cos x = 2\ 3$, and x is in quadrant I.

If $\cos x = -\ 1\ 2$, and x is in quadrant III.

---

a) 3 2 b) $-\ 1\ 2$ c) $-\ 3$

If $\tan x = -8$, and x is in quadrant IV.

For the following exercises, find the values of the six trigonometric functions if the conditions provided hold.

$\cos(2\theta) = 3\ 5$ and $90^\circ \leq \theta \leq 180^\circ$

$\cos\theta = -\ 2\ 5\ 5$,$\sin\theta = 5\ 5$,$\tan\theta = -\ 1\ 2$,$\csc\theta = 5$,$\sec\theta = -\ 5\ 2$,$\cot\theta = -2$

$\cos(2\theta) = 1\ 2$ and $180^\circ \leq \theta \leq 270^\circ$

For the following exercises, simplify to one trigonometric expression.

2sin(π 4)2cos(π 4)

2sin(π 2)

4sin(π 8)cos(π 8)

For the following exercises, find the exact value using half-angle formulas.

sin(π 8)

2− 2 2

cos(− 11π 12)

sin(11π 12)

2− 3 2

cos(7π 8)

tan(5π 12)

$2+\ 3$

$\tan(\ -\ 3\pi\ 12\ )$

$\tan(\ -\ 3\pi\ 8\ )$

$-1-\ 2$

For the following exercises, find the exact values of a) $\sin(\ x\ 2\ )$, b) $\cos(\ x\ 2\ )$, and c) $\tan(\ x\ 2\ )$ without solving for x, when $0^{\circ} \le x \le\ 360^{\circ}$

If $\tan x = -\ 4\ 3$, and x is in quadrant IV.

If $\sin x = -\ 12\ 13$, and x is in quadrant III.

---

a) $3\ 13\ 13$ b) $-\ 2\ 13\ 13$ c) $-\ 3\ 2$

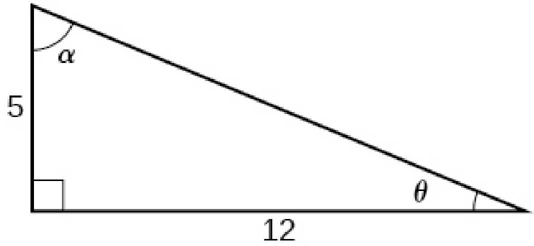
If $\csc x = 7$, and x is in quadrant II.

If $\sec x = -4$, and x is in quadrant II.

---

a) $10\ 4$ b) $6\ 4$ c) $15\ 3$

For the following exercises, use [link] to find the requested half and double angles.



Find sin(2θ),cos(2θ), and tan(2θ).

Find sin(2α),cos(2α), and tan(2α).

---

120 169 ,– 119 169 ,– 120 119

Find sin(θ 2),cos(θ 2), and tan(θ 2).

Find sin(α 2),cos(α 2), and tan(α 2).

---

2 13 13 , 3 13 13 , 2 3

For the following exercises, simplify each expression. Do not evaluate.

cos 2 (28 ∘)− sin 2 (28 ∘)

$2\cos^2(37^\circ)-1$

$\cos(74^\circ)$

$1-2\sin^2(17^\circ)$

$\cos^2(9x)-\sin^2(9x)$

$\cos(18x)$

$4\sin(8x)\cos(8x)$

$6\sin(5x)\cos(5x)$

$3\sin(10x)$

For the following exercises, prove the identity given.

$(\sin t-\cos t)^2=1-\sin(2t)$

$\sin(2x)=-2\sin(-x)\cos(-x)$

$-2\sin(-x)\cos(-x)=-2(-\sin(x)\cos(x$

$)) = \sin(2x)$

$\cot x - \tan x = 2\cot(2x)$

$1 + \cos(2\theta) \sin(2\theta) \tan 2 \theta = \tan\theta$

$\sin(2\theta)\ 1 + \cos(2\theta) \tan 2 \theta = 2\sin(\theta)\cos(\theta)$
$1 + \cos 2 \theta - \sin 2 \theta \tan 2 \theta = 2\sin(\theta)\cos(\theta)$
$2 \cos 2 \theta \tan 2 \theta = \sin(\theta) \cos\theta \tan 2 \theta = \tan \theta$
$\tan 2 \theta = \tan 3\theta$

For the following exercises, rewrite the expression with an exponent no higher than 1.

$\cos 2 (5x)$

$\cos 2 (6x)$

$1 + \cos(12x)\ 2$

$\sin 4 (8x)$

$\sin 4 (3x)$

$3 + \cos(12x) - 4\cos(6x)\ 8$

$\cos 2x \sin 4x$

$\cos 4x \sin 2x$

$2 + \cos(2x) - 2\cos(4x) - \cos(6x)\ 32$

$\tan 2x \sin 2x$

Technology

For the following exercises, reduce the equations to powers of one, and then check the answer graphically.

$\tan 4x$

$3 + \cos(4x) - 4\cos(2x)\ 3 + \cos(4x) + 4\cos(2x)$

$\sin 2(2x)$

$\sin 2x \cos 2x$

$1-\cos(4x)\ 8$

$\tan 2\ x\sin x$

$\tan 4\ x\ \cos 2\ x$

$3+\cos(4x)-4\cos(2x)\ 4(\cos(2x)+1)$

$\cos 2\ x\sin(\ 2x\ )$

$\cos 2\ (\ 2x\ )\sin x$

$(\ 1+\cos(\ 4x\ )\ )\sin x\ 2$

$\tan 2\ (\ x\ 2\ )\sin x$

For the following exercises, algebraically find an equivalent function, only in terms of sinx and/or cosx, and then check the answer by graphing both equations.

$\sin(4x)$

$4\sin x\cos x(\ \cos 2\ x-\ \sin 2\ x\ )$

cos(4x)

Extensions

For the following exercises, prove the identities.

sin(2x)= 2tanx 1+ tan 2 x

2tanx 1+ tan 2 x = 2sinx cosx 1+ sin 2 x cos 2 x = 2sinx cosx cos 2 x+ sin 2 x cos 2 x = 2sinx cosx . cos 2 x 1 =2sinxcosx=sin(2x)

cos(2α)= 1− tan 2 α 1+ tan 2 α

tan(2x)= 2sinxcosx 2 cos 2 x−1

2sinxcosx 2 cos 2 x−1 = sin(2x) cos(2x) =tan(2x)

(sin 2 x−1) 2 =cos(2x)+ sin 4 x

sin(3x)=3sinx cos 2 x− sin 3 x

---

sin(x+2x)=sinxcos(2x)+sin(2x)cosx =sinx(

$\cos^2 x - \sin^2 x) + 2\sin x\cos x\cos x = \sin x\cos^2 x - \sin^3 x + 2\sin x\cos^2 x = 3\sin x\cos^2 x - \sin^3 x$

$\cos(3x) = \cos^3 x - 3\sin^2 x\cos x$

$1 + \cos(2t)\ \sin(2t) - \cos t = 2\cos t\ 2\sin t - 1$

$1 + \cos(2t)\ \sin(2t) - \cos t = 1 + 2\cos^2 t - 1\ 2\sin t\cos t - \cos t = 2\cos^2 t\ \cos t(2\sin t - 1) = 2\cos t\ 2\sin t - 1$

$\sin(16x) = 16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

$\cos(16x) = (\cos^2(4x) - \sin^2(4x) - \sin(8x))(\cos^2(4x) - \sin^2(4x) + \sin(8x))$

$(\cos^2(4x) - \sin^2(4x) - \sin(8x))(\cos^2(4x) - \sin^2(4x) + \sin(8x)) =$

$\cos(8x) - \sin(8x))(\cos(8x) + \sin(8x))$

$\cos^2(8x) - \sin^2(8x)$

Glossary

double-angle formulas
: identities derived from the sum formulas for sine, cosine, and tangent in which the angles are equal

half-angle formulas
: identities derived from the reduction formulas and used to determine half-angle values of trigonometric functions

reduction formulas
: identities derived from the double-angle formulas and used to reduce the power of a trigonometric function

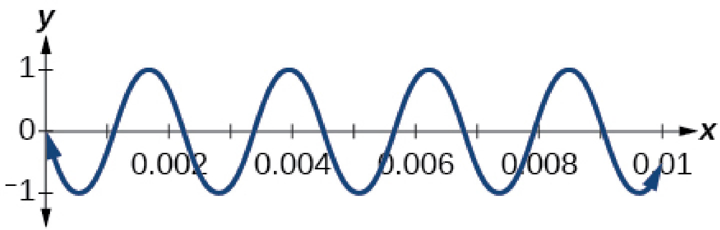
Sum-to-Product and Product-to-Sum Formulas

In this section, you will:

- Express products as sums.
- Express sums as products.

The UCLA marching band (credit: Eric Chan, Flickr).

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions. For example, [link] represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.



Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the product-to-sum formulas, which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for cosine. If we add the two equations, we get:

$$\begin{array}{r} \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \\ + \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta) \end{array}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

Given a product of cosines, express as a sum.

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\frac{3x}{2}$.

We begin by writing the formula for the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned}2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)&=(2)\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right]\\&=\left[\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\right]\\&=\cos 2x+\cos 5x\end{aligned}$$

Use the product-to-sum formula to write the

product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

$\frac{1}{2}(\cos 6\theta + \cos 2\theta)$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for sine. If we add the sum and difference identities, we get:

$$\begin{array}{rl} \sin(\alpha+\beta) = & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ + \quad \sin(\alpha-\beta) = & \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \hline \sin(\alpha+\beta)+\sin(\alpha-\beta) = & 2\sin\alpha\cos\beta \end{array}$$

Then, we divide by 2 to isolate the product of cosine and sine:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no

products: $\sin(4\theta)\cos(2\theta)$.

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin(4\theta)\cos(2\theta) = \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)] = \frac{1}{2}[\sin(6\theta)+\sin(2\theta)]$$

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

$\frac{1}{2}(\sin 2x+\sin 2y)$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of cosine is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{array}{rl} & \cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta \\ - & \cos(\alpha+\beta) = -(\cos\alpha\cos\beta-\sin\alpha\sin\beta) \\ \hline & \cos(\alpha-\beta \end{array}$$

$) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$

Then, we divide by 2 to isolate the product of sines:
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

The Product-to-Sum Formulas

The **product-to-sum formulas** are as follows:

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$
$$\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta - 5\theta) + \cos(3\theta + 5\theta)]$$
$$= \frac{1}{2}$$

$[\cos(2\theta)+\cos(8\theta)]$ Use even-odd identity.

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

$\frac{-2-\sqrt{3}}{4}$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The sum-to-product formulas allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for sine. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\alpha+\beta=\frac{u+v}{2}+\frac{u-v}{2} = \frac{2u}{2} = u \qquad \alpha-\beta=\frac{u+v}{2}-\frac{u-v}{2} = \frac{2v}{2} = v$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u+\sin v]$$
Substitute for $(\alpha+\beta)$ and $(\alpha-\beta)$ $2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)=\sin u+\sin v$

The other sum-to-product identities are derived similarly.

Sum-to-Product Formulas

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta)-\sin(2\theta)$.

We begin by writing the formula for the difference of sines.

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta)-\sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right) \\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right) \\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta)+\sin(\theta)$.

$2\sin(2\theta)\cos(\theta)$

Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15^\circ)-\cos(75^\circ)$.

We begin by writing the formula for the difference of cosines.

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\cos(15^\circ)-\cos(75^\circ)=-2\sin\left(\frac{15^\circ+75^\circ}{2}$$

$)\sin(\frac{15^\circ - 75^\circ}{2})$ $=$
$-2\sin(45^\circ)\sin(-30^\circ)$
$= -2(\frac{\sqrt{2}}{2})(-\frac{1}{2})$
$= \frac{\sqrt{2}}{2}$

Proving an Identity

Prove the identity:
$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} = -\tan t$

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \frac{-2\sin(\frac{4t+2t}{2})\sin(\frac{4t-2t}{2})}{2\sin(\frac{4t+2t}{2})\cos(\frac{4t-2t}{2})} \\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t} \\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t} \\ &= -\frac{\sin t}{\cos t} \\ &= -\tan t\end{aligned}$$

Analysis

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the

other side.

Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc 2\,\theta - 2 = \cos(2\theta) \sin 2\,\theta$
.

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned} \cos(2\theta) \sin 2\,\theta &= 1 - 2 \sin 2\,\theta \sin 2\,\theta \\ &= 1 \sin 2\,\theta - 2 \sin 2\,\theta \sin 2\,\theta \\ &= \csc 2\,\theta - 2 \end{aligned}$$

Verify the identity $\tan\theta\cot\theta - \cos 2\,\theta = \sin 2\,\theta$.

$\tan\theta\cot\theta - \cos 2\,\theta = (\sin\theta \cos\theta)(\cos\theta \sin\theta) - \cos 2\,\theta = 1 - \cos 2\,\theta = \sin 2\,\theta$

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

Key Equations

Product-to-sum Formulas	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
Sum-to-product Formulas	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha$

	$-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Key Concepts

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See [link], [link], and [link].
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See [link].
- Trigonometric expressions are often simpler to evaluate using the formulas. See [link].
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we

choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See [link] and [link].

Section Exercises

Verbal

Starting with the product to sum formula $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$, explain how to determine the formula for $\cos\alpha\sin\beta$.

Substitute α into cosine and β into sine and evaluate.

Explain two different methods of calculating $\cos(195^\circ)\cos(105^\circ)$, one of which uses the product to sum. Which method is easier?

Explain a situation where we would convert an equation from a sum to a product and give an example.

Answers will vary. There are some equations

that involve a sum of two trig expressions where when converted to a product are easier to solve. For example: $\sin(3x)+\sin x \cos x = 1$. When converting the numerator to a product the equation becomes: $2\sin(2x)\cos x \cos x = 1$

Explain a situation where we would convert an equation from a product to a sum, and give an example.

Algebraic

For the following exercises, rewrite the product as a sum or difference.

$16\sin(16x)\sin(11x)$

$8(\cos(5x)-\cos(27x))$

$20\cos(36t)\cos(6t)$

$2\sin(5x)\cos(3x)$

$\sin(2x)+\sin(8x)$

$10\cos(5x)\sin(10x)$

$\sin(-x)\sin(5x)$

$\frac{1}{2}(\cos(6x)-\cos(4x))$

$\sin(3x)\cos(5x)$

For the following exercises, rewrite the sum or difference as a product.

$\cos(6t)+\cos(4t)$

$2\cos(5t)\cos t$

$\sin(3x)+\sin(7x)$

$\cos(7x)+\cos(-7x)$

$2\cos(7x)$

$\sin(3x)-\sin(-3x)$

$\cos(3x)+\cos(9x)$

$2\cos(6x)\cos(3x)$

$\sin h-\sin(3h)$

For the following exercises, evaluate the product for the following using a sum or difference of two functions.

$\cos(45^\circ)\cos(15^\circ)$

$1\ 4\ (1+\ 3)$

$\cos(45^\circ)\sin(15^\circ)$

$\sin(-345^\circ)\sin(-15^\circ)$

$1\ 4\ (3\ -2)$

$\sin(195^\circ)\cos(15^\circ)$

$\sin(-45^\circ)\sin(-15^\circ)$

$1\ 4\ (\ 3\ -1\ )$

For the following exercises, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

$\cos(\ 23^\circ\ )\sin(\ 17^\circ\ )$

$2\sin(\ 100^\circ\ )\sin(\ 20^\circ\ )$

$\cos(\ 80^\circ\ )-\cos(\ 120^\circ\ )$

$2\sin(-100^\circ)\sin(-20^\circ)$

$\sin(\ 213^\circ\ )\cos(\ 8^\circ\ )$

$1\ 2\ (\sin(221^\circ)+\sin(205^\circ))$

$2\cos(56^\circ)\cos(47^\circ)$

For the following exercises, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

$\sin(76^\circ)+\sin(14^\circ)$

$2\cos(31^\circ)$

$\cos(58^\circ) - \cos(12^\circ)$

$\sin(101^\circ) - \sin(32^\circ)$

$2\cos(66.5^\circ)\sin(34.5^\circ)$

$\cos(100^\circ) + \cos(200^\circ)$

$\sin(-1^\circ) + \sin(-2^\circ)$

$2\sin(-1.5^\circ)\cos(0.5^\circ)$

For the following exercises, prove the identity.

$\cos(a+b)\ \cos(a-b)\ =\ 1-\text{tanatanb}$
$1+\text{tanatanb}$

$4\sin(3x)\cos(4x) = 2\sin(7x) - 2\sin x$

$2\sin(7x) - 2\sin x = 2\sin(4x+3x) - 2\sin(4x$
$-3x) =$

$2(\sin(4x)\cos(3x)+\sin(3x)\cos(4x))-2(\sin(4x)\cos(3x)-\sin$
$2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))-2\sin(4x)\cos(3x)+2s$
$4\sin(3x)\cos(4x)$

$6\cos(8x)\sin(2x)\ \sin(-6x)\ = -3\sin(10x$
$)\csc(6x)+3$

$\sin x+\sin(3x)=4\sin x\ \cos\ 2\ x$

$\sin x+\sin(3x)=2\sin(4x\ 2)\cos(-2x\ 2)=$
$2\sin(2x)\cos x=2(2\sin x\cos x)\cos x=$
$4\sin x\ \cos\ 2\ x$

$2(\cos\ 3\ x-\cos x\ \sin\ 2\ x)=\cos(3x)+\cos x$

$2\tan x\cos(3x)=\sec x(\sin(4x)-\sin(2x))$

$2\tan x\cos(3x)=\ 2\sin x\cos(3x)\ \cos x\ =$
$2(.5(\sin(4x)-\sin(2x)))\ \cos x$
$=\ 1\ \cos x\ (\sin(4x)-\sin(2x)\ )=\sec x(\sin(4x$
$)-\sin(2x))$

$\cos(a+b)+\cos(a-b)=2\cos a\cos b$

Numeric

For the following exercises, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

$\cos(58^\circ)+\cos(12^\circ)$

$2\cos(35^\circ)\cos(23^\circ)$, 1.5081

$\sin(2^\circ)-\sin(3^\circ)$

$\cos(44^\circ)-\cos(22^\circ)$

$-2\sin(33^\circ)\sin(11^\circ), -0.2078$

$\cos(176^\circ)\sin(9^\circ)$

$\sin(-14^\circ)\sin(85^\circ)$

$\frac{1}{2}\left(\cos(99^\circ)-\cos(71^\circ)\right), -0.2410$

Technology

For the following exercises, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

$2\sin(2x)\sin(3x)=\cos x-\cos(5x)$

$\cos(10\theta)+\cos(6\theta)\ \cos(6\theta)-\cos(10\theta)$
$=\cot(2\theta)\cot(8\theta)$

It is and identity.

$\sin(3x)-\sin(5x)\ \cos(3x)+\cos(5x)\ =\tan x$

$2\cos(2x)\cos x+\sin(2x)\sin x=2\sin x$

It is not an identity, but 2 cos 3 x is.

$\sin(2x)+\sin(4x)\ \sin(2x)-\sin(4x)\ =-\tan($
$3x)\cot x$

For the following exercises, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

$\sin(9t) - \sin(3t) \cos(9t) + \cos(3t)$

$\tan(3t)$

$2\sin(8x)\cos(6x) - \sin(2x)$

$\sin(3x) - \sin x \sin x$

$2\cos(2x)$

$\cos(5x) + \cos(3x) \sin(5x) + \sin(3x)$

$\sin x\cos(15x) - \cos x\sin(15x)$

$-\sin(14x)$

Extensions

For the following exercises, prove the following sum-to-product formulas.

$\sin x - \sin y = 2\sin(x - y\ 2)\cos(x + y\ 2)$

$\cos x + \cos y = 2\cos(x + y\ 2)\cos(x - y\ 2)$

Start with $\cos x+\cos y$. Make a substitution and let $x=\alpha+\beta$ and let $y=\alpha-\beta$, so $\cos x+\cos y$ becomes
$\cos(\alpha+\beta)+\cos(\alpha-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$+\cos\alpha\cos\beta+\sin\alpha\sin\beta=2\cos\alpha\cos\beta$

Since $x=\alpha+\beta$ and $y=\alpha-\beta$, we can solve for α and β in terms of *x* and *y* and substitute in for $2\cos\alpha\cos\beta$ and get $2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

For the following exercises, prove the identity.

$$\frac{\sin(6x)+\sin(4x)}{\sin(6x)-\sin(4x)}=\tan(5x)\cot x$$

$$\frac{\cos(3x)+\cos x}{\cos(3x)-\cos x}=-\cot(2x)\cot x$$

$\frac{\cos(3x)+\cos x}{\cos(3x)-\cos x}=\frac{2\cos(2x)\cos x}{-2\sin(2x)\sin x}=-\cot(2x)\cot x$

$$\frac{\cos(6y)+\cos(8y)}{\sin(6y)-\sin(4y)}=\cot y\cos(7y)\sec(5y)$$

$$\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)}=\tan y$$

$\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)}=\frac{-2\sin(3y)\sin(-y)}{2\sin(3y)\cos y}=2\sin(3y)$

$)\sin(y)\ 2\sin(3y)\cos y \ = \tan y$

$\sin(10x) - \sin(2x)\ \cos(10x) + \cos(2x)$
$= \tan(4x)$

$\cos x - \cos(3x) = 4\ \sin\ 2\ x\cos x$

$\cos x - \cos(3x) = -2\sin(2x)\sin(- x) =$
$2(2\sin x\cos x)\sin x = 4\ \sin\ 2\ x\cos x$

$(\cos(2x) - \cos(4x))\ 2\ +\ (\sin(4x) + \sin(2x))\ 2\ = 4$
$\sin\ 2\ (3x)$

$\tan(\pi\ 4\ - t) =\ 1 - \tan t\ 1 + \tan t$

$\tan(\pi\ 4\ - t) =\ \tan(\pi\ 4) - \tan t\ 1 + \tan(\pi\ 4$
$)\tan(t)\ =\ 1 - \tan t\ 1 + \tan t$

Glossary

product-to-sum formula
: a trigonometric identity that allows the writing of a product of trigonometric functions as a sum or difference of trigonometric functions

sum-to-product formula

a trigonometric identity that allows, by using substitution, the writing of a sum of trigonometric functions as a product of trigonometric functions

Solving Trigonometric Equations

In this section, you will:

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff.

Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is

valid. The period of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$, where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi)$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

From the unit circle, we know that

$$\cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solving for all possible values of *t* means that solutions include angles beyond the period of 2π. From [link], we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Given a trigonometric equation, solve using algebra.

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5, 0 \le \theta < 2\pi$.

Use algebraic techniques to solve the equation.
$2\cos\theta - 3 = -5$ $2\cos\theta = -2$ $\cos\theta = -1$ $\theta = \pi$

Solve exactly the following linear equation on

the interval $[0,2\pi)$: $2\sin x+1=0$.
$x=\frac{7\pi}{6},\frac{11\pi}{6}$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π, not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$, unless, of course, a problem places its own restrictions on the domain.

Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0, 0 \le \theta < 2\pi$.

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$2\sin^2\theta - 1 = 0 \quad 2\sin^2\theta = 1 \quad \sin^2\theta = \frac{1}{2} \quad \sqrt{\sin^2\theta} = \pm\sqrt{\frac{1}{2}} \quad \sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2, 0 \le \theta < 4\pi$.

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\csc\theta = -2 \quad \frac{1}{\sin\theta} = -2 \quad \sin\theta = -\frac{1}{2} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1, 0 \le \theta < 2\pi$.

Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4} \qquad \theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$, we have two solutions:

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$$

Find all solutions for $\tan x = \sqrt{3}$.

$\frac{\pi}{3} \pm \pi k$

Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x+3)=5+\tan x, 0\le x<2\pi$.

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$2(\tan x)+2(3) = 5+\tan x \quad 2\tan x+6 = 5+\tan x \quad 2\tan x-\tan x = 5-6 \quad \tan x = -1$$

There are two angles on the unit circle that have a tangent value of -1: $\theta=\frac{3\pi}{4}$ and $\theta=\frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given

problem.

Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}($. The calculator is ready for the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$$\sin^{-1}(0.8) \approx 0.9273$$

The solution is

$$\theta \approx 0.9273 \pm 2\pi k$$

The angle measurement in degrees is

$$\theta \approx 53.1^\circ \quad \theta \approx 180^\circ - 53.1^\circ \approx 126.9^\circ$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is

the range of the inverse sine. The other angle is obtained by using $\pi - \theta$. Thus, the additional solution is $\approx 2.2143 \pm 2\pi k$

Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

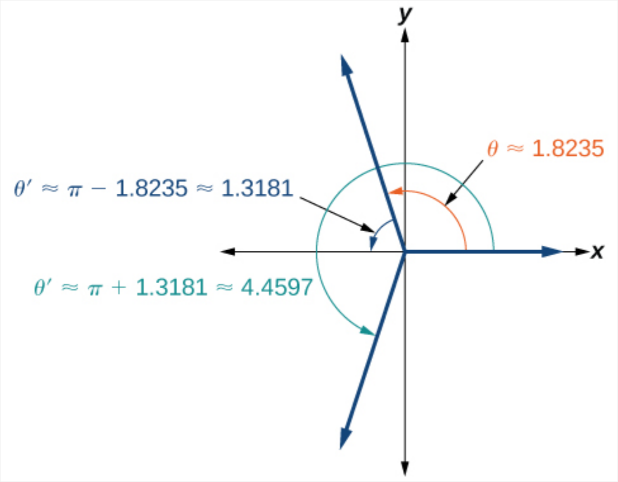
We can begin with some algebra.

$$\sec\theta = -4 \quad \frac{1}{\cos\theta} = -4 \quad \cos\theta = -\frac{1}{4}$$

Check that the MODE is in radians. Now use the inverse cosine function.

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235 \quad \theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, 1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See [link].



So, we also need to find the measure of the angle in quadrant III. In quadrant II, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Solve $\cos\theta = -0.2$.

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a quadratic equation may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos 2\,\theta + 3\cos\theta - 1 = 0, 0 \le \theta < 2\pi$.

We begin by using substitution and replacing $\cos\,\theta$ with x. It is not necessary to use

substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} = \frac{-3 \pm \sqrt{13}}{2}$$

Replace x with cosθ, and solve.

$$\cos\theta = \frac{-3 \pm \sqrt{13}}{2} \quad \theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right)$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\frac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\theta = \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \approx 1.26$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\theta = 2\pi - \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \approx 5.02$$

Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta$

$+3=0, 0\leq\theta\leq 2\pi.$

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta=u$, or imagine it, as we factor:

$$2\sin^2\theta-5\sin\theta+3=0 \quad (2\sin\theta-3)(\sin\theta-1)=0$$

Now set each factor equal to zero.

$$2\sin\theta-3=0 \quad 2\sin\theta=3 \quad \sin\theta=\frac{3}{2} \quad \sin\theta-1=0 \quad \sin\theta=1$$

Next solve for θ: $\sin\theta\neq\frac{3}{2}$, as the range of the sine function is $[-1,1]$. However, $\sin\theta=1$, giving the solution $\theta=\frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Solve $\sin^2\theta=2\cos\theta+2, 0\leq\theta\leq 2\pi.$ [Hint: Make a substitution to express the equation only in terms of cosine.]

$\cos\theta=-1, \theta=\pi$

Solving a Trigonometric Equation Using Algebra

Solve exactly:
$2\sin^2\theta+\sin\theta=0; 0\le\theta<2\pi$

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta=x$. The equation becomes $2x^2+x=0$. We begin by factoring:

$$2x^2+x=0 \quad x(2x+1)=0$$

Set each factor equal to zero.

$$x=0 \quad (2x+1)=0 \quad x=-\frac{1}{2}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\sin\theta=0 \quad \theta=0,\pi \quad \sin\theta=-\frac{1}{2} \quad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

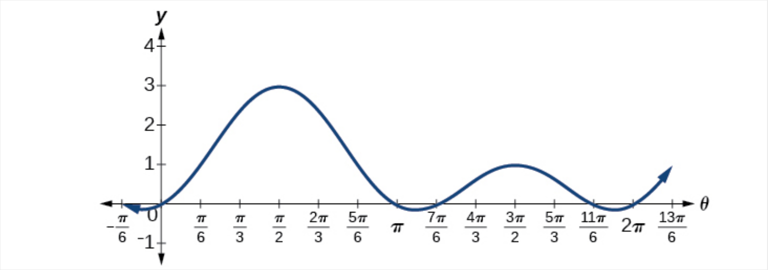
The solutions within the domain $0\le\theta<2\pi$ are $\theta=0,\pi,\frac{7\pi}{6},\frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$2\sin^2\theta+\sin\theta=0 \quad \sin\theta(2\sin\theta+1)=0 \quad \sin\theta=0 \quad \theta=0,\pi \quad 2\sin\theta+1=0 \quad 2\sin\theta=-1 \quad \sin\theta=-\frac{1}{2} \quad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

Analysis

We can see the solutions on the graph in [link]. On the interval $0 \le \theta < 2\pi$, the graph crosses the *x*-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.



We can verify the solutions on the unit circle in [link] as well.

Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0, 0 \le \theta < 2\pi$.

We can factor using grouping. Solution values of θ can be found on the unit circle.

$(2\sin\theta - 1)(\sin\theta - 1) = 0$ $2\sin\theta - 1 = 0$ $\sin\theta = \frac{1}{2}$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\sin\theta = 1$ $\theta = \frac{\pi}{2}$

Solve the quadratic equation $2\cos^2\theta + \cos\theta = 0$.

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$

Notice that the left side of the equation is the difference formula for cosine.

$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$ $\cos(x - 2x) = \frac{\sqrt{3}}{2}$ Difference formula for cosine $\cos(-x) = \frac{\sqrt{3}}{2}$ Use the negative angle identity. $\cos x = \frac{\sqrt{3}}{2}$

From the unit circle in [link], we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-

angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$, then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$, then $\theta = 0 \pm 2\pi k$.

Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta, 0 \le \theta < 2\pi$.

If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta + 3 &= 2\sin^2\theta \\ 3\cos\theta + 3 &= 2(1 - \cos^2\theta) \\ 3\cos\theta + 3 &= 2 - 2\cos^2\theta \\ 2\cos^2\theta + 3\cos\theta + 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta + 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3} \\ \cos\theta + 1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y=\sin(2x)$ is a horizontal compression by a factor of 2 of the function $y=\sin x$. On an interval of 2π, we can graph two periods of $y=\sin(2x)$, as opposed to one cycle of $y=\sin x$. This compression of the graph leads us to believe there may be twice as many *x*-intercepts or solutions to $\sin(2x)=0$ compared to $\sin x=0$. This information will help us solve the equation.

Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x)=\frac{1}{2}$ on $[0,2\pi)$.

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha)=\frac{1}{2}$, we know α is in quadrants I and IV. While $\theta=\cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta=\frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$2x = \frac{\pi}{3} + 2\pi = \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3}$$

so $x = \frac{7\pi}{6}$.

One more rotation yields

$$2x = \frac{\pi}{3} + 4\pi = \frac{\pi}{3} + \frac{12\pi}{3} = \frac{13\pi}{3}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$2x = \frac{5\pi}{3} + 2\pi = \frac{5\pi}{3} + \frac{6\pi}{3} = \frac{11\pi}{3}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$2x = \frac{5\pi}{3} + 4\pi = \frac{5\pi}{3} + \frac{12\pi}{3} = \frac{17\pi}{3}$$

$x = \frac{17\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

Our solutions are $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $\sin(nx)=c$, we must go around the unit circle n times.

Solving Right Triangle Problems

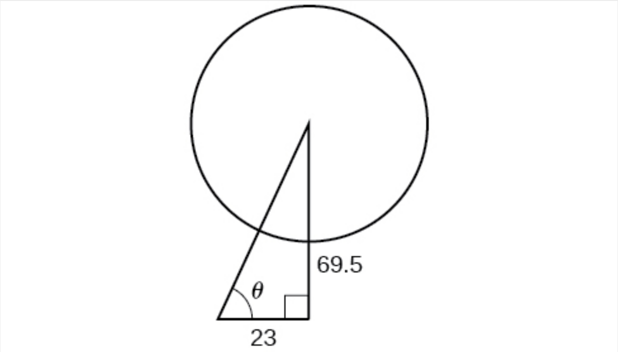
We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem, $a^2+b^2=c^2$, and model an equation to fit a situation.

Using the Pythagorean Theorem to Model an Equation

Use the Pythagorean Theorem, and the properties of right triangles to model an equation that fits the problem.

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris

wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See [link].



Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \quad (23)^2 + (69.5)^2 \approx 5359$$
$$5359 \approx 73.2 \text{ m}$$

The angle of elevation is θ, formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\tan\theta = \frac{69.5}{23} \quad \tan^{-1}\left(\frac{69.5}{23}\right) \approx 1.2522$$
$$\approx 71.69^\circ$$

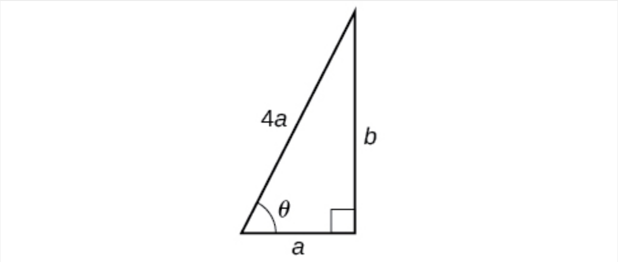
The angle of elevation is approximately 71.7°,

and the length of the cable is 73.2 meters.

Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "*a*" feet from the wall, the length of the ladder will be 4*a* feet. See [link].



The side adjacent to θ is *a* and the hypotenuse is 4a. Thus,

$$\cos\theta = \frac{a}{4a} = \frac{1}{4} \quad \cos^{-1}\left(\frac{1}{4}\right) \approx 75.5^\circ$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$a^2 + b^2 = (4a)^2 \quad b^2 = (4a)^2 - a^2 \quad b^2 = 16a^2 - a^2 \quad b^2 = 15a^2 \quad b = a\sqrt{15}$$

Thus, the ladder touches the wall at a $\sqrt{15}$ feet from the ground.

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations,

we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See [link], [link], and [link].

- Equations involving a single trigonometric function can be solved or verified using the unit circle. See [link], [link], and [link], and [link].
- We can also solve trigonometric equations using a graphing calculator. See [link] and [link].
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See [link], [link], [link], and [link].
- We can also use the identities to solve trigonometric equation. See [link], [link], and [link].
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See [link].
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See [link].

Section Exercises

Verbal

Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x) = -5$.

When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

If the sine or cosine function has a coefficient of

one, isolate the term on one side of the equals sign. If the number it is set equal to has an absolute value less than or equal to one, the equation has solutions, otherwise it does not. If the sine or cosine does not have a coefficient equal to one, still isolate the term but then divide both sides of the equation by the leading coefficient. Then, if the number it is set equal to has an absolute value greater than one, the equation has no solution.

Algebraic

For the following exercises, find all solutions exactly on the interval $0 \leq \theta < 2\pi$.

$2\sin\theta = -\ 2$

$2\sin\theta = \ 3$

$\pi\ 3\ ,\ 2\pi\ 3$

$2\cos\theta = 1$

$2\cos\theta = -\ 2$

$3\pi\ 4\ ,\ 5\pi\ 4$

$\tan\theta = -1$

$\tan x = 1$

$\pi\ 4\ ,\ 5\pi\ 4$

$\cot x + 1 = 0$

$4\ \sin\ 2\ x - 2 = 0$

$\pi\ 4\ ,\ 3\pi\ 4\ ,\ 5\pi\ 4\ ,\ 7\pi\ 4$

$\csc\ 2\ x - 4 = 0$

For the following exercises, solve exactly on $[0,2\pi)$.

$2\cos\theta = \ 2$

$\pi\ 4\ ,\ 7\pi\ 4$

$2\cos\theta = -1$

$2\sin\theta = -1$

$7\pi\ 6$, $11\pi\ 6$

$2\sin\theta = -\ 3$

$2\sin(\ 3\theta\ ) = 1$

$\pi\ 18$, $5\pi\ 18$, $13\pi\ 18$, $17\pi\ 18$, $25\pi\ 18$, $29\pi\ 18$

$2\sin(\ 2\theta\ ) =\ 3$

$2\cos(\ 3\theta\ ) = -\ 2$

$3\pi\ 12$, $5\pi\ 12$, $11\pi\ 12$, $13\pi\ 12$, $19\pi\ 12$, $21\pi\ 12$

$\cos(\ 2\theta\ ) = -\ 3\ 2$

$2\sin(\ \pi\theta\ ) = 1$

$1\ 6$, $5\ 6$, $13\ 6$, $17\ 6$, $25\ 6$, $29\ 6$, $37\ 6$

$2\cos(\ \pi\ 5\ \theta\ ) = \ 3$

For the following exercises, find all exact solutions on $[\ 0,2\pi\ )$.

$\sec(x)\sin(x) - 2\sin(x) = 0$

$0,\ \pi\ 3\ ,\pi,\ 5\pi\ 3$

$\tan(x) - 2\sin(x)\tan(x) = 0$

$2\ \cos\ 2\ t + \cos(\ t\ ) = 1$

$\pi\ 3\ ,\pi,\ 5\pi\ 3$

$2\ \tan\ 2\ (t) = 3\sec(t)$

$2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

$\pi\ 3\ ,\ 3\pi\ 2\ ,\ 5\pi\ 3$

$\cos\ 2\ \theta = \ 1\ 2$

$\sec^2 x = 1$

$0, \pi$

$\tan^2(x) = -1 + 2\tan(-x)$

$8\sin^2(x) + 6\sin(x) + 1 = 0$

$\pi - \sin^{-1}\left(-\frac{1}{4}\right), \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi + \sin^{-1}\left(-\frac{1}{4}\right)$

$\tan^5(x) = \tan(x)$

For the following exercises, solve with the methods shown in this section exactly on the interval $[0, 2\pi)$.

$\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

$\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{\pi}{3} - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{2\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \pi - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{4\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{5\pi}{3} - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right)$

$\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

$\cos(2x)\cos x+\sin(2x)\sin x=1$

0

$6\sin(2t)+9\sin t=0$

$9\cos(2\theta)=9\cos^2\theta-4$

$\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

$\sin(2t)=\cos t$

$\cos(2t)=\sin t$

$\frac{3\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$

$\cos(6x)-\cos(3x)=0$

For the following exercises, solve exactly on the interval $[0,2\pi)$. Use the quadratic formula if the equations do not factor.

$\tan^2 x-\sqrt{3}\tan x=0$

$0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$

$\sin^2 x + \sin x - 2 = 0$

$\sin^2 x - 2\sin x - 4 = 0$

There are no solutions.

$5\cos^2 x + 3\cos x - 1 = 0$

$3\cos^2 x - 2\cos x - 2 = 0$

$\cos^{-1}\left(\frac{1}{3}\left(1- 7\right)\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\left(1- 7\right)\right)$

$5\sin^2 x + 2\sin x - 1 = 0$

$\tan^2 x + 5\tan x - 1 = 0$

$\tan^{-1}\left(\frac{1}{2}\left(29 - 5\right)\right), \pi + \tan^{-1}\left(\frac{1}{2}\left(- 29 - 5\right)\right), \pi + \tan^{-1}\left(\frac{1}{2}\left(29 - 5\right)\right), 2\pi + \tan^{-1}\left(\frac{1}{2}\left(- 29 - 5\right)\right)$

$\cot 2x = -\cot x$

$-\tan 2x - \tan x - 2 = 0$

There are no solutions.

For the following exercises, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

$\sin 2x - \cos 2x - \sin x = 0$

$\sin 2x + \cos 2x = 0$

There are no solutions.

$\sin(2x) - \sin x = 0$

$\cos(2x) - \cos x = 0$

$0, \frac{2\pi}{3}, \frac{4\pi}{3}$

$2\tan x 2 - \sec 2x - \sin 2x = \cos 2x$

$1-\cos(2x)=1+\cos(2x)$

$\pi\ 4\ ,\ 3\pi\ 4\ ,\ 5\pi\ 4\ ,\ 7\pi\ 4$

$\sec\ 2\ x=7$

$10\sin x\cos x=6\cos x$

$\sin\ -1\ (\ 3\ 5\ ),\ \pi\ 2\ ,\pi-\ \sin\ -1\ (\ 3\ 5\ ),\ 3\pi\ 2$

$-3\sin t=15\cos t\sin t$

$4\ \cos\ 2\ x-4=15\cos x$

$\cos\ -1\ (\ -\ 1\ 4\ ),2\pi-\ \cos\ -1\ (\ -\ 1\ 4\ )$

$8\ \sin\ 2\ x+6\sin x+1=0$

$8\ \cos\ 2\ \theta=3-2\cos\theta$

$\pi\ 3\ ,\ \cos\ -1\ (\ -\ 3\ 4\ ),2\pi-\ \cos\ -1\ (\ -\ 3\ 4\ ),$
$5\pi\ 3$

$6\cos^2 x + 7\sin x - 8 = 0$

$12\sin^2 t + \cos t - 6 = 0$

$\cos^{-1}\left(\frac{3}{4}\right), \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(\frac{3}{4}\right)$

$\tan x = 3\sin x$

$\cos^3 t = \cos t$

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Graphical

For the following exercises, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

$6\sin^2 x - 5\sin x + 1 = 0$

$8\cos^2 x - 2\cos x - 1 = 0$

$\frac{\pi}{3}, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi-\cos^{-1}\left(-\frac{1}{4}\right), \frac{5\pi}{3}$

$100\tan^2 x+20\tan x-3=0$

$2\cos^2 x-\cos x+15=0$

There are no solutions.

$20\sin^2 x-27\sin x+7=0$

$2\tan^2 x+7\tan x+6=0$

$\pi+\tan^{-1}(-2), \pi+\tan^{-1}\left(-\frac{3}{2}\right), 2\pi+\tan^{-1}(-2), 2\pi+\tan^{-1}\left(-\frac{3}{2}\right)$

$130\tan^2 x+69\tan x-130=0$

Technology

For the following exercises, use a calculator to find all solutions to four decimal places.

$\sin x=0.27$

$2\pi k+0.2734, 2\pi k+2.8682$

$\sin x=-0.55$

$\tan x=-0.34$

$\pi k-0.3277$

$\cos x=0.71$

For the following exercises, solve the equations algebraically, and then use a calculator to find the values on the interval $[0,2\pi)$. Round to four decimal places.

$\tan^2 x+3\tan x-3=0$

$0.6694, 1.8287, 3.8110, 4.9703$

$6\tan^2 x+13\tan x=-6$

$\tan^2 x-\sec x=1$

1.0472,3.1416,5.2360

$\sin^2 x - 2\cos^2 x = 0$

$2\tan^2 x + 9\tan x - 6 = 0$

0.5326,1.7648,3.6742,4.9064

$4\sin^2 x + \sin(2x)\sec x - 3 = 0$

Extensions

For the following exercises, find all solutions exactly to the equations on the interval $[0,2\pi)$.

$\csc^2 x - 3\csc x - 4 = 0$

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right), \frac{3\pi}{2}$

$\sin^2 x - \cos^2 x - 1 = 0$

$\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

$\frac{\pi}{2}, \frac{3\pi}{2}$

$3 \sec 2 x+2+ \sin 2 x- \tan 2 x+ \cos 2 x=0$

$\sin 2 x-1+2\cos(2x)- \cos 2 x=1$

There are no solutions.

$\tan 2 x-1- \sec 3 x\cos x=0$

$\sin(2x) \sec 2 x =0$

$0, \pi 2 ,\pi, 3\pi 2$

$\sin(2x) 2 \csc 2 x =0$

$2 \cos 2 x- \sin 2 x-\cos x-5=0$

There are no solutions.

$1 \sec 2 x +2+ \sin 2 x+4 \cos 2 x=4$

Real-World Applications

An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

7.2 ∘

If a loading ramp is placed next to a truck, at a height of 4 feet, and the ramp is 15 feet long, what angle does the ramp make with the ground?

If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

5.7 ∘

A woman is watching a launched rocket currently 11 miles in altitude. If she is standing 4 miles from the launch pad, at what angle is she looking up from horizontal?

An astronaut is in a launched rocket currently

15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

82.4 ∘

A woman is standing 8 meters away from a 10-meter tall building. At what angle is she looking to the top of the building?

A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

31.0 ∘

A 20-foot tall building has a shadow that is 55 feet long. What is the angle of elevation of the sun?

A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

$88.7°$

A spotlight on the ground 3 meters from a 2-meter tall man casts a 6 meter shadow on a wall 6 meters from the man. At what angle is the light?

A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

$59.0°$

For the following exercises, find a solution to the following word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

A person does a handstand with his feet touching a wall and his hands 1.5 feet away from the wall. If the person is 6 feet tall, what angle do his feet make with the wall?

A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle

do her feet make with the wall?

36.9°

A 23-foot ladder is positioned next to a house. If the ladder slips at 7 feet from the house when there is not enough traction, what angle should the ladder make with the ground to avoid slipping?

Chapter Review Exercises

Solving Trigonometric Equations with Identities

For the following exercises, find all solutions exactly that exist on the interval $[0,2\pi)$.

$\csc^2 t=3$

$\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),\pi-\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),\pi+\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),2\pi-\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$\cos^2 x=\frac{1}{4}$

$2\sin\theta = -1$

$\frac{7\pi}{6}, \frac{11\pi}{6}$

$\tan x \sin x + \sin(-x) = 0$

$9\sin\omega - 2 = 4\sin^2\omega$

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right)$

$1 - 2\tan(\omega) = \tan^2(\omega)$

For the following exercises, use basic identities to simplify the expression.

$\sec x \cos x + \cos x - \frac{1}{\sec x}$

1

$\sin^3 x + \cos^2 x \sin x$

For the following exercises, determine if the given identities are equivalent.

$\sin 2\ x + \sec 2\ x - 1 = (1 - \cos 2\ x)(1 + \cos 2\ x)\ \cos 2\ x$

Yes

$\tan 3\ x\ \csc 2\ x\ \cot 2\ x \cos x \sin x = 1$

Sum and Difference Identities

For the following exercises, find the exact value.

$\tan(\ 7\pi\ 12\ )$

$-2-\ 3$

$\cos(\ 25\pi\ 12\ )$

$\sin(70^\circ)\cos(25^\circ) - \cos(70^\circ)\sin(25^\circ)$

$2\ 2$

$\cos(83^\circ)\cos(23^\circ) + \sin(83^\circ)\sin(23^\circ)$

For the following exercises, prove the identity.

cos(4x)−cos(3x)cosx= sin 2 x−4 cos 2 x sin 2 x

---

cos(4x)−cos(3x)cosx = cos(2x+2x)−cos(x +2x)cosx = cos(2x)cos(2x)−sin(2x)sin(2x)−cosxcos(2x)cosx +sinxsin(2x)cosx = (cos 2 x− sin 2 x) 2 −4 cos 2 x sin 2 x− cos 2 x(cos 2 x− sin 2 x)+sinx(2)sinxcosxcosx = (cos 2 x− sin 2 x) 2 −4 cos 2 x sin 2 x− cos 2 x(cos 2 x− sin 2 x)+2 sin 2 x cos 2 x = cos 4 x−2 cos 2 x sin 2 x + sin 4 x−4 cos 2 x sin 2 x− cos 4 x+ cos 2 x sin 2 x+2 sin 2 x cos 2 x = sin 4 x−4 cos 2 x sin 2 x+ cos 2 x sin 2 x = sin 2 x(sin 2 x+ cos 2 x)−4 cos 2 x sin 2 x = sin 2 x−4 cos 2 x sin 2 x

cos(3x)− cos 3 x=−cosx sin 2 x−sinxsin(2x)

For the following exercise, simplify the expression.

tan(1 2 x)+tan(1 8 x) 1−tan(1 8 x)tan(1 2 x)

---

tan(5 8 x)

For the following exercises, find the exact value.

$\cos\left(\sin^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

$\tan\left(\sin^{-1}(0) + \sin^{-1}\left(\frac{1}{2}\right)\right)$

---

3 3

Double-Angle, Half-Angle, and Reduction Formulas

For the following exercises, find the exact value.

Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos\theta = -\frac{1}{3}$ and θ is in the interval $\left[\frac{\pi}{2}, \pi\right]$.

Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec\theta = -\frac{5}{3}$ and θ is in the interval $\left[\frac{\pi}{2}, \pi\right]$.

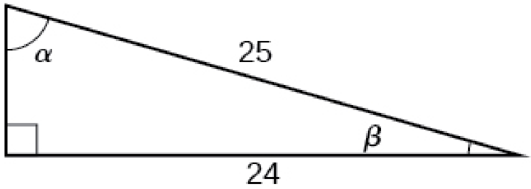
$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$

$\sin\left(\frac{7\pi}{8}\right)$

$\sec\left(\frac{3\pi}{8}\right)$

$2(2+2)$

For the following exercises, use [link] to find the desired quantities.



$\sin(2\beta), \cos(2\beta), \tan(2\beta), \sin(2\alpha), \cos(2\alpha), \text{and } \tan(2\alpha)$

$\sin(\frac{\beta}{2}), \cos(\frac{\beta}{2}), \tan(\frac{\beta}{2}), \sin(\frac{\alpha}{2}), \cos(\frac{\alpha}{2}), \text{and } \tan(\frac{\alpha}{2})$

$\frac{\sqrt{2}}{10}, \frac{7\sqrt{2}}{10}, \frac{1}{7}, \frac{3}{5}, \frac{4}{5}, \frac{3}{4}$

For the following exercises, prove the identity.

$\frac{2\cos(2x)}{\sin(2x)} = \cot x - \tan x$

$\cot x \cos(2x) = -\sin(2x) + \cot x$

$\cot x \cos(2x) = \cot x(1 - 2\sin^2 x) = \cot x - \frac{\cos x}{\sin x}(2)\sin^2 x = -2\sin x \cos x + \cot x = -\sin(2x) + \cot x$

For the following exercises, rewrite the expression with no powers.

$\cos 2 x \sin 4 (2x)$

$\tan 2 x \sin 3 x$

$10\sin x - 5\sin(3x) + \sin(5x)\ 8(\cos(2x) + 1)$

Sum-to-Product and Product-to-Sum Formulas

For the following exercises, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

$\cos(\pi\ 3)\sin(\pi\ 4)$

$2\sin(2\pi\ 3)\sin(5\pi\ 6)$

$3\ 2$

$2\cos(\pi\ 5)\cos(\pi\ 3)$

For the following exercises, evaluate the sum by using a product formula. Write the exact answer.

$\sin(\pi\ 12) - \sin(7\pi\ 12)$

$-\frac{\sqrt{2}}{2}$

$\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{7\pi}{12}\right)$

For the following exercises, change the functions from a product to a sum or a sum to a product.

$\sin(9x)\cos(3x)$

$\frac{1}{2}\left(\sin(6x)+\sin(12x)\right)$

$\cos(7x)\cos(12x)$

$\sin(11x)+\sin(2x)$

$2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$

$\cos(6x)+\cos(5x)$

Solving Trigonometric Equations

For the following exercises, find all exact solutions on the interval $[0,2\pi)$.

$\tan x+1=0$

$3\pi\ 4\ ,\ 7\pi\ 4$

$2\sin(2x)+\ 2\ =0$

For the following exercises, find all exact solutions on the interval $[\ 0,2\pi\ )$.

$2\ \sin\ 2\ x-\sin x=0$

$0,\ \pi\ 6\ ,\ 5\pi\ 6\ ,\pi$

$\cos\ 2\ x-\cos x-1=0$

$2\ \sin\ 2\ x+5\sin x+3=0$

$3\pi\ 2$

$\cos x-5\sin(\ 2x\ )=0$

$1\ \sec\ 2\ x\ +2+\ \sin\ 2\ x+4\ \cos\ 2\ x=0$

No solution

For the following exercises, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0,2\pi)$. Round to four decimal places.

$3\cot^2 x+\cot x=1$

$\csc^2 x-3\csc x-4=0$

0.2527,2.8889,4.7124

For the following exercises, graph each side of the equation to find the approximate solutions on the interval $[0,2\pi)$.

$20\cos^2 x+21\cos x+1=0$

$\sec^2 x-2\sec x=15$

1.3694,1.9106,4.3726,4.9137

Practice Test

For the following exercises, simplify the given expression.

$\cos(-x)\sin x \cot x + \sin 2x$

1

$\sin(-x)\cos(-2x) - \sin(-x)\cos(-2x)$

$\csc(\theta)\cot(\theta)(\sec 2\theta - 1)$

$\sec(\theta)$

$\cos 2(\theta)\sin 2(\theta)(1 + \cot 2(\theta))(1 + \tan 2(\theta))$

For the following exercises, find the exact value.

$\cos(7\pi\ 12)$

$2 - 6\ 4$

$\tan(3\pi\ 8)$

$\tan(\sin^{-1} (2\ 2)+ \tan^{-1} 3)$

$-\ 2 - 3$

$2\sin(\pi\ 4)\sin(\pi\ 6)$

$\cos(4\pi\ 3 + \theta)$

$-\ 1\ 2 \cos(\theta) - 3\ 2 \sin(\theta)$

$\tan(- \pi\ 4 + \theta)$

For the following exercises, simplify each expression. Do not evaluate.

$\cos^2 (32^\circ) \tan^2 (32^\circ)$

$1 - \cos(64^\circ)\ 2$

$\cot(\theta\ 2)$

For the following exercises, find all exact solutions to the equation on $[0,2\pi)$.

$\cos 2x - \sin 2x - 1 = 0$

$0, \pi$

$\cos 2x = \cos x 4 \sin 2x + 2\sin x - 3 = 0$

$\cos(2x) + \sin 2x = 0$

$\pi 2, 3\pi 2$

$2 \sin 2x - \sin x = 0$

Rewrite the expression as a product instead of a sum: $\cos(2x) + \cos(-8x)$.

$2\cos(3x)\cos(5x)$

For the following exercise, rewrite the product as a sum or difference.

$8\cos(15x)\sin(3x)$

For the following exercise, rewrite the sum or difference as a product.

$2(\sin(8\theta) - \sin(4\theta))$

$4\sin(2\theta)\cos(6\theta)$

Find all solutions of $\tan(x) - 3 = 0$.

Find the solutions of $\sec 2x - 2\sec x = 15$ on the interval $[0, 2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

$x = \cos{-1}(15)$

For the following exercises, find all solutions exactly on the interval $0 \le \theta \le \pi$

$2\cos(\theta\ 2) = 1$

$3\cot(y) = 1$

$\pi 3$

Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cot\theta = -\ 3\ 4$ and θ is on the interval $[\pi\ 2, \pi]$.

Find $\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = \frac{7}{25}$ and θ is in quadrant IV.

$\frac{3}{5}, -\frac{4}{5}, -\frac{3}{4}$

Rewrite the expression $\sin^4 x$ with no powers greater than 1.

For the following exercises, prove the identity.

$\tan^3 x - \tan x \sec^2 x = \tan(-x)$

$\tan^3 x - \tan x \sec^2 x = \tan x(\tan^2 x - \sec^2 x)$
$= \tan x(\tan^2 x - (1 + \tan^2 x)) = \tan x(\tan^2 x - 1 - \tan^2 x) = -\tan x = \tan(-x) = \tan(-x)$

$\sin(3x) - \cos x \sin(2x) = \cos^2 x \sin x - \sin^3 x$

$\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$

$\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \frac{2\sin x\cos x}{\sin x} - \frac{2\cos^2 x - 1}{\cos x} = 2\cos x - 2\cos x + \frac{1}{\cos x} = \frac{1}{\cos x}$
$= \sec x = \sec x$

Plot the points and find a function of the form

$y = A\cos(Bx + C) + D$ that fits the given data.

x	0	1	2	3	4	5
y	-2	2	-2	2	-2	2

The displacement h(t) in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 1\ 4 \sin(120\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Amplitude: 14 , period: 160 , frequency: 60 Hz

A woman is standing 300 feet away from a 2000-foot building. If she looks to the top of the building, at what angle above horizontal is she looking? A bored worker looks down at her from the 15th floor (1500 feet above her). At what angle is he looking down at her? Round to the nearest tenth of a degree.

Two frequencies of sound are played on an

instrument governed by the equation $n(t) = 8\cos(20\pi t)\cos(1000\pi t)$. What are the period and frequency of the “fast” and “slow” oscillations? What is the amplitude?

Amplitude: 8, fast period: 1500 , fast frequency: 500 Hz, slow period: 110 , slow frequency: 10 Hz

The average monthly snowfall in a small village in the Himalayas is 6 inches, with the low of 1 inch occurring in July. Construct a function that models this behavior. During what period is there more than 10 inches of snowfall?

A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

$D(t) = 20(0.9086)t \cos(4\pi t)$, 31 second

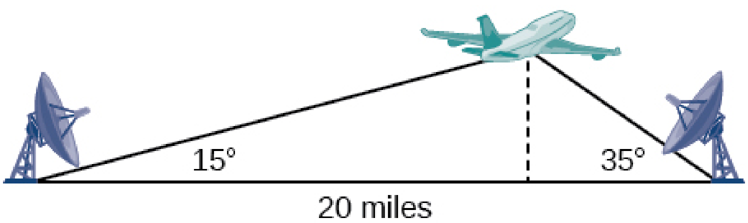
Water levels near a glacier currently average 9 feet, varying seasonally by 2 inches above and

below the average and reaching their highest point in January. Due to global warming, the glacier has begun melting faster than normal. Every year, the water levels rise by a steady 3 inches. Find a function modeling the depth of the water t months from now. If the docks are 2 feet above current water levels, at what point will the water first rise above the docks?

Non-right Triangles: Law of Sines
In this section, you will:

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

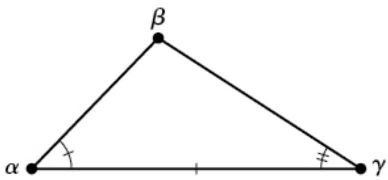
Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in [link] that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.

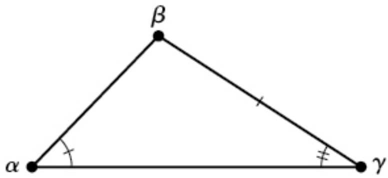


Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

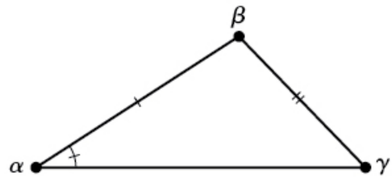
Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See [link].


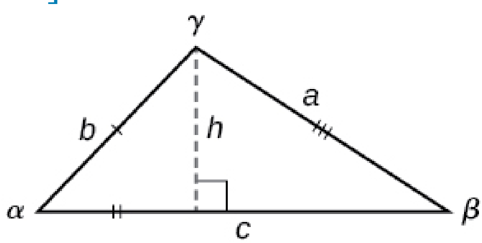
2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See [link].


3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is

not between the known sides. See [link].



Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let’s see how this statement is derived by considering the triangle shown in [link].



Using the right triangle relationships, we know that $\sin \alpha = \frac{h}{b}$ and $\sin \beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b \sin \alpha \text{ and } h = a \sin \beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b \sin \alpha &= a \sin \beta \\ \left(\frac{1}{ab}\right)(b \sin \alpha) &= (a \sin \beta)\left(\frac{1}{ab}\right) && \text{Multiply both sides by } \frac{1}{ab}. \\ \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

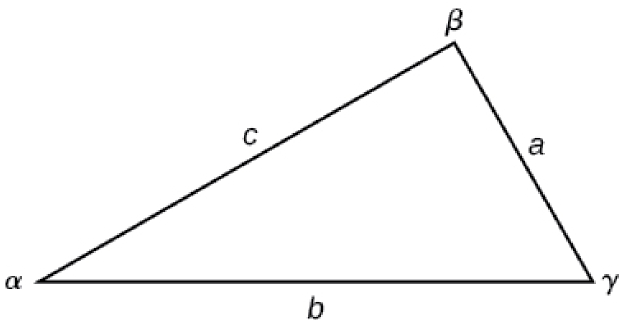
$\sin \alpha \; a = \sin \gamma \; c$ and $\sin \beta \; b = \sin \gamma \; c$

Collectively, these relationships are called the **Law of Sines**.

$\sin \alpha \; a = \sin \beta \; b = \sin \gamma \; c$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See [link].

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.



Law of Sines

Given a triangle with angles and opposite sides labeled as in [link], the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to

opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

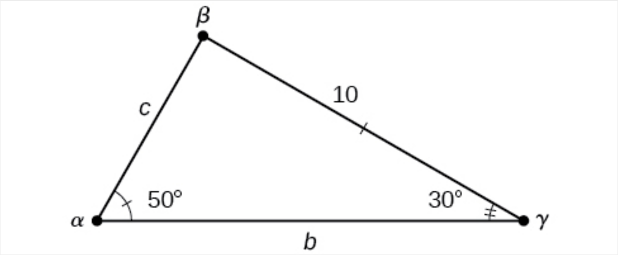
$\sin\alpha\ a = \sin\beta\ b = \sin\gamma\ c$

$a\ \sin\alpha = b\ \sin\beta = c\ \sin\gamma$

To solve an oblique triangle, use any pair of applicable ratios.

Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in [link] to the nearest tenth.



The three angles must add up to 180 degrees. From this, we can determine that

$\beta = 180° - 50° - 30° \quad = 100°$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50°$ and its corresponding

side $a=10$. We can use the following proportion from the Law of Sines to find the length of c.

$\frac{\sin(50^\circ)}{10} = \frac{\sin(30^\circ)}{c}$
$c\frac{\sin(50^\circ)}{10} = \sin(30^\circ)$ Multiply both sides by c.
$c = \sin(30^\circ)\frac{10}{\sin(50^\circ)}$
Multiply by the reciprocal to isolate c.
$c \approx 6.5270$

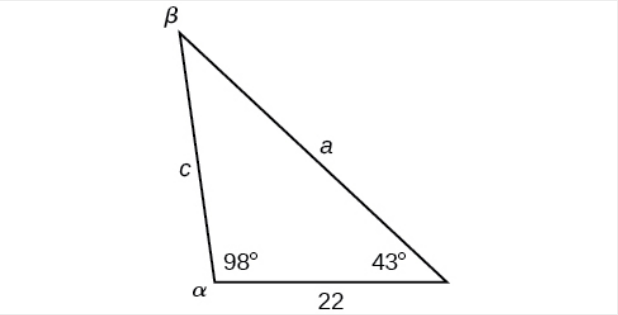
Similarly, to solve for b, we set up another proportion.

$\frac{\sin(50^\circ)}{10} = \frac{\sin(100^\circ)}{b}$
$b\sin(50^\circ) = 10\sin(100^\circ)$
Multiply both sides by b. $b = \frac{10\sin(100^\circ)}{\sin(50^\circ)}$
Multiply by the reciprocal to isolate b.
$b \approx 12.8558$

Therefore, the complete set of angles and sides is

$\alpha = 50^\circ$ $a = 10$
$\beta = 100^\circ$ $b \approx 12.8558$
$\gamma = 30^\circ$ $c \approx 6.5270$

Solve the triangle shown in [link] to the nearest tenth.



$\alpha = 98^\circ$ $a = 34.6181$ $\beta = 39^\circ$ $b = 22$
$\gamma = 43^\circ$ $c = 23.8415$

Note: Most of these problems will involve using a calculated value to calculate a future value. For simplicity, we will write the rounded values in the formula. When doing the calculations on your calculator, you should keep all digits and round only the final answer to the desired accuracy.

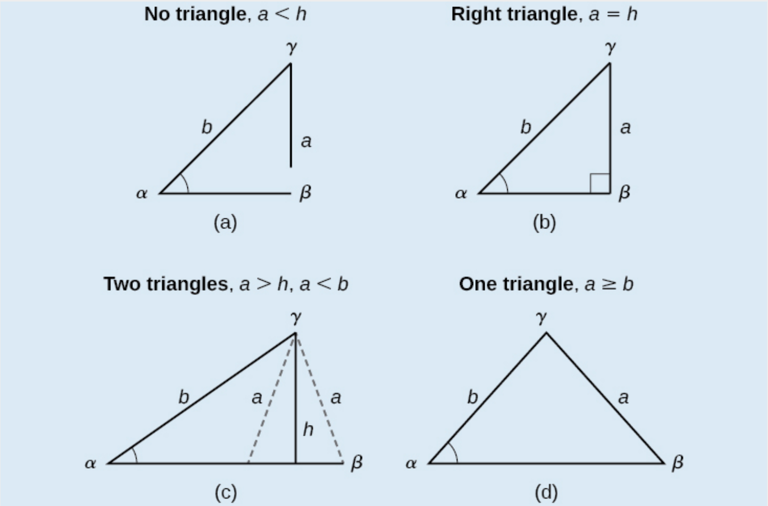
Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an **ambiguous case**. Triangles classified

as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Possible Outcomes for SSA Triangles

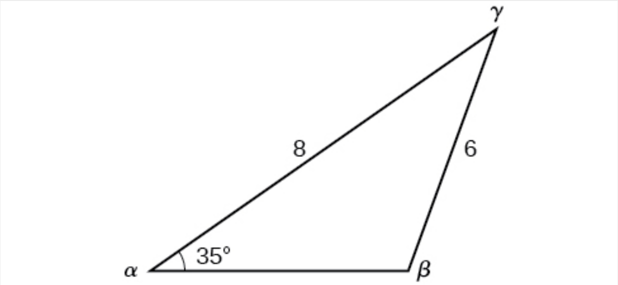
Oblique triangles in the category SSA may have four different outcomes. [link] illustrates the solutions with the known sides a and b and known angle α.



Solving an Oblique SSA Triangle

Solve the triangle in [link] for the missing side

and find the missing angle measures to the nearest tenth.



Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

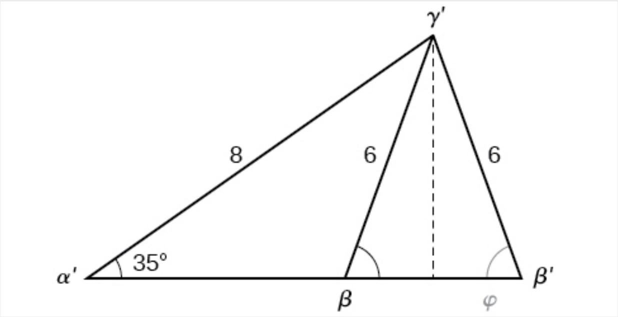
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \quad \frac{\sin(35^\circ)}{6} = \frac{\sin\beta}{8} \quad \frac{8\sin(35^\circ)}{6} = \sin\beta$$

We know sine is positive in QI and QII, so this equation

QI: $\beta = \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right) \quad \beta_1 \approx 49.8864^\circ$

QII: $180^\circ - \beta = 180^\circ - \sin^{-1}\left(\frac{8\sin(35^\circ)}{6}\right) \quad \beta_2 \approx 130.1136^\circ$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. Do both solutions work? Let’s investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have [link]. It appears that there may be a second triangle that will fit the given criteria.



Remember that the sine function is positive in both the first and second quadrants so both β_1 and β_2 are possible answers. Solving for γ, in QII we have
$\gamma_2 = 180° - 35° - 130.1136° \approx 14.8864°$

We can then use these measurements to solve the other triangle in QI. Since γ_1 is supplementary to the sum of α_1 and β_1, we have
$\gamma_1 = 180° - 35° - 49.8864° \approx 95.1136°$

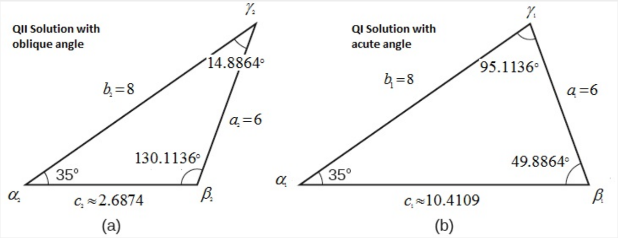
Now we need to find c_2 and c_1

We have
$\frac{c_2}{\sin(14.8864°)} = \frac{6}{\sin(35°)}$ $c = \frac{6\sin(14.8864°)}{\sin(35°)} \approx 2.6874$

Finally,
$\frac{c_1}{\sin(95.1136°)} = \frac{6}{\sin(35°)}$ $c = \frac{6\sin(95.1136°)}{\sin(35°)} \approx 10.4190$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an

opposite side of 6, as shown below.



We need to remember to look for 2 possible solutions and find both! It is easier to redraw the picture and label the QI and QII angles in the triangle and then finish solving that triangle so we don't accidentally use a value from the other triangle in our next calculation.

Note: If we calculate h we get $h=8\sin(35\circ)\approx 4.5886$. The value $a=6$ and $b=8$ so $h<a<b$. From possible outcomes for SSA triangles, we can see that we should expect 2 triangles!
Sometimes is is easier to find h first and determine how many triangles we have before we start!

Given $\alpha=80°, a=120,$ and $b=121,$ find the missing side and angles. If there is more than one possible solution, show both.

Solution 1

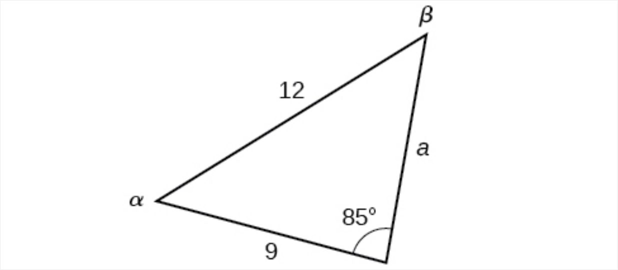
$\alpha_1 = 80^\circ$ $a_1 = 120$ $\beta_1 \approx 83.2237^\circ$ $b_1 = 121$
$\gamma_1 \approx 16.7766^\circ$ $c_1 \approx 35.1548$

Solution 2

$\alpha_2 = 80^\circ$ $a_2 = 120$ $\beta_2 \approx 96.7763^\circ$ $b_2 = 121$
$\gamma_2 \approx 3.2237^\circ$ $c_2 \approx 6.8523$

Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in [link], solve for the unknown side and angles. Round your answers to the nearest tenth.



In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for

β.

$$\frac{\sin(85°)}{12} = \frac{\sin \beta}{9} \quad \text{Isolate the unknown.}$$
$$\frac{9\sin(85°)}{12} = \sin \beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right) \quad \beta \approx 48.3438°$$

We also need to consider the QII solution. In this case, if we subtract β from $180°$, we find that there may be a second possible solution. Thus, $\beta = 180° - 48.3438° \approx 131.6562°$. To check the solution, subtract both angles, $131.6562°$ and $85°$, from $180°$. This gives

$$\alpha = 180° - 85° - 131.6562° \approx -36.6562°,$$

which is impossible, and so $\beta \approx 48.3438°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3438° \approx 46.6562°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.6562°)}{a} \quad a\frac{\sin(85°)}{12} = \sin(46.7°) \qquad a = \frac{12\sin(46.6562°)}{\sin(85°)} \approx 8.7603$$

The complete set of solutions for the given triangle is

$\alpha \approx 46.6562^\circ$ $a \approx 8.7603$
$\beta \approx 48.3438^\circ$ $b = 9$ $\gamma = 85^\circ$ $c = 12$

Given $\alpha = 80^\circ, a = 100,\ b = 10,$ find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

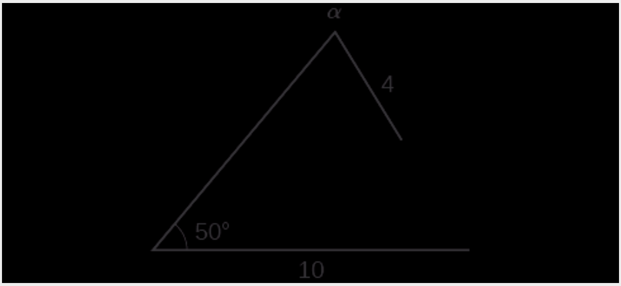
$\beta \approx 5.6517^\circ, \gamma \approx 94.3483^\circ, c \approx 101.2504$

Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Using the given information, we can solve for the angle opposite the side of length 10. See [link].

$\sin\alpha\ 10 = \sin(50^\circ)\ 4 \quad \sin\alpha = 10\sin(50^\circ)\ 4$
$\sin\alpha \approx 1.915$



We can stop here without finding the value of α. Because the range of the sine function is $[-1,1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

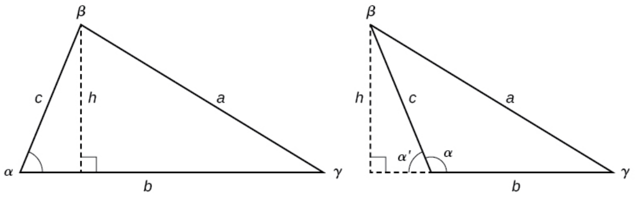
Determine the number of triangles possible given $a=31$, $b=26$, $\beta=48°$.

two

Finding the Area of an Oblique Triangle

Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $\text{Area} = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in [link], one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin \alpha = \frac{h}{c}$ or $c \sin \alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.



Thus,

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}b(c \sin \alpha)$$

Similarly,

$$\text{Area} = \frac{1}{2}a(b \sin \gamma) = \frac{1}{2}a(c \sin \beta)$$

Area of an Oblique Triangle

The formula for the area of an oblique triangle is given by

$$\text{Area} = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90, b = 52,$ and angle $\gamma = 102°.$ Round the area to the nearest integer.

Using the formula, we have

$$\text{Area} = \frac{1}{2}ab\sin\gamma \quad \text{Area} = \frac{1}{2}(90)(52)\sin(102°) \quad \text{Area} \approx 2289 \text{ square units}$$

Find the area of the triangle given $\beta = 42°,$ $a = 7.2\text{ ft},$ $c = 3.4\text{ ft}.$ Round the area to the nearest tenth.

---

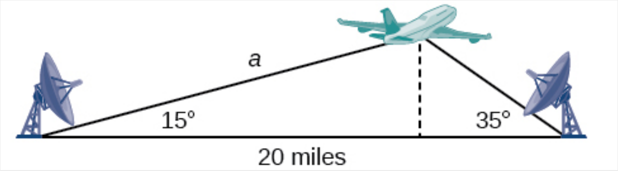
about 8.2 square feet

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in [link]. Round the altitude to the nearest tenth of a mile.



To find the elevation of the aircraft, we first

find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$
$$a\sin(130°) = 20\sin(35°)$$
$$a = \frac{20\sin(35°)}{\sin(130°)}$$
$$a \approx 14.9750$$

The distance from one station to the aircraft is about 14.9750 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\sin(15°) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(15°) = \frac{h}{a}$$
$$\sin(15°) = \frac{h}{14.9750}$$
$$h = 14.9750\sin(15°)$$
$$h \approx 3.8758$$

The aircraft is at an altitude of approximately 3.9 miles.

The bearing from fire tower A to tower B fire tower is N70ºE. The two fire towers are 30 km apart. A fire is spotted by rangers in each tower. The bearing to the fire from tower A is N85ºE and from tower B is S75ºE. Find the

distance of the fire to each tower.

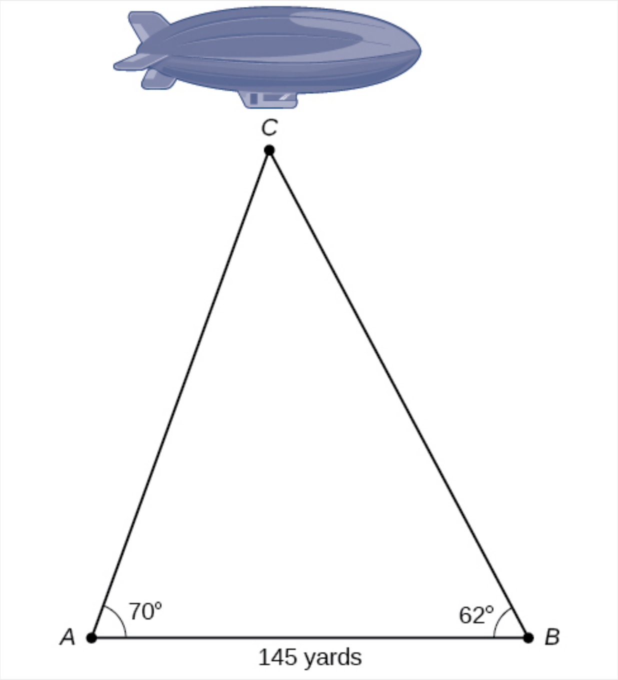
tower A to fire: 50.3108 km tower B to fire: 22.7021 km

The bearing from fire tower A to tower B fire tower is N70ºE. The two fire towers are 30 km apart. A fire is spotted by rangers in each tower. The bearing to the fire from tower A is N85ºE and from tower B is S75ºE. Find the distance of the fire to each tower.

tower A to fire: 50.3108 km tower B to fire: 22.7021 km

The diagram shown in [link] represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the

distance between the viewing points of the two end zones is 145 yards.



161.9 yd.

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$ $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$
Area for oblique triangles	$\text{Area} = \frac{1}{2} bc \sin \alpha = \frac{1}{2} ac \sin \beta = \frac{1}{2} ab \sin \gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See [link].

- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See [link] and [link].
- The Law of Sines can be used to solve triangles with given criteria. See [link].
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See [link].
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See [link].

Section Exercises

Verbal

Describe the altitude of a triangle.

The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a 90° angle.

Compare right triangles and oblique triangles.

When can you use the Law of Sines to find a missing angle?

When the known values are the side opposite the missing angle and another side and its opposite angle.

In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

What type of triangle results in an ambiguous case?

A triangle with two given sides and a non-included angle.

Algebraic

For the following exercises, assume α is opposite side a, β is opposite side $b,$ and γ is opposite side $c.$ Solve each triangle, if possible. Round each answer to 4 decimal places.

$\alpha = 43^\circ, \gamma = 69^\circ, a = 20$

$\alpha = 35^\circ, \gamma = 73^\circ, c = 20$

$\beta = 72^\circ, a \approx 11.9957, b \approx 19.8902$

$\alpha = 60^\circ,\ \beta = 60^\circ, \gamma = 60^\circ$

$a = 4,\ \alpha = \ 60^\circ\ , \beta = 100^\circ$

$\gamma = 20^\circ, b \approx 4.5486, c \approx 1.5797$

$b = 10, \beta = 95^\circ, \gamma = \ 30^\circ$

For the following exercises, use the Law of Sines to solve for the missing side for each oblique triangle. Round each answer to the nearest hundredth. Assume that angle A is opposite side a, angle B is opposite side b, and angle C is opposite side c.

Find side b when $A = 37^\circ,\ B = 49^\circ, c = 5.$

$b \approx 3.78$

Find side a when $A=132°, C=23°, b=10.$

Find side c when $B=37°, C=21°, b=23.$

$c \approx 13.70$

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

$\alpha=119°, a=14, b=26$

$\gamma=113°, b=10, c=32$

---

one triangle, $\alpha \approx 50.3°, \beta \approx 16.7°, a \approx 26.7$

$b=3.5,\ c=5.3,\ \gamma=\ 80°$

$a=12,\ c=17,\ \alpha=\ 35°$

---

two triangles, $\gamma_1 \approx 54.3°,\ \beta_1 \approx 90.7°,\ b_1 \approx 20.9$ $\gamma_2 \approx 125.7°,\ \beta_2 \approx 19.3°,\ b_2 \approx 6.9$

$a=20.5,\ b=35.0,\ \beta=\ 25^\circ$

$a=7, c=9,\ \alpha=\ 43^\circ$

---

two triangles, $\beta 1\ \approx 75.7^\circ,\ \ \gamma 1\ \approx 61.3^\circ$, $b 1 \approx 9.9\ \ \beta 2\ \approx 18.3^\circ$, $\gamma 2\ \approx 118.7^\circ$, $b 2\ \approx 3.2$

$a=7, b=3, \beta=24^\circ$

$b=13, c=5, \gamma=\ 10^\circ$

---

two triangles, $\alpha 1\ \approx 143.2^\circ$, $\beta 1\ \approx 26.8^\circ$, $a 1 \approx 17.3$, $\alpha 2\ \approx 16.8^\circ$, $\beta 2\ \approx 153.2^\circ$, $a 2\ \approx 8.3$

$a=2.3, c=1.8, \gamma=28^\circ$

$\beta=119^\circ, b=8.2, a=11.3$

---

no triangle possible

For the following exercises, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

Find angle A when $a = 24, b = 5, B = 22°$.

Find angle A when $a = 13, b = 6, B = 20°$.

$A \approx 47.8°$ or $A' \approx 132.2°$

Find angle B when $A = 12°, a = 2, b = 9$.

For the following exercises, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

$a = 5, c = 6, \beta = 35°$

8.6

$b = 11, c = 8, \alpha = 28°$

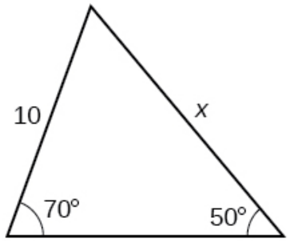
$a = 32, b = 24, \gamma = 75°$

370.9

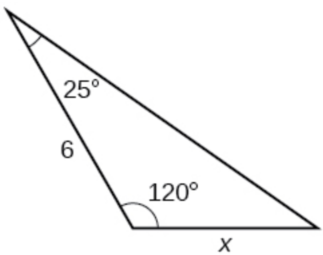
$a = 7.2, b = 4.5, \gamma = 43°$

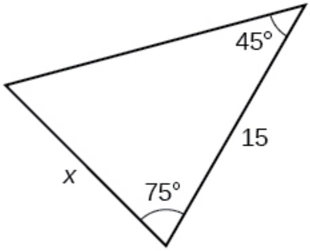
Graphical

For the following exercises, find the length of side x. Round to 4 decimal places.

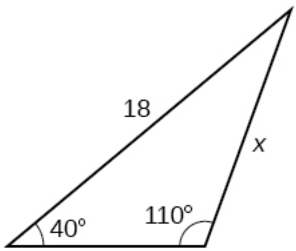


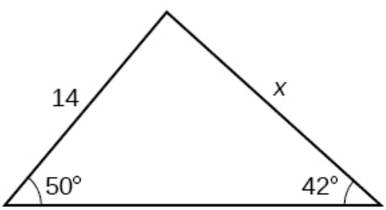
12.2668



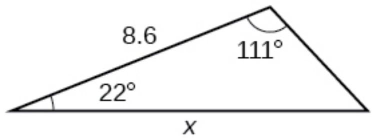


12.2474

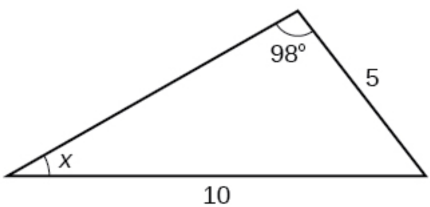




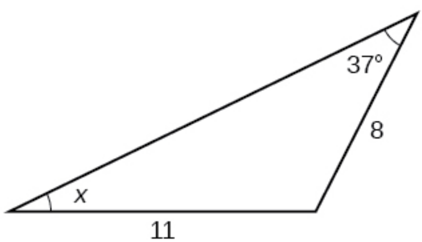
16.0277

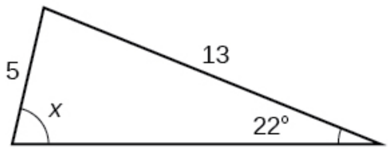


For the following exercises, find the measure of angle x, if possible. Round to the nearest tenth.

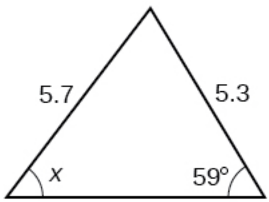


29.6786°

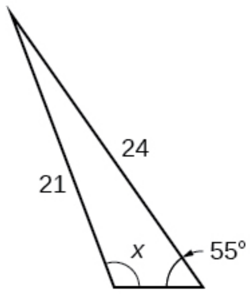




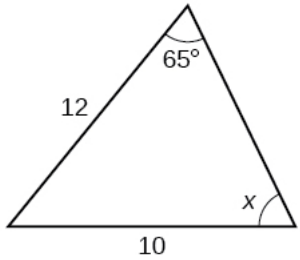
$x = 76.9003°$ or $x = 103.0997°$



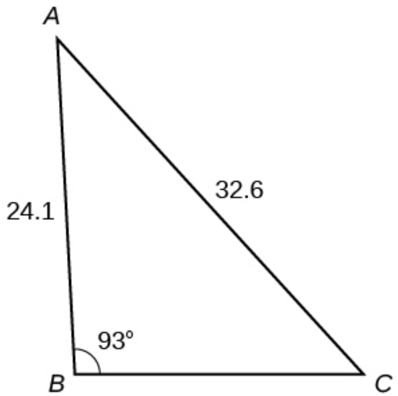
Notice that x is an obtuse angle.



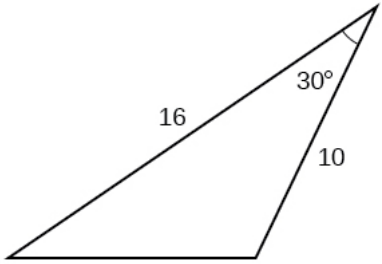
$110.5814°$

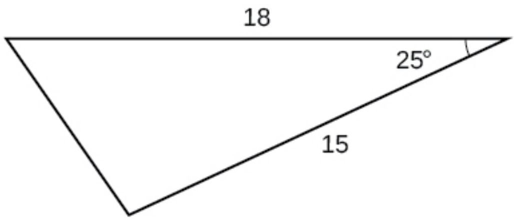


For the following exercises, find the area of each triangle. Round each answer to 4 decimal places.

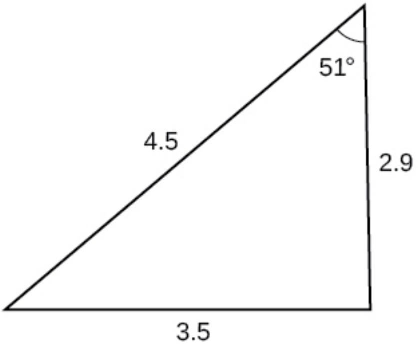


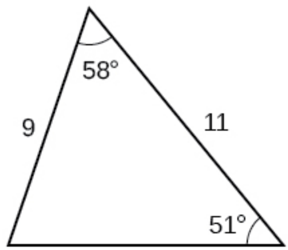
$A \approx 39.4174$, $C \approx 47.5826$, $BC \approx 20.7283$



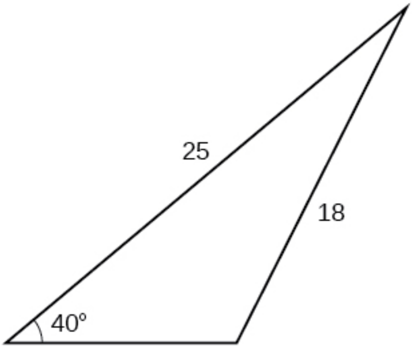


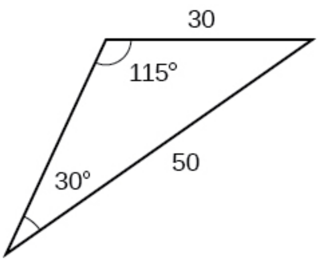
57.0535





41.9783

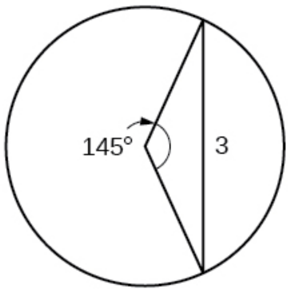




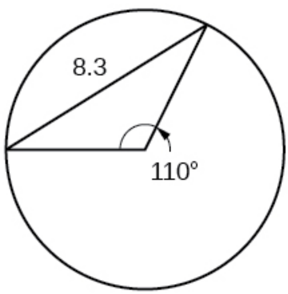
430.1823

Extensions

Find the radius of the circle in [link]. Round to the nearest tenth.

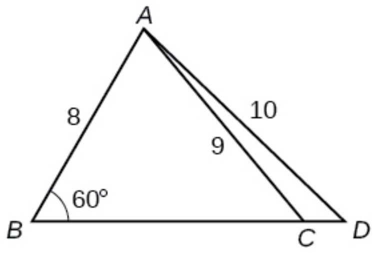


Find the diameter of the circle in [link]. Round to the nearest tenth.

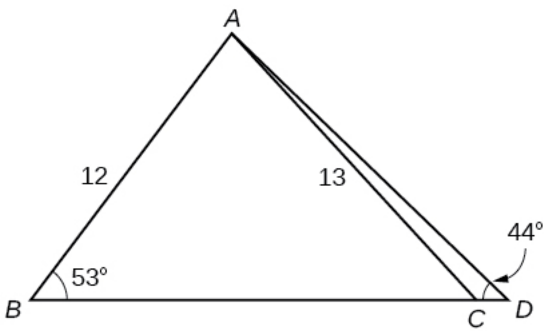


10.1

Find $m\angle ADC$ in [link]. Round to the nearest tenth.

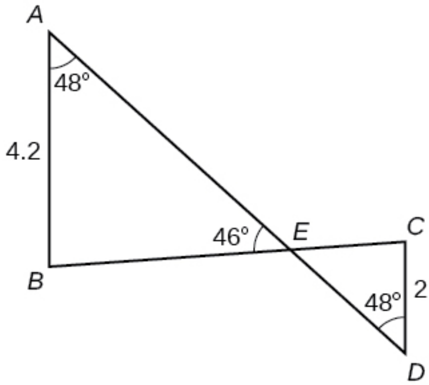


Find AD in [link]. Round to the nearest tenth.

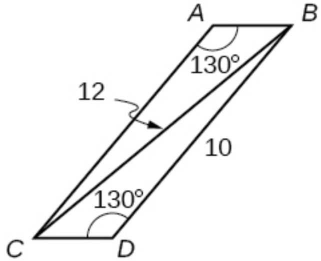


$AD \approx 13.8$

Solve both triangles in [link]. Round each answer to the nearest tenth.

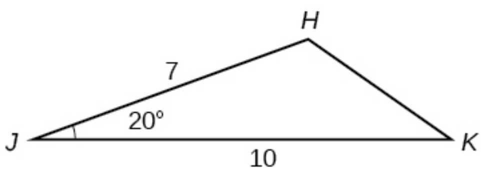


Find AB in the parallelogram shown in [link].

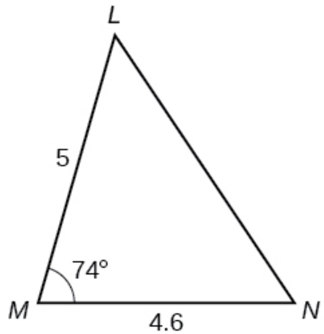


$AB \approx 2.8$

Solve the triangle in [link]. (Hint: Draw a perpendicular from H to JK). Round each answer to the nearest tenth.

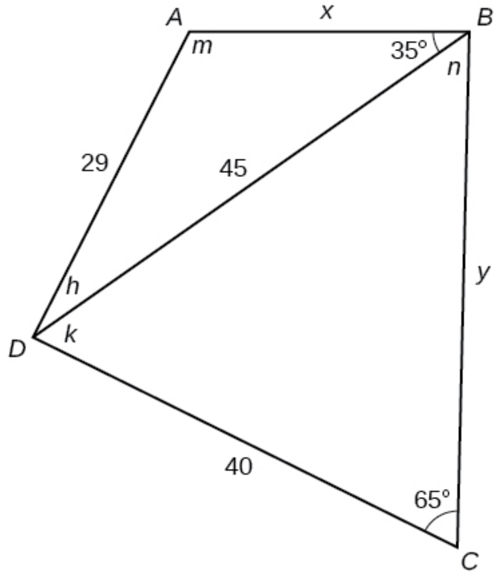


Solve the triangle in [link]. (Hint: Draw a perpendicular from N to LM). Round each answer to the nearest tenth.



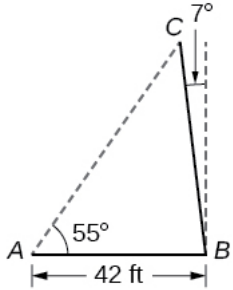
$L\approx 49.7,\ N\approx 56.3,\ LN\approx 5.8$

In [link], $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.



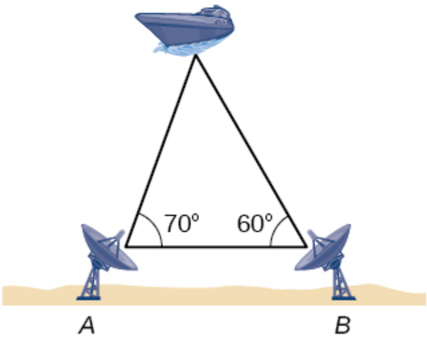
Real-World Applications

A pole leans away from the sun at an angle of 7° to the vertical, as shown in [link]. When the elevation of the sun is 55°, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.

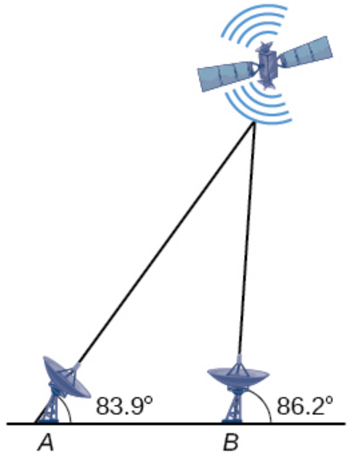


51.4 feet

To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in [link]. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.

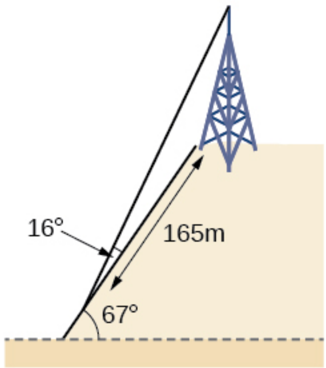


[link] shows a satellite orbiting Earth. The satellite passes directly over two tracking stations A and $B,$ which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°,$ respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.

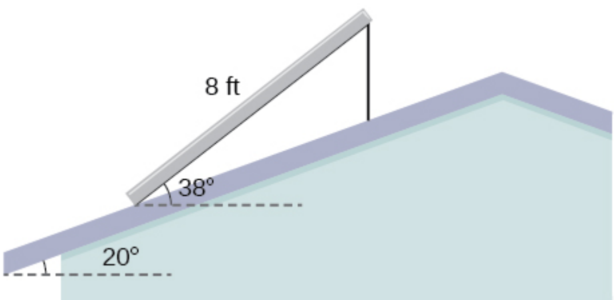


The distance from the satellite to station A is approximately 1716 miles. The satellite is approximately 1706 miles above the ground.

A communications tower is located at the top of a steep hill, as shown in [link]. The angle of inclination of the hill is $67°$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16°$. Find the length of the cable required for the guy wire to the nearest whole meter.



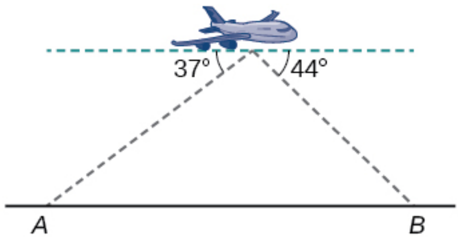
The roof of a house is at a 20° angle. An 8-foot solar panel is to be mounted on the roof and should be angled 38° relative to the horizontal for optimal results. (See [link]). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.



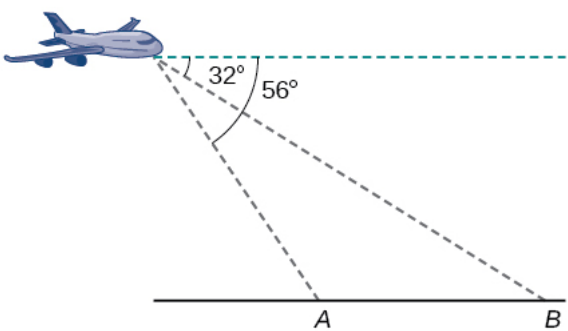
2.6 ft

Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to

an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44°, as shown in [link]. Find the distance of the plane from point A to the nearest tenth of a kilometer.



A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in [link]. Find the distance of the plane from point A to the nearest tenth of a kilometer.



5.6 km

In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

---

371 ft

Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of angle BAC is determined to be 101°, and the measure of angle ACB is determined to be 53°. What is the distance from A to B, rounded to the nearest whole meter?

A man and a woman standing 3 1 2 miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

5936 ft

Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man’s shadow to the top of his head of 28°. A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from

the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

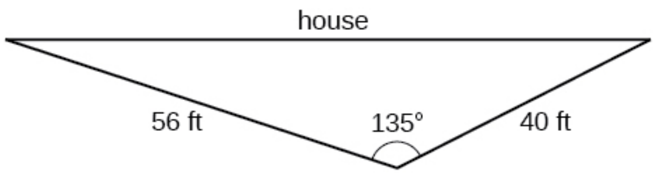
24.1 ft

Three cities, $A, B,$ and $C,$ are located so that city A is due east of city $B.$ If city C is located 35° west of north from city B and is 100 miles from city A and 70 miles from city $B,$ how far is city A from city $B?$ Round the distance to the nearest tenth of a mile.

Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

19,056 ft^2

Brian’s house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in [link].

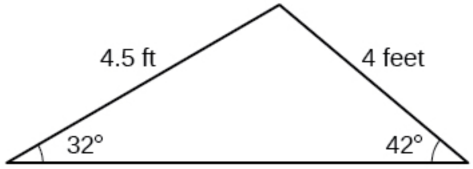


The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

445,624 square miles

A yield sign measures 30 inches on all three sides. What is the area of the sign?

Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in [link].



8.65 ft^2

Glossary

altitude
: a perpendicular line from one vertex of a triangle to the opposite side, or in the case of an obtuse triangle, to the line containing the opposite side, forming two right triangles

ambiguous case
: a scenario in which more than one triangle is a valid solution for a given oblique SSA triangle

Law of Sines
: states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side

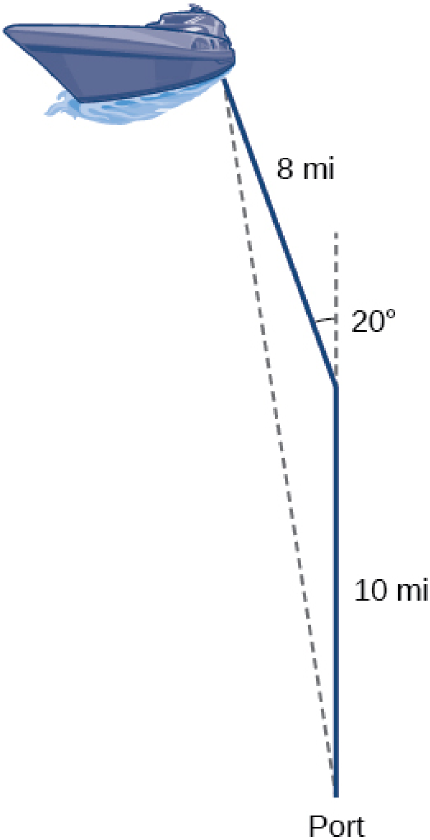
oblique triangle
: any triangle that is not a right triangle

Non-right Triangles: Law of Cosines

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in [link]. How far from port is the boat?

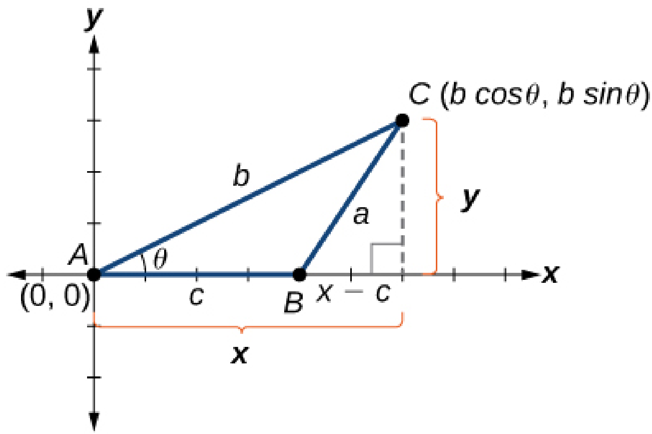


Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the **Law of Cosines**, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the **Generalized Pythagorean Theorem**, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the *x*-axis, and vertex C located at some point (x,y) in the plane, as illustrated in [link]. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.



We can drop a perpendicular from C to the *x*-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$\cos\theta = \frac{x(\text{adjacent})}{b(\text{hypotenuse})}$ and $\sin\theta = \frac{y(\text{opposite})}{b(\text{hypotenuse})}$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x,y) point located at C has coordinates $(b\cos\theta,\ b\sin\theta)$. Using the side $(x-c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x-c)^2 + y^2 \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y. \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1. \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2. \\
a^2 &= b^2 + c^2 - 2bc\cos\theta
\end{aligned}$$

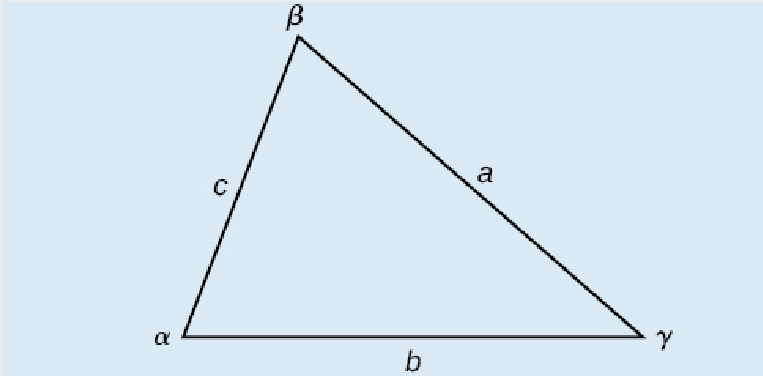
The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

Law of Cosines

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in [link], with angles $\alpha, \beta,$ and $\gamma,$ and opposite corresponding sides $a, b,$ and $c,$ respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc \cos \alpha \quad b^2 = a^2 + c^2 - 2ac \cos \beta \quad c^2 = a^2 + b^2 - 2ab \cos \gamma$$



To solve for a missing side measurement, the corresponding opposite angle measure is needed. When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

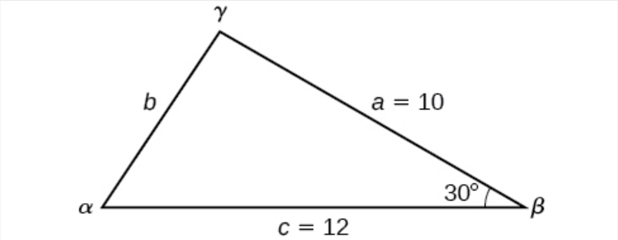
Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side.
3. Apply the Law of Sines or Law of Cosines to find the measure of another angle. If using

Law of Sines, find the smallest angle first.
4. Compute the measure of the remaining angle by subtracting the 2 known angles from 180.

Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in [link].



First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ)$$

Substitute the measurements for the known quantities.
$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right)$
Evaluate the cosine and begin to simplify. $b^2 = 244 - 120\sqrt{3}$ $b = \sqrt{244 - 120\sqrt{3}}$
Use the square root property. $b \approx 6.0128$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. When using Law of sines we should find the smallest angle first. This way we know if will be acute and don't have to worry about the Quadrant II solution. Solving for angle α, we have

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$ $\frac{\sin\alpha}{10} = \frac{\sin(30°)}{6.0128}$
$\sin\alpha = \frac{10\sin(30°)}{6.0128}$
Multiply both sides of the equation by 10.
$\alpha = \sin^{-1}\left(\frac{10\sin(30°)}{6.0128}\right)$
Find the inverse sine of $\frac{10\sin(30°)}{6.0108}$.
$\alpha \approx 56.2591°$

The other possibility for α would be $\alpha = 180° - 56.2591° \approx 123.7409°$. In the original diagram, α is adjacent to the longest side, so α is an acute angle and, therefore, $123.7409°$ does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between $0°$ and $180°$. Proceeding

with $\alpha \approx 56.2591^{\circ}$, we can then find the third angle of the triangle.

$\gamma = 180^{\circ} - 30^{\circ} - 56.2590^{\circ} \approx 93.7409^{\circ}$

The complete set of angles and sides is

$\alpha \approx 56.2591^{\circ}$ $a = 10$ $\beta = 30^{\circ}$ $b \approx 6.0128$

$\gamma \approx 93.7409^{\circ}$ $c = 12$

Note About Round Off error

When calculating values from a previously calculated values, keep all digits and round only in the last step. It is sometimes easier to store the values in your calculator and recall the stored values when typing in the next step into your calculator.

Find the missing side and angles of the given triangle: $\alpha = 30^{\circ}$, $b = 12$, $c = 24$.

$a \approx 14.8718$, $\beta \approx 23.7940^{\circ}$, $\gamma \approx 126.2060^{\circ}$.

Given two sides and the angle between them

(SSS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the measure of the largest angle. (If we find the largest angle first, we know all the other angles must be acute.)
3. Apply the Law of Sines or Law of Cosines to find the measure of another angle. If using Law of Sines, find the smallest angle first.
4. Compute the measure of the remaining angle by subtracting the 2 known angles from 180.

Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a=20$, side $b=25$, and side $c=18$.

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha$$ Substitute the appropriate measurements.

$$400 = 625 + 324 - 900\cos\alpha$$

Simplify in each step.

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha$$

Isolate $\cos\alpha$.

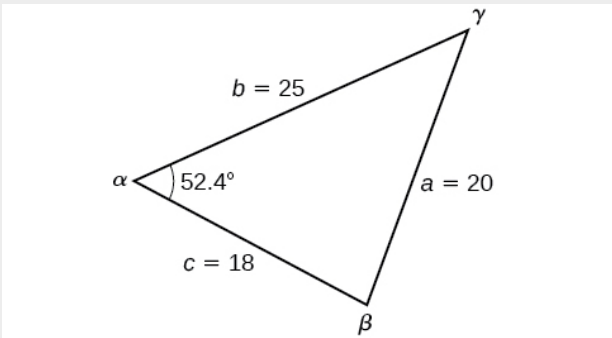
$$\frac{-549}{-900} = \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{549}{900}\right)$$

Find the inverse cosine.

$$\alpha \approx 52.4105^\circ$$

See [link].



Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Given $a=5, b=7,$ and $c=10,$ find the missing angles.

$\alpha \approx 27.6604°$, $\beta \approx 40.5358°$, $\gamma \approx 111.8037°$

Solving Applied Problems Using the Law of Cosines

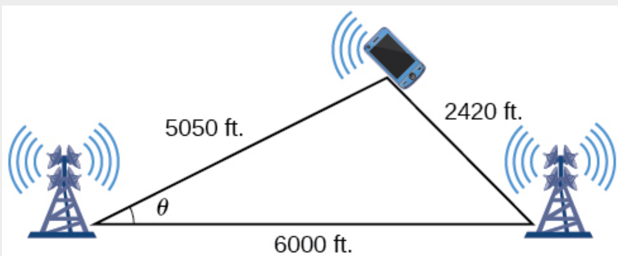
Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the

cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

For simplicity, we start by drawing a diagram similar to [link] and labeling our given information.



Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a=2420, b=5050,$ and $c=6000$. Thus, θ corresponds to the opposite side $a=2420$.

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

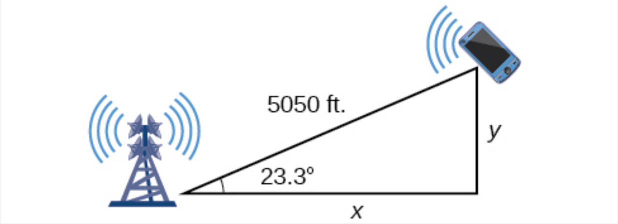
$$(2420)^2 = (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta$$

$$(2420)^2 - (5050)^2 - (6000)^2 = -2(5050)(6000)\cos\theta$$

$$\frac{(2420)^2 - (5050)^2 - (6000)^2}{-2(5050)(6000)} = \cos\theta$$

$$\cos\theta \approx 0.9183$$
$$\theta \approx \cos^{-1}(0.9183)$$
$$\theta \approx 23.3281^\circ$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in [link]. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.



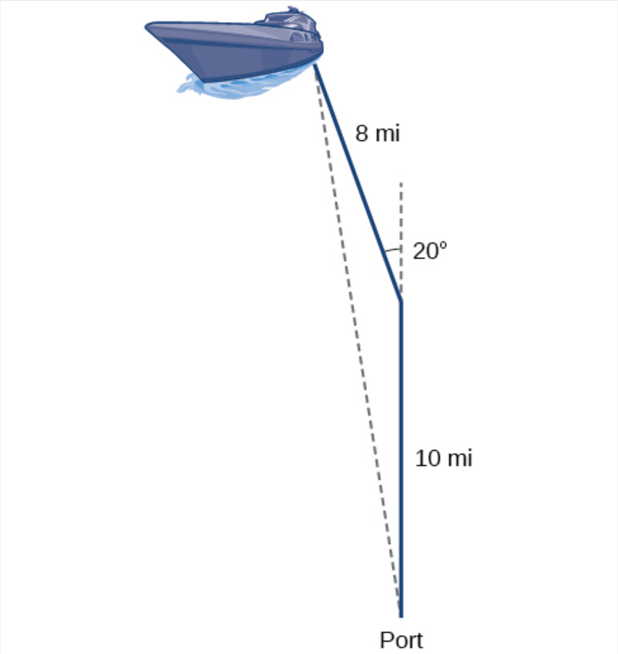
Using the angle $\theta = 23.3281^\circ$ and the basic trigonometric identities, we can find the solutions. Thus

$$\cos(23.3281^\circ) = \frac{x}{5050}$$
$$x = 5050\cos(23.3281^\circ)$$
$$x \approx 4638.15 \text{ feet}$$
$$\sin(23.3281^\circ) = \frac{y}{5050}$$
$$y = 5050\sin(23.3281^\circ)$$
$$y \approx 1999.7770 \text{ feet}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles North, turns 20 degrees West, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in [link].



The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find

the missing side of the obtuse triangle—the distance of the boat to the port.

$$x^2 = 8^2 + 10^2 - 2(8)(10)\cos(160^\circ)$$
$$x = \sqrt{8^2 + 10^2 - 2(8)(10)\cos(160^\circ)}$$
$$x \approx 17.7299 \text{ miles}$$

The boat is about 17.7299 miles from port.

Example: Two ships leave a port at 9am, One travels at a bearing of N59ºW at 12 mph, and the other travels at a bearing of S63ºW at 14 mph.

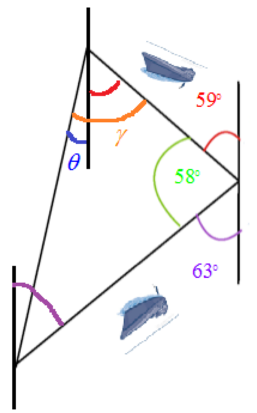
a. Approximate how far apart the ships are at noon that day.

b. What is the bearing of the port from each ship at noon?

c. What is the bearing of the second ship from the first ship?

Solution

First, we draw a picture.



If the ships leave at 9 am and arrive at noon, then they traveled for 3 hours. We know distance = rate * time. The distance the first ship traveled is $12\,\text{mph}\cdot 3\,\text{hours} = 36\,\text{miles}$. The distance the second ship traveled is $14\,\text{mph}\cdot 3\,\text{hours} = 42\,\text{miles}$

We can find the angle between the two ships by using the fact that their bearings plus this angle make a straight line. $\beta = 180^\circ - 59^\circ - 63^\circ = 58^\circ$

Using the law of cosines, we can find x.

$$x^2 = a^2 + c^2 - 2ac\cos\beta$$

$$x^2 = 36^2 + 42^2 - 2(36)(42)\cos(58^\circ)$$

Substitute the appropriate measurements.

$$x = \sqrt{36^2 + 42^2 - 2(36)(42)\cos(58^\circ)}$$

Solve for x. $x = 38.1175$ miles

b. The bearing is asking what direction does each

ship need to go to get back to the port. Using alternate interior angle theorem, we can see the bearing of the first ship to the port is S 59° E The bearing of the second ship to the port is N 63° E

c. To find the bearing from the first ship to the second ship, we need to find the angle at γ . Again, we can use Law of Cosines.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$42^2 = 36^2 + 38.17753456^2 - 2(36)(38.17753456)\cos\gamma$$

Substitute the appropriate measurements.

$$42^2 - 36^2 - 38.17753456^2 = -2(36)(38.17753456)\cos\alpha$$ Simplify in each step.

$$\frac{42^2 - 36^2 - 38.17753456^2}{-2(36)(38.17753456)} = \cos\gamma$$ Isolate cos γ.

$$\gamma = \cos^{-1}\left(\frac{42^2 - 36^2 - 38.17753456^2}{-2(36)(38.17753456)}\right)$$ Find the inverse cosine.

$$\gamma \approx 68.9006^\circ$$

Then the bearing angle we need is

$\theta = \gamma - 59^\circ = 68.9006^\circ - 59^\circ = 9.9006^\circ$

So the bearing from the first shop to the second ship is S 9.9006° W

Example: A squirrel found 3 nuts. He buries the first one in the location where he found it. He then runs 14 ft in the direction N23ºE and buries the second. He turns and runs 18 ft in

the direction of N60ºW and buries the last nut. a) How far is the third nut from the first nut? b) What is the bearing of the third nut FROM the first nut? c) What is the bearing of the first nut FROM the third nut? d) What is the bearing of the first nut FROM the second nut? e) What is the bearing from the third nut to the second nut?

a) 24.1127 ft b) N 24.8107° W c)S 24.8107° E d)S 23° W e) S 60° E

a) A ship travels 60 mi due east adjusts his course 12° north then travels 80 mi in that direction. How far is the ship from its point of departure?
b. A ship travels 60 mi due east adjusts his course N 12° E then travels 80 mi in that direction. How far is the ship from its point of departure?
c. A ship travels 60 mi due east adjusts his course N 12° W then travels 80 mi in that direction. How far is the ship from its point of departure?

a. 139.2488 mi. b. 109.5260 mi. c. 89.4653

mi.

Using Heron’s Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron’s formula instead of finding the height. Heron of Alexandria was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

Heron’s Formula

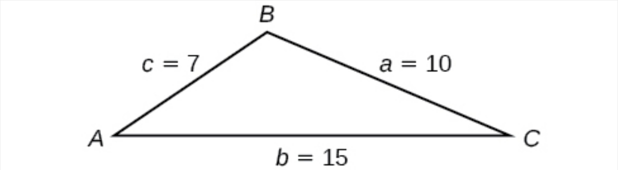
Heron’s formula finds the area of oblique triangles in which sides $a, b,$ and c are known.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Using Heron’s Formula to Find the Area of a Given Triangle

Find the area of the triangle in [link] using Heron’s formula.



First, we calculate s.
$s= (a+b+c)\ 2\ s= (10+15+7)\ 2\ =16$

Then we apply the formula.
$\text{Area}= s(s-a)(s-b)(s-c)\ \text{Area}= 16(16-10)(16-15)(16-7)\ \text{Area}\approx 29.4$

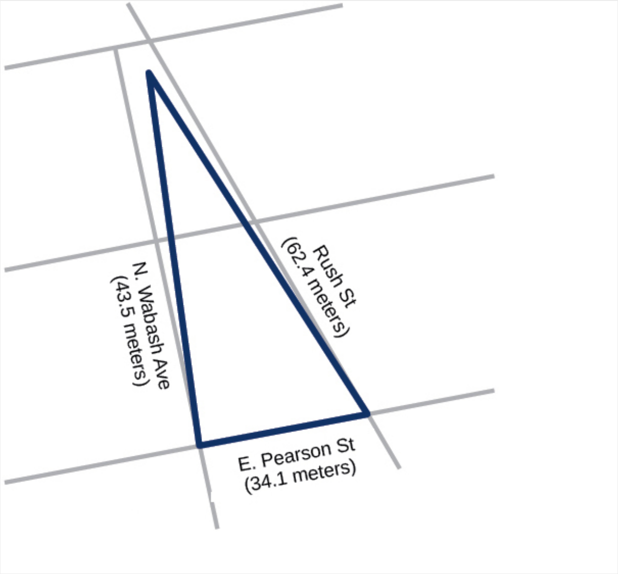
The area is approximately 29.4 square units.

Use Heron’s formula to find the area of a triangle with sides of lengths $a=29.7$ ft, $b=42.3$ ft, and $c=38.4$ ft.

Area = 552 square feet

Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? Below is a view of the city property.



Find the measurement for s, which is one-half

of the perimeter.
$s = (62.4 + 43.5 + 34.1)\ 2\ s = 70\ \text{m}$

Apply Heron's formula.
$\text{Area} = 70(70 - 62.4)(70 - 43.5)(70 - 34.1)$
$\text{Area} = 506{,}118.2\ \text{Area} \approx 711.4$

The developer has about 711.4 square meters.

Find the area of a triangle given $a = 4.38$ ft ,$b = 3.79$ ft, and $c = 5.22$ ft.

about 8.15 square feet

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See [link] and [link].
- The Law of Cosines is useful for many types of

applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See [link] and [link].
- When solving a SSS case, it is best to find the largest angle first with using Law Of Cosines. When using Law of Sines to find the second angle in a triangle, it is best to find the smallest angle first.
- Heron’s formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron’s formula. See [link] and See [link].

Section Exercises

Verbal

If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

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two sides and the angle opposite the missing side.

If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

Explain what s represents in Heron's formula.

---

s is the semi-perimeter, which is half the perimeter of the triangle.

Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

When must you use the Law of Cosines instead of the Pythagorean Theorem?

The Law of Cosines must be used for any oblique (non-right) triangle.

Algebraic

For the following exercises, assume α is opposite side a, β is opposite side $b,$ and γ is opposite side $c.$ If possible, solve each triangle. Round to 4 decimal places.

$\gamma = 41.2^{\circ}, a = 2.49, b = 3.13$

$\alpha = 120^\circ, b = 6, c = 7$

$a = 11.2694,\ \gamma = \ 32.5429 \circ,\ \beta = \ 27.4571 \circ$

$\beta = 58.7^\circ, a = 10.6, c = 15.7$

$\gamma = 115^\circ, a = 18, b = 23$

$c = 34.6833,\ \alpha = \ 18.0575 \circ,\ \beta = \ 36.9425 \circ$

$\alpha = 119^\circ, a = 26, b = 14$

$\gamma = 113^\circ, b = 10, c = 32$

$\beta = \ 16.7178 \circ,\ \alpha = \ 50.2822 \circ,\ a = 26.7402$

$\beta = 67^\circ, a = 49, b = 38$

$\alpha = 43.1^\circ, a = 184.2, b = 242.8$

$c = 257.3284,\ \beta = \ 64.2430 \circ,\ \gamma = \ 72.6570 \circ$

$\alpha = 36.6^\circ, a = 186.2, b = 242.2$

$\beta = 50^\circ, a = 105, b = 45$

---

not possible

$a = 42, b = 19, c = 30;$

$a = 14,\ b = 13,\ c = 20;$

$\alpha = 40.3149^\circ,\ \beta = 44.1674^\circ,\ \gamma = 95.5177^\circ$

$a = 16, b = 31, c = 20;$

$a = 13,\ b = 22,\ c = 28;$ find angle A.

$\alpha = 26.8686^\circ,\ \beta = 49.8920^\circ,\ \gamma = 103.8920^\circ$

$a = 108,\ b = 132,\ c = 160;$ find angle C.

For the following exercises, solve the triangle. Round to the nearest tenth.

$A=35°, b=8, c=11$

$B\approx 45.9°, C\approx 99.1°, a\approx 6.4$

$B=88°, a=4.4, c=5.2$

$C=121°, a=21, b=37$

$A\approx 20.6°, B\approx 38.4°, c\approx 51.1$

$a=13, b=11, c=15$

$a=3.1, b=3.5, c=5$

$A\approx 37.8°, B\approx 43.8, C\approx 98.3°$

$a=51, b=25, c=29$

For the following exercises, use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest

tenth.

177.56 in^2

Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

$a = \frac{1}{2}$ m, $b = \frac{1}{3}$ m, $c = \frac{1}{4}$ m

---

0.04 m^2

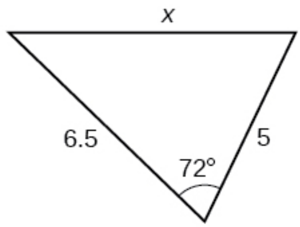
$a = 12.4$ ft, $b = 13.7$ ft, $c = 20.2$ ft

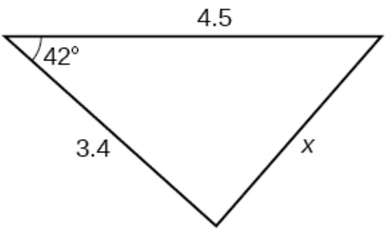
$a = 1.6$ yd, $b = 2.6$ yd, $c = 4.1$ yd

0.91 yd^2

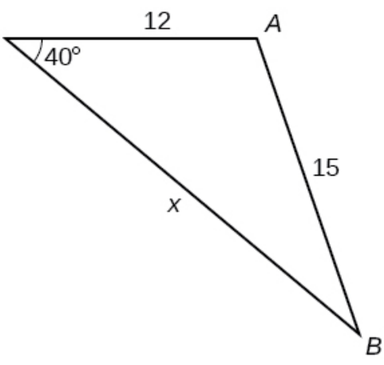
Graphical

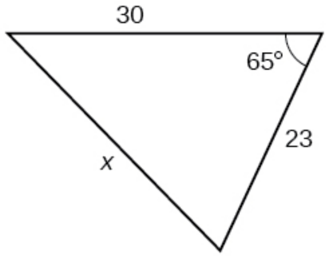
For the following exercises, find the length of side x. Round to the nearest tenth.



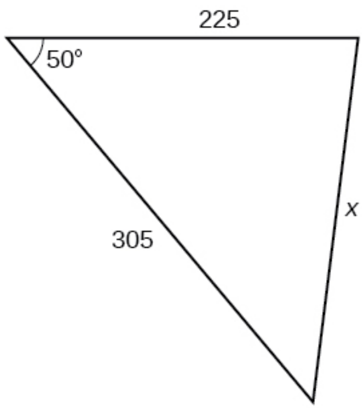


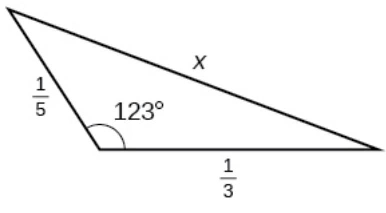
3.0





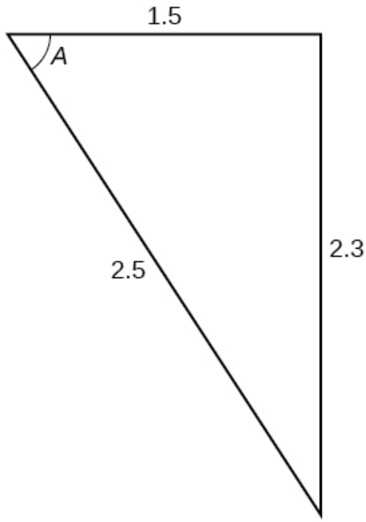
29.1

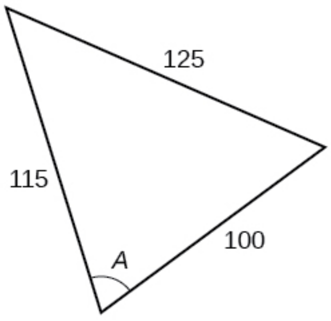




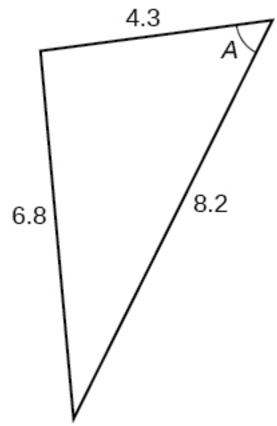
0.5

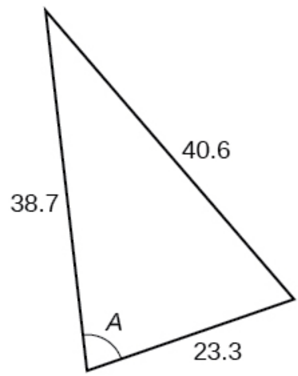
For the following exercises, find the measurement of angle A.





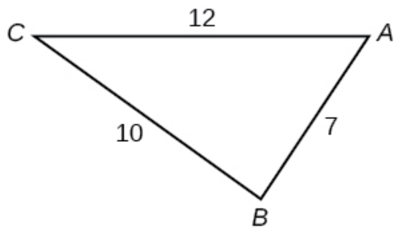
$70.7°$



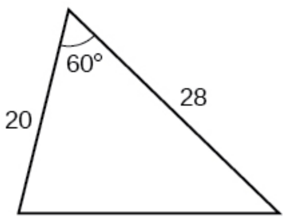


77.4°

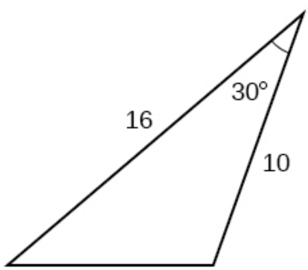
Find the measure of each angle in the triangle shown in [link]. Round to the nearest tenth.

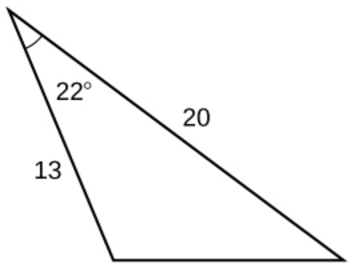


For the following exercises, solve for the unknown side. Round to the nearest tenth.

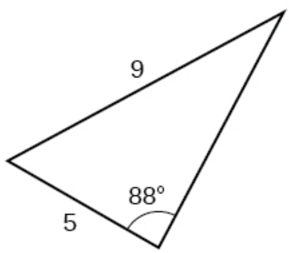


25.0

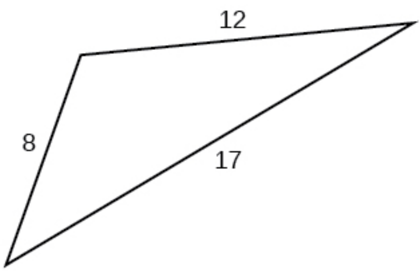




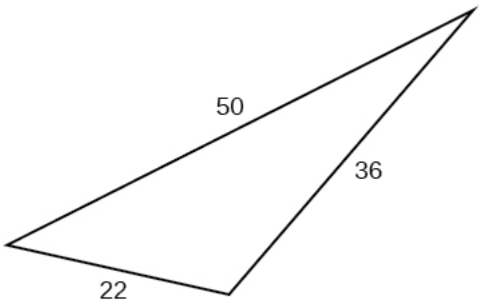
9.3

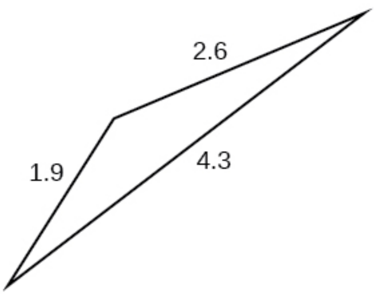


For the following exercises, find the area of the triangle. Round to the nearest hundredth.

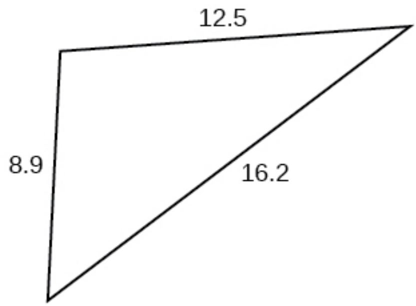


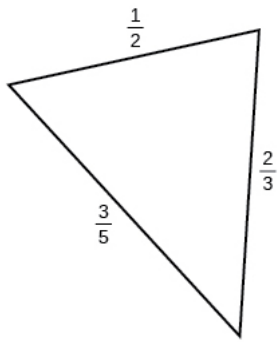
43.52





1.41





0.14

Extensions

A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

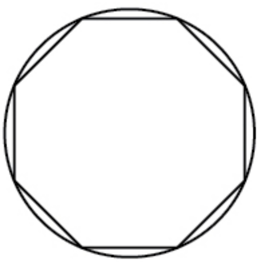
The sides of a parallelogram are 11 feet and 17

feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

18.3

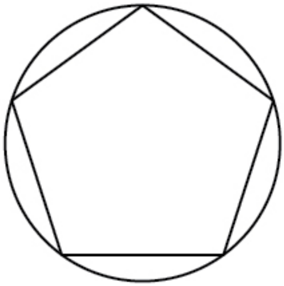
The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

A regular octagon is inscribed in a circle with a radius of 8 inches. (See [link].) Find the perimeter of the octagon.



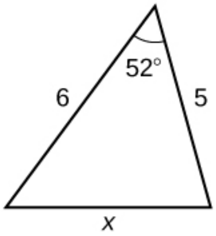
48.98

A regular pentagon is inscribed in a circle of radius 12 cm. (See [link].) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.



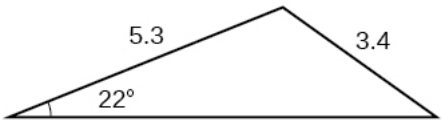
For the following exercises, suppose that $x^2 = 25 + 36 - 60\cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.

Draw the triangle.

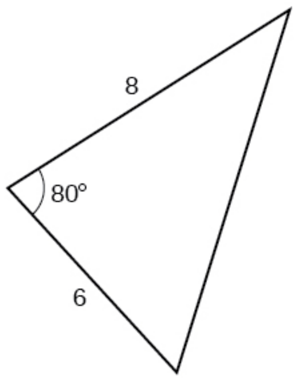


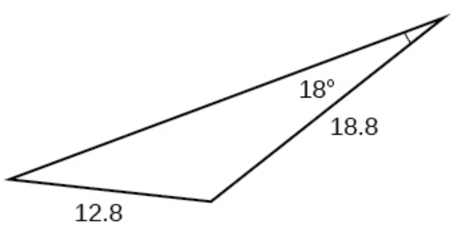
Find the length of the third side.

For the following exercises, find the area of the triangle.



7.62

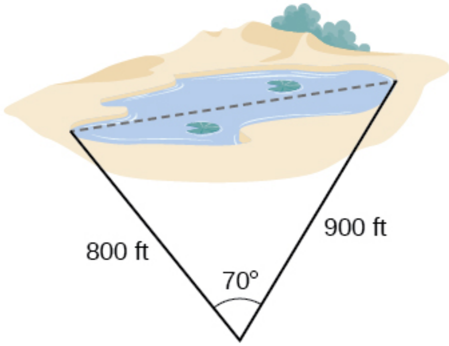




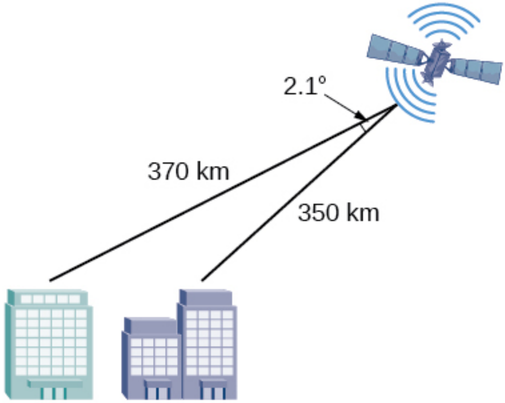
85.1

Real-World Applications

A surveyor has taken the measurements shown in [link]. Find the distance across the lake. Round answers to the nearest tenth.



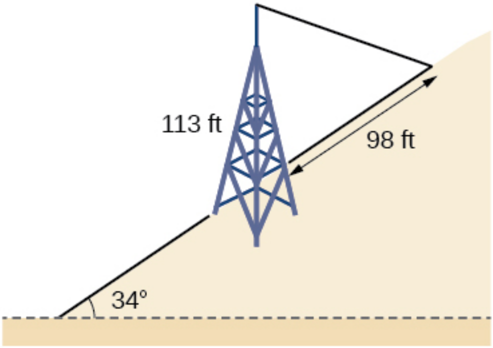
A satellite calculates the distances and angle shown in [link] (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.



24.0 km

An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in [link]. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

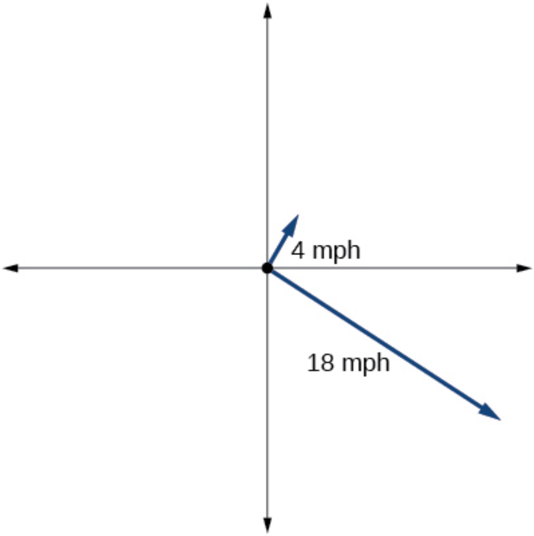


99.9 ft

Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a

heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

The graph in [link] represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 123° and the second boat is traveling at 4 miles per hour at a heading of 30°. Find the distance between the two boats after 2 hours.



37.3 miles

A triangular swimming pool measures 40 feet

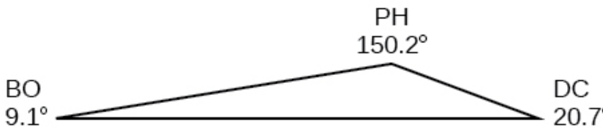
on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

2371 miles

Los Angeles is 1,744 miles from Chicago, Chicago is 714 miles from New York, and New York is 2,451 miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.



Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

599.8 miles

A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

65.4 cm^2

The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

---

468 ft^2

Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

Glossary

Law of Cosines
: states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle

Generalized Pythagorean Theorem

an extension of the Law of Cosines; relates the sides of an oblique triangle and is used for SAS and SSS triangles

Polar Form of Complex Numbers

In this section, you will:

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

“God made the integers; all else is the work of man.” This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers:

translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre’s Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

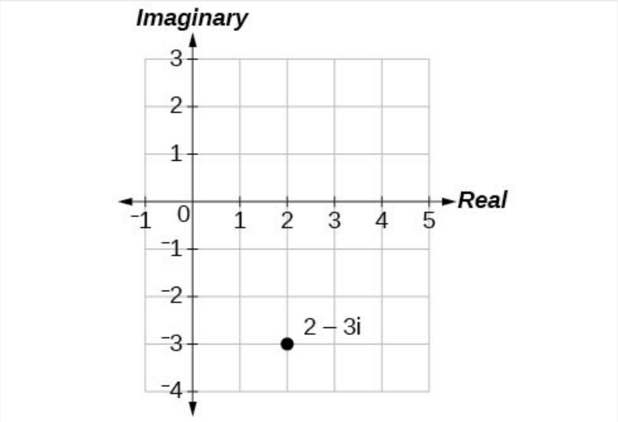
Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

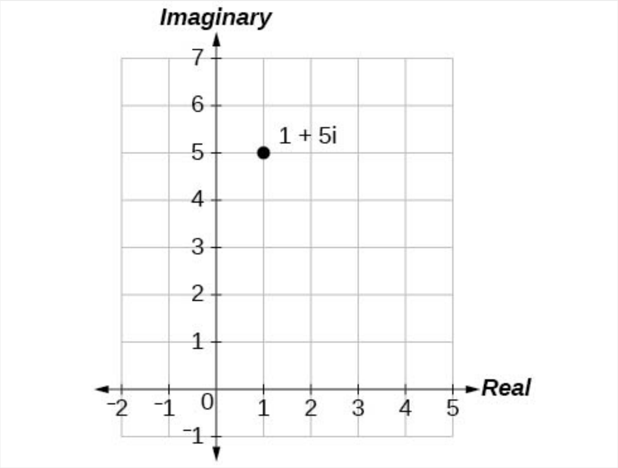
Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See [link].

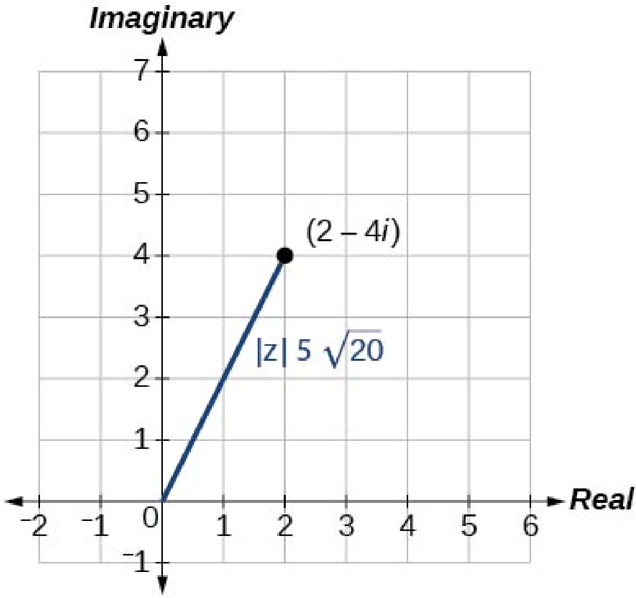


Plot the point $1+5i$ in the complex plane.



Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in [link], shows $|z|$.



Absolute Value of a Complex Number

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$$|z| = \sqrt{x^2 + y^2}$$

It is the distance from the origin to the point (x,y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, (0,0).

Finding the Absolute Value of a Complex

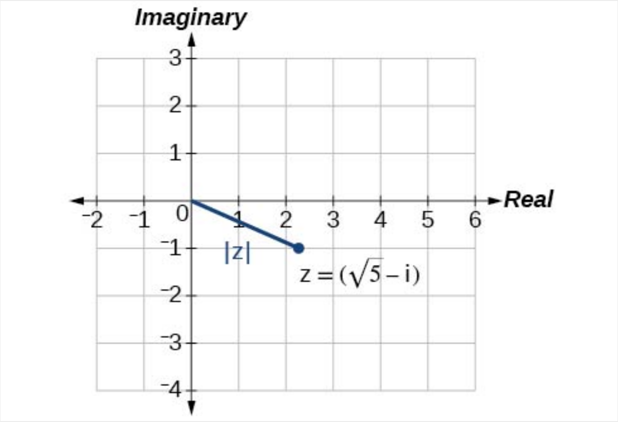
Number with a Radical

Find the absolute value of $z = 5 - i$.

Using the formula, we have

$| z | = x 2 + y 2 \; | z | = (5) 2 + (-1) 2 \; | z | = 5+1 \; | z | = 6$

See [link].



Find the absolute value of the complex number $z = 12 - 5i$.

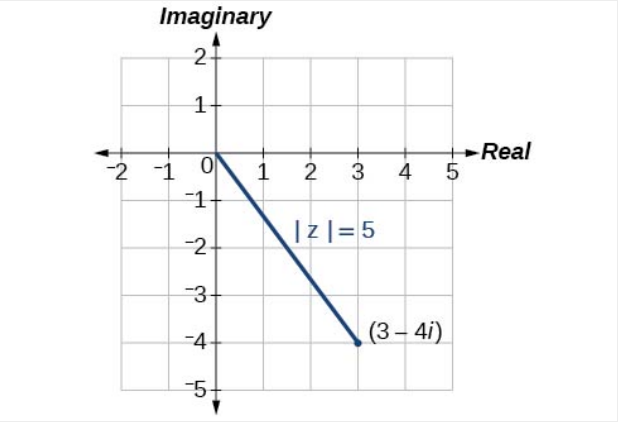
13

Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $| z |$.

Using the formula, we have
$| z | = x\ 2 + y\ 2\ | z | = (3)\ 2 + (-4)\ 2\ | z | = 9 + 16\ | z | = 25\ | z | = 5$

The absolute value z is 5. See [link].



Given $z = 1 - 7i$, find $| z |$.

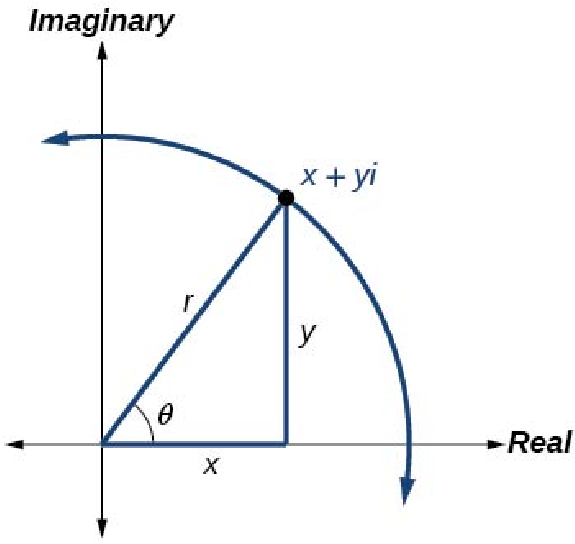
$| z | = 50 = 5\ 2$

Writing Complex Numbers in Polar Form

The **polar form of a complex number** expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z=x+yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$x=r\cos\theta \quad y=r\sin\theta \quad r=\sqrt{x^2+y^2}$$

We review these relationships in [link].



We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x,y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar

coordinates). Substituting, we have

$$z = x + yi \quad z = r\cos\theta + (r\sin\theta)i \quad z = r(\cos\theta + i\sin\theta)$$

Polar Form of a Complex Number

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2}$$

Making a direct substitution, we have

$$z = x + yi \quad z = (r\cos\theta) + i(r\sin\theta) \quad z = r(\cos\theta + i\sin\theta)$$

where r is the **modulus** and θ is the **argument**. We often use the abbreviation $r\text{cis}\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Expressing a Complex Number Using Polar Coordinates

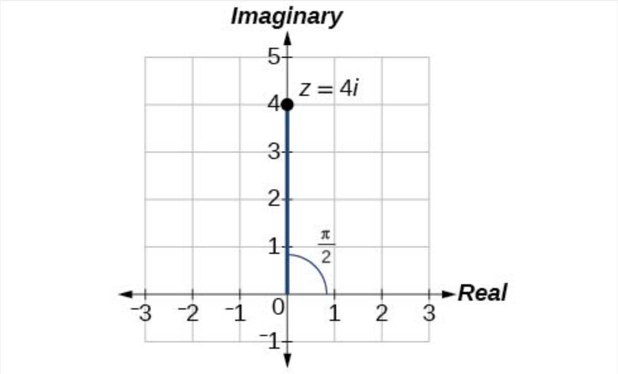
Express the complex number $4i$ using polar coordinates.

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2 + y^2} \quad r = \sqrt{0^2 + 4^2} \quad r = \sqrt{16} \quad r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex

number $z=0+4i$ can be written as $z=4\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\text{cis}\left(\frac{\pi}{2}\right)$. See [link].



Express $z=3i$ as $r\text{cis}\theta$ in polar form.

$z=3\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$

Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

First, find the value of r.

$r= x 2 + y 2 r= (-4) 2 +(4 2) r= 32$
$r=4 2$

Find the angle θ using the formula:
$\cos\theta= x r \cos\theta= -4 4 2 \cos\theta= - 1 2 \theta= \cos -1 (- 1 2)= 3\pi 4$

Thus, the solution is $4 2 \text{cis}(3\pi 4)$.

Write $z= 3 +i$ in polar form.

$z=2(\cos(\pi 6)+i\sin(\pi 6))$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z=r(\cos\theta+i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then,

multiply through by r.

Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:
$z=12(\cos(\pi\ 6)+i\sin(\pi\ 6))$

We begin by evaluating the trigonometric expressions.
$\cos(\pi\ 6)=\ 3\ 2$ and $\sin(\pi\ 6)=\ 1\ 2$

After substitution, the complex number is
$z=12(\ 3\ 2\ +\ 1\ 2\ i\ )$

We apply the distributive property:
$z=12(\ 3\ 2\ +\ 1\ 2\ i\ )\quad =(\ 12\ )\ 3\ 2\ +(\ 12\ )\ 1\ 2\ i\quad =6\ 3\ +6i$

The rectangular form of the given point in complex form is $6\ 3\ +6i$.

Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r=13$ and $\tan\theta=\frac{5}{12}$.

If $\tan\theta=\frac{5}{12}$, and $\tan\theta=\frac{y}{x}$, we first determine $r=\sqrt{x^2+y^2}=\sqrt{12^2+5^2}=13$. We then find $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.

$$z=13(\cos\theta+i\sin\theta)=13\left(\frac{12}{13}+\frac{5}{13}i\right)$$
$$=12+5i$$

The rectangular form of the given number in complex form is $12+5i$.

Convert the complex number to rectangular form:

$$z=4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

$z=2\sqrt{3}-2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham De Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

Products of Complex Numbers in Polar Form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80°) + i\sin(80°))$ and $z_2 = 2(\cos(145°) + i\sin(145°))$.

Follow the formula

$$z_1 z_2 = 4 \cdot 2[\cos(80^\circ + 145^\circ) + i\sin(80^\circ + 145^\circ)]$$
$$z_1 z_2 = 8[\cos(225^\circ) + i\sin(225^\circ)]$$
$$z_1 z_2 = 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$z_1 z_2 = 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$z_1 z_2 = -4\sqrt{2} - 4i\sqrt{2}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

Quotients of Complex Numbers in Polar Form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2),\ z_2 \neq 0$$

Notice that the moduli are divided, and the angles are subtracted.

Given two complex numbers in polar form, find the quotient.

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213°) + i\sin(213°))$ and $z_2 = 4(\cos(33°) + i\sin(33°))$.

Using the formula, we have

$$\frac{z_1}{z_2} = \frac{2}{4}[\cos(213° - 33°) + i\sin(213° - 33°)]$$
$$\frac{z_1}{z_2} = \frac{1}{2}[\cos(180°) + i\sin(180°)]$$
$$\frac{z_1}{z_2} = \frac{1}{2}[-1 + 0i]$$
$$\frac{z_1}{z_2} = -\frac{1}{2} + 0i$$
$$\frac{z_1}{z_2} = -\frac{1}{2}$$

Find the product and the quotient of $z_1 = 2\ 3$ $(\cos(150°) + i\sin(150°))$ and $z_2 = 2(\cos(30°) + i\sin(30°))$.

$z_1 z_2 = -4\ 3$; $z_1 z_2 = -\ 3\ 2 + 3\ 2\ i$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \quad z^n = r^n \operatorname{cis}(n\theta)$$

where n is a positive integer.

Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i) 5$ using De Moivre's Theorem.

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$r= x 2 + y 2 \; r= (1) 2 + (1) 2 \; r= 2$

Then we find θ. Using the formula $\tan\theta= y x$ gives

$\tan\theta= 1 1 \; \tan\theta=1 \; \theta= \pi 4$

Use De Moivre's Theorem to evaluate the expression.

$(a+bi) n = r n [\cos(n\theta)+i\sin(n\theta)] \; (1+i) 5 = (2) 5 [\cos(5\cdot \pi 4)+i\sin(5\cdot \pi 4)] \; (1+i) 5 =4 2 [\cos(5\pi 4)+i\sin(5\pi 4)] \; (1+i) 5 =4 2 [- 2 2 +i(- 2 2)] \; (1+i) 5 =-4-4i$

Finding Roots of Complex Numbers in Polar Form

To find the *n*th root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding nth roots of complex numbers in polar form.

The *n*th Root Theorem

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Finding the *n*th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When

$k=0$, we have
$z^{\frac{1}{3}}=2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$

When $k=1$, we have
$z^{\frac{1}{3}}=2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right]$ Add $\frac{2(1)\pi}{3}$ to each angle. $z^{\frac{1}{3}}=2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)$

When $k=2$, we have
$z^{\frac{1}{3}}=2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right]$ Add $\frac{2(2)\pi}{3}$ to each angle. $z^{\frac{1}{3}}=2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$

Remember to find the common denominator to simplify fractions in situations like this one. For $k=1$, the angle simplification is
$\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3}=\frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)=\frac{2\pi}{9}+\frac{6\pi}{9}=\frac{8\pi}{9}$

Find the four fourth roots of $16(\cos(120°)+i\sin(120°))$.

$z_0=2(\cos(30°)+i\sin(30°))$

$z_1=2(\cos(120°)+i\sin(120°))$

$z_2=2(\cos(210°)+i\sin(210°))$

$z_3 = 2(\cos(300°) + i\sin(300°))$

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the *x*-axis as the *real* axis and the *y*-axis as the *imaginary* axis. See [link].
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See [link] and [link].
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta, y = r\sin\theta$, and $r = \sqrt{x^2

$+ y^2$. Then, $z = r(\cos\theta + i\sin\theta)$. See [link] and [link].

- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See [link] and [link].
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See [link].
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See [link].
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See [link].
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See [link].

Section Exercises

Verbal

A complex number is $a + bi$. Explain each part.

a is the real part, *b* is the imaginary part, and $i = \sqrt{-1}$

What does the absolute value of a complex number represent?

How is a complex number converted to polar form?

Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta$.

How do we find the product of two complex numbers?

What is De Moivre's Theorem and what is it used for?

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ It is used to simplify polar form when a number has been raised to a power.

Algebraic

For the following exercises, find the absolute value of the given complex number.

$5+3i$

$-7+i$

$5\ 2$

$-3-3i$

$2\ -6i$

38

$2i$

$2.2-3.1i$

14.45

For the following exercises, write the complex number in polar form.

$2+2i$

$8-4i$

$4\ 5\ \text{cis}(\ 333.4^\circ\ )$

$-\ 1\ 2\ -\ 1\ 2\ i$

$3\ +i$

$2\text{cis}(\ \pi\ 6\ )$

$3i$

For the following exercises, convert the complex number from polar to rectangular form.

$z=7\text{cis}(\ \pi\ 6\ )$

$7\ 3\ 2\ +i\ 7\ 2$

$z=2\text{cis}(\ \pi\ 3\ )$

$z=4\text{cis}(\ 7\pi\ 6\ )$

$-2\ 3\ -2i$

$z=7\text{cis}(\ 25^\circ\ )$

$z=3\text{cis}(\ 240^\circ\ )$

$-1.5-i\ 3\ 3\ 2$

$z=\ 2\ \text{cis}(\ 100^\circ\ )$

For the following exercises, find $z\ 1\ z\ 2$ in polar form.

$z\ 1\ =2\ 3\ \text{cis}(\ 116^\circ\ );\ z\ 2\ =2\text{cis}(\ 82^\circ\ )$

$4\ 3\ \text{cis}(\ 198^\circ\ )$

$z\ 1\ =\ 2\ \text{cis}(\ 205^\circ\ );\ z\ 2\ =2\ 2\ \text{cis}(\ 118^\circ\ )$

$z\ 1\ =3\text{cis}(\ 120^\circ\ );\ z\ 2\ =\ 1\ 4\ \text{cis}(\ 60^\circ\ )$

$3\ 4\ \text{cis}(\ 180^\circ\ )$

$z\ 1\ = 3\text{cis}(\ \pi\ 4\ );\ z\ 2\ = 5\text{cis}(\ \pi\ 6\ )$

$z\ 1\ =\ 5\ \text{cis}(\ 5\pi\ 8\ );\ z\ 2\ =\ 15\ \text{cis}(\ \pi\ 12\ )$

$5\ 3\ \text{cis}(\ 17\pi\ 24\ )$

$z\ 1\ = 4\text{cis}(\ \pi\ 2\ );\ z\ 2\ = 2\text{cis}(\ \pi\ 4\ )$

For the following exercises, find z 1 z 2 in polar form.

$z\ 1\ = 21\text{cis}(\ 135^\circ\ );\ z\ 2\ = 3\text{cis}(\ 65^\circ\ )$

$7\text{cis}(\ 70^\circ\ )$

$z\ 1\ =\ 2\ \text{cis}(\ 90^\circ\ );\ z\ 2\ = 2\text{cis}(\ 60^\circ\ )$

$z\ 1\ = 15\text{cis}(\ 120^\circ\ );\ z\ 2\ = 3\text{cis}(\ 40^\circ\ )$

$5\text{cis}(\ 80^\circ\ )$

$z_1 = 6\text{cis}\left(\frac{\pi}{3}\right);\ z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

$z_1 = 5\sqrt{2}\,\text{cis}(\pi);\ z_2 = \sqrt{2}\,\text{cis}\left(\frac{2\pi}{3}\right)$

$5\text{cis}\left(\frac{\pi}{3}\right)$

$z_1 = 2\text{cis}\left(\frac{3\pi}{5}\right);\ z_2 = 3\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find the powers of each complex number in polar form.

Find z^3 when $z = 5\text{cis}(45°)$.

$125\text{cis}(135°)$

Find z^4 when $z = 2\text{cis}(70°)$.

Find z^2 when $z = 3\text{cis}(120°)$.

$9\text{cis}(240°)$

Find z^2 when $z = 4\text{cis}\left(\frac{\pi}{4}\right)$.

Find z 4 when z=cis(3π 16).

---

cis(3π 4)

Find z 3 when z=3cis(5π 3).

For the following exercises, evaluate each root.

Evaluate the cube root of z when z=27cis(240°).

3cis(80°),3cis(200°),3cis(320°)

Evaluate the square root of z when z=16cis(100°).

Evaluate the cube root of z when z=32cis(2π 3).

---

2 4 3 cis(2π 9),2 4 3 cis(8π 9),2 4 3 cis(14π 9)

Evaluate the square root of z when z=32cis(π).

Evaluate the cube root of z when $z = 8\text{cis}\left(\frac{7\pi}{4}\right)$.

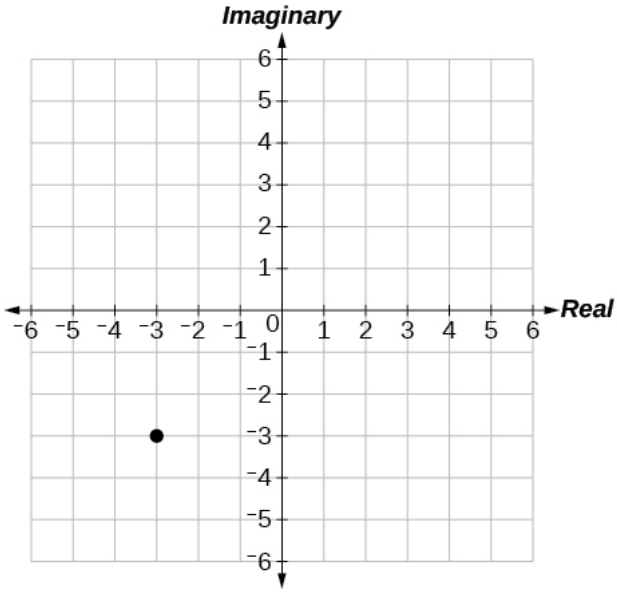
$2\ 2\ \text{cis}\left(\frac{7\pi}{8}\right), 2\ 2\ \text{cis}\left(\frac{15\pi}{8}\right)$

Graphical

For the following exercises, plot the complex number in the complex plane.

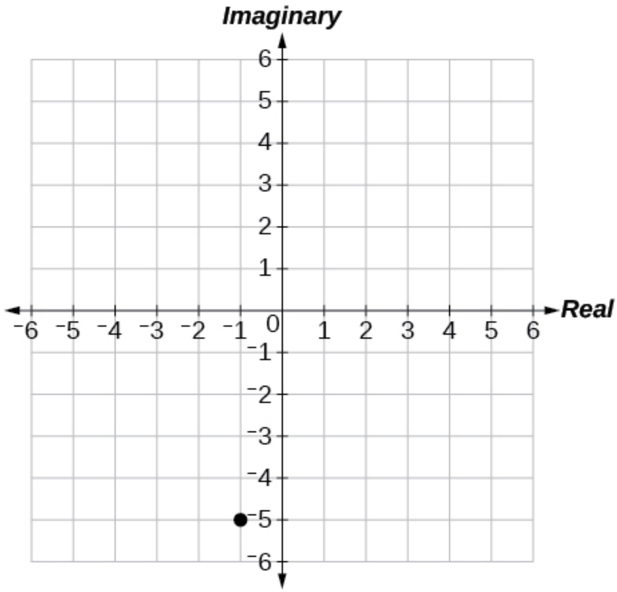
$2 + 4i$

$-3 - 3i$



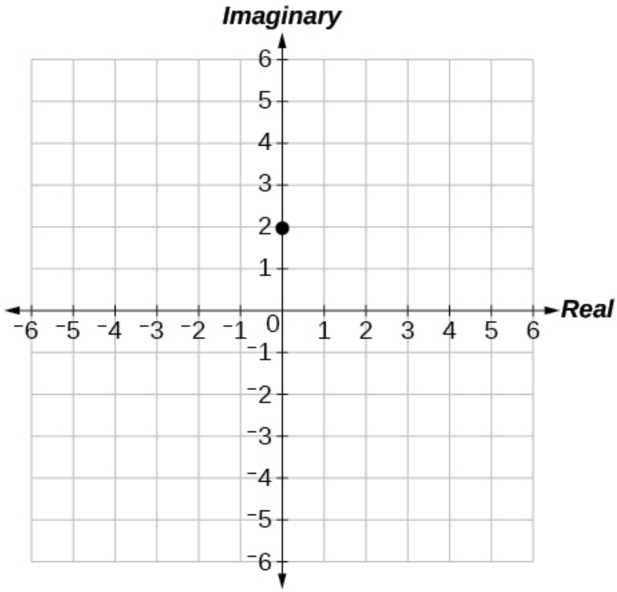
$5-4i$

$-1-5i$



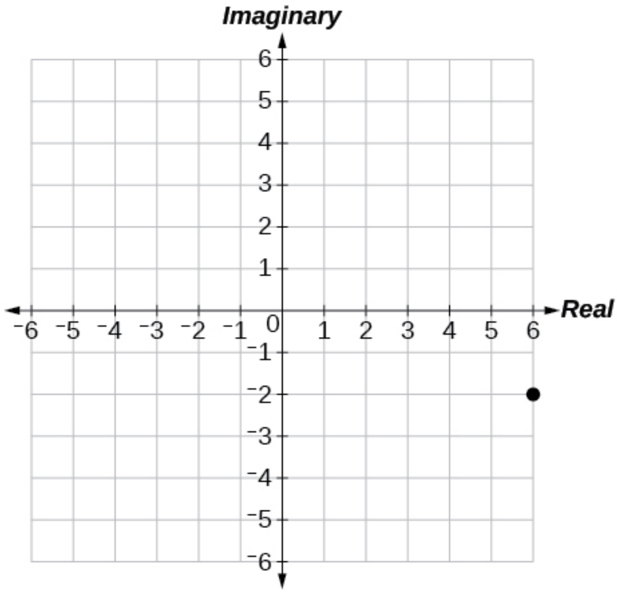
$3 + 2i$

$2i$



-4

$6-2i$



$-2+i$

$1-4i$

[missing_resource: CNX_Precalc_Figure_08_05_210.jpg]

Technology

For the following exercises, find all answers rounded to the nearest hundredth.

Use the rectangular to polar feature on the graphing calculator to change $5+5i$ to polar

form.

Use the rectangular to polar feature on the graphing calculator to change $3-2i$ to polar form.

$3.61\ e\ -0.59i$

Use the rectangular to polar feature on the graphing calculator to change $-3-8i$ to polar form.

Use the polar to rectangular feature on the graphing calculator to change $4\text{cis}(120^\circ)$ to rectangular form.

$-2+3.46i$

Use the polar to rectangular feature on the graphing calculator to change $2\text{cis}(45^\circ)$ to rectangular form.

Use the polar to rectangular feature on the graphing calculator to change $5\text{cis}(210^\circ)$ to rectangular form.

$-4.33-2.50i$

Glossary

argument
: the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

De Moivre's Theorem
: formula used to find the nth power or *n*th roots of a complex number; states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the angles by n

modulus
: the absolute value of a complex number, or the distance from the origin to the point (x,y); also called the amplitude

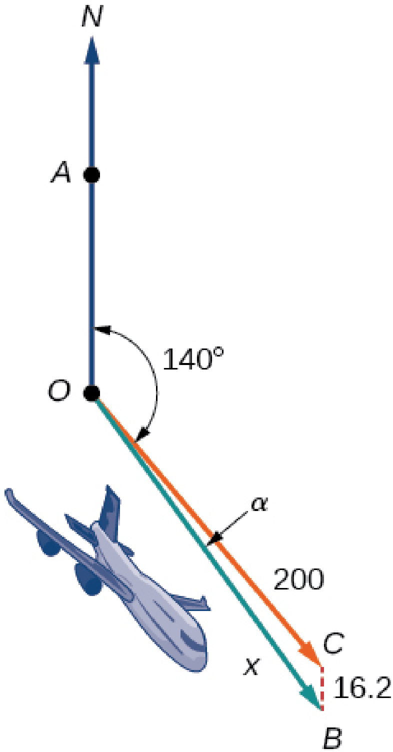
polar form of a complex number
: a complex number expressed in terms of an angle θ and its distance from the origin r; can be found by using conversion formulas $x=r\cos\theta, y=r\sin\theta$, and $r=\sqrt{x^2+y^2}$

Vectors

In this section you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of $\mathbf{v}$.
- Perform operations with vectors in terms of $\mathbf{i}$ and $\mathbf{j}$.
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in [link]. What are the ground speed and actual bearing of the plane?



Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

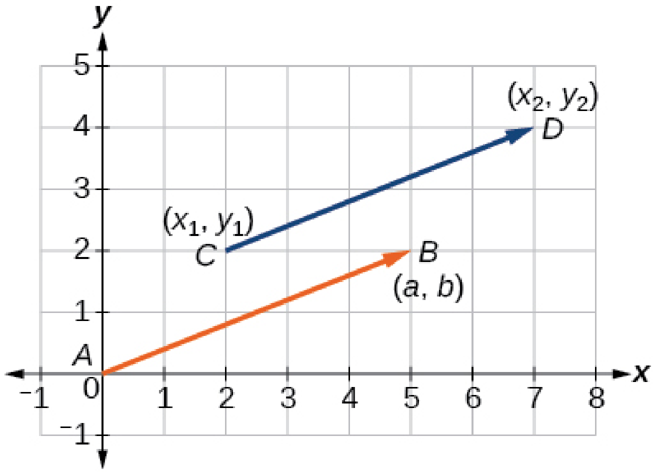
A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as **v, u, w,** $\vec{\mathbf{v}}$, $\vec{\mathbf{u}}$, $\vec{\mathbf{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0,0)$ and terminal point (a,b), a vector may be represented as $\langle a,b\rangle$.

This last symbol $\langle a,b\rangle$ has special significance. It is called the **standard position**. The position vector has an initial point $(0,0)$ and a terminal point (a,b). To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is

found by calculating
$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle a, b \rangle$

In [link], we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.



Properties of Vectors
A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0,0)$ and is identified by its terminal point (a,b).

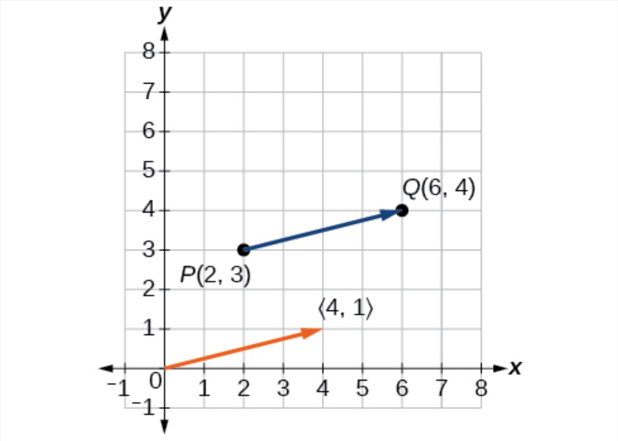
Find the Position Vector

Consider the vector whose initial point is P(2,3) and terminal point is Q(6,4). Find the position vector.

The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$\mathrm{v} = \langle 6-2, 4-3 \rangle = \langle 4,1 \rangle$$

The position vector begins at (0,0) and terminates at (4,1). The graphs of both vectors are shown in [link].



We see that the position vector is $\langle 4,1 \rangle$.

Drawing a Vector with the Given Criteria

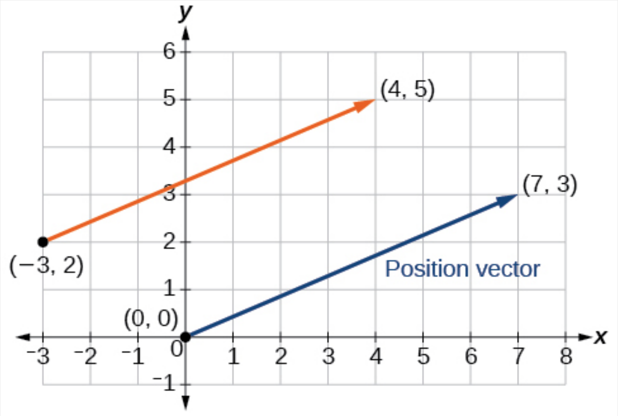
and Its Equivalent Position Vector

Find the position vector given that vector **v** has an initial point at $(-3,2)$ and a terminal point at $(4,5)$, then graph both vectors in the same plane.

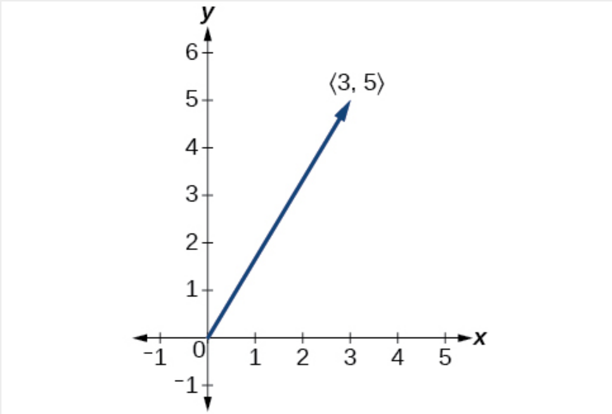
The position vector is found using the following calculation:

$$v = \langle 4-(-3), 5-2 \rangle \quad = \langle 7,3 \rangle$$

Thus, the position vector begins at $(0,0)$ and terminates at $(7,3)$. See [link].



Draw a vector **v** that connects from the origin to the point (3,5).



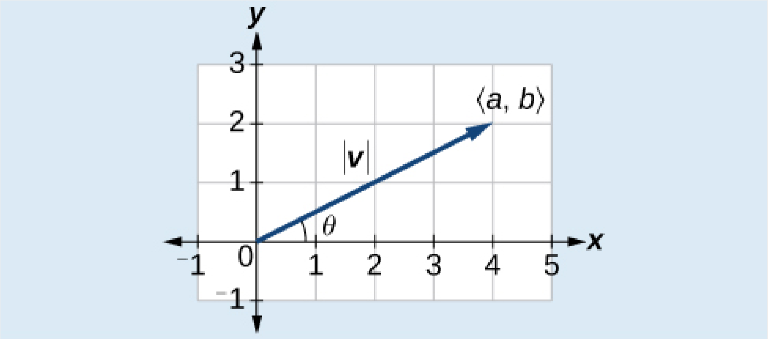
Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

Magnitude and Direction of a Vector

Given a position vector $\mathbf{v} = \langle a, b \rangle$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$. The direction is

equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in [link].



Two vectors ***v*** and ***u*** are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8,1)$ and terminal point $Q(-2,-5)$. Draw the vector.

First, find the position vector.

$$u = \langle -2,-(-8), -5-1 \rangle \quad = \langle 6,-6 \rangle$$

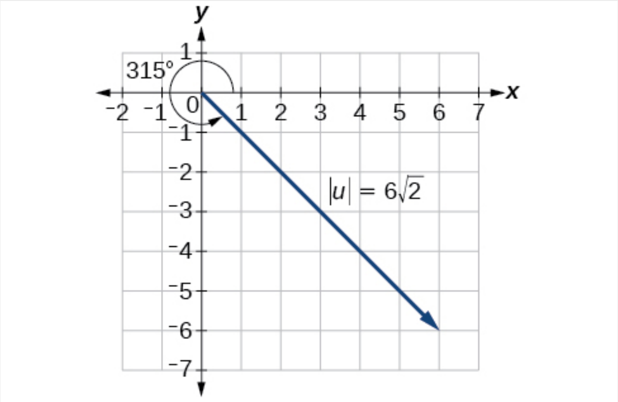
We use the Pythagorean Theorem to find the magnitude.

$|u| = \sqrt{(6)^2 + (-6)^2} = \sqrt{72} = 6\sqrt{2}$

The direction is given as

$\tan\theta = \frac{-6}{6} = -1 \Rightarrow \theta = \tan^{-1}(-1) = -45°$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45° + 360° = 315°$. See [link].



Showing That Two Vectors Are Equal

Show that vector ***v*** with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector ***u*** with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as ***v*** and ***u***. Next, find

the magnitude and direction of each vector.

As shown in [link], draw the vector v starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector u with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for ***v*** and ***u***. We have

$$v = \langle -1-5, 2-(-3)\rangle \quad = \langle -6, 5\rangle$$
$$u = \langle -7-(-1), 2-(-3)\rangle \quad = \langle -6, 5\rangle$$

Since the position vectors are the same, ***v*** and ***u*** are the same.

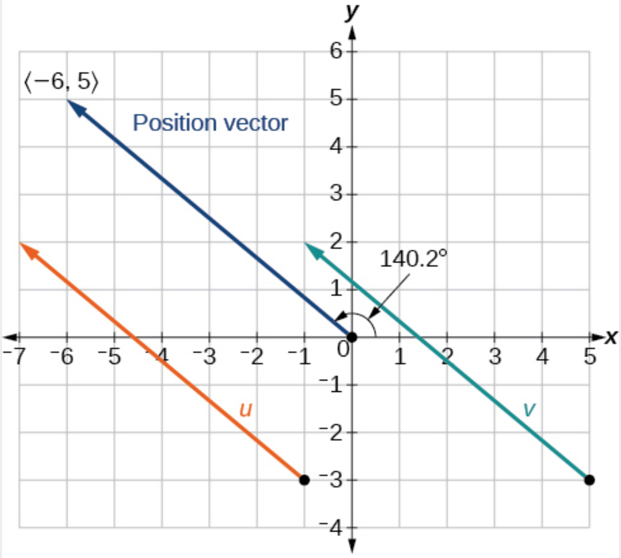
An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$|v| = (-1-5)\ 2 + (2-(-3))\ 2 = (-6)\ 2 + (5)\ 2 = 36+25 = 61$ $|u| = (-7-(-1))\ 2 + (2-(-3))\ 2 = (-6)\ 2 + (5)\ 2 = 36+25 = 61$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$\tan\theta = -\ 5\ 6 \Rightarrow \theta = \tan\ -1\ (-\ 5\ 6\ ) = -39.8^\circ$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^\circ + 180^\circ = 140.2^\circ$.

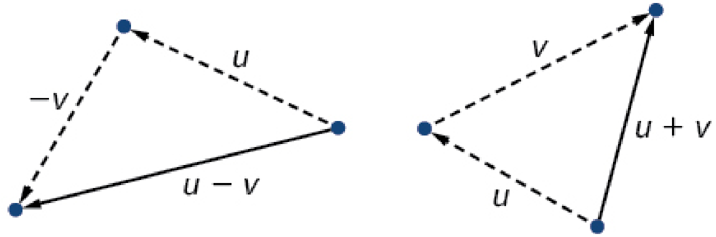


Performing Vector Addition and Scalar Multiplication

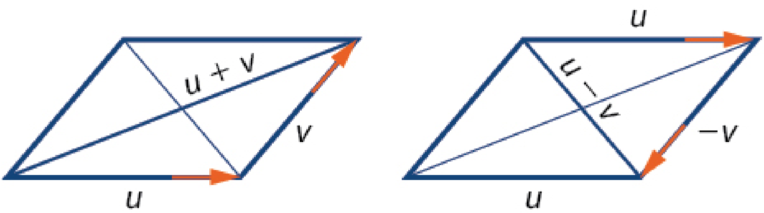
Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $\mathbf{u} = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in

the plane. The sum of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, or **vector addition**, produces a third vector $\boldsymbol{u}+\ \boldsymbol{v}$, the **resultant** vector.

To find $\boldsymbol{u}\ +\ \boldsymbol{v}$, we first draw the vector $\boldsymbol{u}$, and from the terminal end of $\boldsymbol{u}$, we drawn the vector $\boldsymbol{v}$. In other words, we have the initial point of $\boldsymbol{v}$ meet the terminal end of $\boldsymbol{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\boldsymbol{u}\ +\ \boldsymbol{v}$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of $\boldsymbol{u}$ to the end of $\boldsymbol{v}$ in a straight path, as shown in [link].



Vector subtraction is similar to vector addition. To find $\boldsymbol{u}\ -\ \boldsymbol{v}$, view it as $\boldsymbol{u}\ +\ (-\boldsymbol{v})$. Adding $-\boldsymbol{v}$ is reversing direction of $\boldsymbol{v}$ and adding it to the end of $\boldsymbol{u}$. The new vector begins at the start of $\boldsymbol{u}$ and stops at the end point of $-\boldsymbol{v}$. See [link] for a visual that compares vector addition and vector subtraction using parallelograms.



Adding and Subtracting Vectors

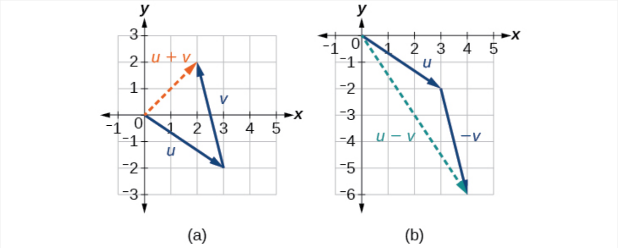
Given $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -1, 4 \rangle$, find two new vectors $\boldsymbol{u} + \boldsymbol{v}$, and $\boldsymbol{u} - \boldsymbol{v}$.

To find the sum of two vectors, we add the components. Thus,
$$\mathrm{u} + \mathrm{v} = \langle 3, -2 \rangle + \langle -1, 4 \rangle = \langle 3 + (-1), -2 + 4 \rangle = \langle 2, 2 \rangle$$

See [link] **(a)**.

To find the difference of two vectors, add the negative components of **v** to **u**. Thus,
$$\mathrm{u} + (-\mathrm{v}) = \langle 3, -2 \rangle + \langle 1, -4 \rangle = \langle 3 + 1, -2 + (-4) \rangle = \langle 4, -6 \rangle$$

See [link] **(b).**
(a) Sum of two vectors (b) Difference of two vectors



Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

Scalar Multiplication

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $\mathbf{v} = \langle a, b \rangle$ by k, we have

$$kv = \langle ka, kb \rangle$$

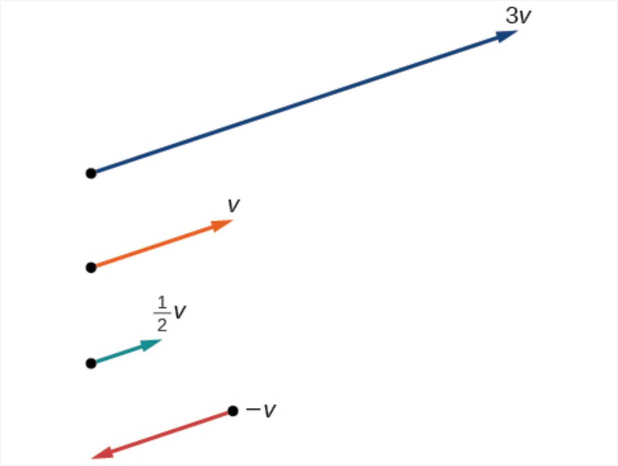
Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Performing Scalar Multiplication

Given vector $\mathbf{v} = \langle 3,1 \rangle$, find $3\boldsymbol{v}$, $\frac{1}{2}\mathbf{v}$, and $-\boldsymbol{v}$.

See [link] for a geometric interpretation. If $\mathbf{v} = \langle 3,1 \rangle$, then

$$3\mathrm{v} = \langle 3\cdot 3, 3\cdot 1 \rangle = \langle 9,3 \rangle \quad \frac{1}{2}\mathrm{v} = \langle \frac{1}{2}\cdot 3, \frac{1}{2}\cdot 1 \rangle = \langle \frac{3}{2}, \frac{1}{2} \rangle \quad -\mathrm{v} = \langle -3, -1 \rangle$$



Analysis

Notice that the vector $3\boldsymbol{v}$ is three times the length of $\boldsymbol{v}$, $\frac{1}{2}\mathbf{v}$ is half the length of $\boldsymbol{v}$, and $-\boldsymbol{v}$ is the same length of $\boldsymbol{v}$, but in the opposite direction.

Find the scalar multiple 3 $\mathbf{u}$ given $\mathbf{u} = \langle 5,4 \rangle$.

$3u = \langle 15,12 \rangle$

Using Vector Addition and Scalar Multiplication to Find a New Vector

Given $\mathbf{u} = \langle 3,-2 \rangle$ and $\mathbf{v} = \langle -1,4 \rangle$, find a new vector $\boldsymbol{w} = 3\boldsymbol{u} + 2\boldsymbol{v}$.

First, we must multiply each vector by the scalar.
$3u = 3\langle 3,-2 \rangle = \langle 9,-6 \rangle$ $2v = 2\langle -1,4 \rangle = \langle -2,8 \rangle$

Then, add the two together.
$w = 3u + 2v = \langle 9,-6 \rangle + \langle -2,8 \rangle = \langle 9-2, -6+8 \rangle = \langle 7,2 \rangle$

So, $\mathbf{w} = \langle 7,2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in [link] that the position vector $\langle 2,3 \rangle$ comes from adding the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. We have $\boldsymbol{v}_1$ with initial point $(0,0)$ and terminal point $(2,0)$.

$$v_1 = \langle 2-0, 0-0 \rangle = \langle 2,0 \rangle$$

We also have $\boldsymbol{v}_2$ with initial point $(0,0)$ and terminal point $(0,3)$.

$$v_2 = \langle 0-0, 3-0 \rangle = \langle 0,3 \rangle$$

Therefore, the position vector is

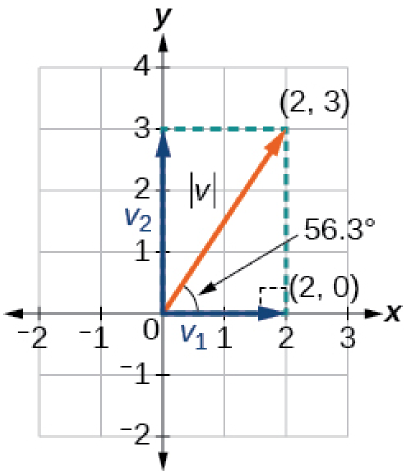
$$v = \langle 2+0, 3+0 \rangle = \langle 2,3 \rangle$$

Using the Pythagorean Theorem, the magnitude of $\boldsymbol{v}_1$ is 2, and the magnitude of $\boldsymbol{v}_2$ is 3. To find the magnitude of $\boldsymbol{v}$, use the formula with the position vector.

$$|v| = |v_1|^2 + |v_2|^2 = 2^2 + 3^2 = 13$$

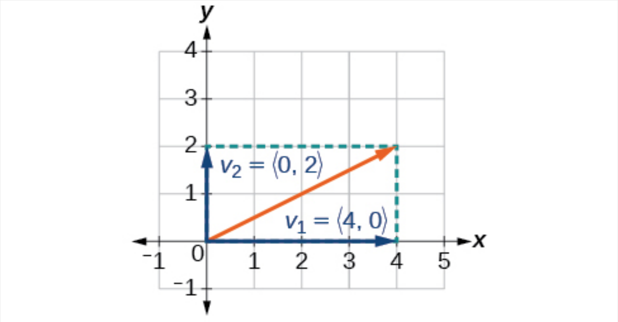
The magnitude of $\boldsymbol{v}$ is 13 . To find the direction, we use the tangent function $\tan\theta = y\ x$.

$$\tan\theta = v_2\ v_1 \quad \tan\theta = 3\ 2 \quad \theta = \tan^{-1}(3\ 2) = 56.3°$$



Thus, the magnitude of **v** is 13 and the direction is 56.3° off the horizontal.

Finding the Components of the Vector
Find the components of the vector **v** with initial point $(3,2)$ and terminal point $(7,4)$.
First find the standard position. $v=\langle 7-3,4-2\rangle=\langle 4,2\rangle$ See the illustration in [link].



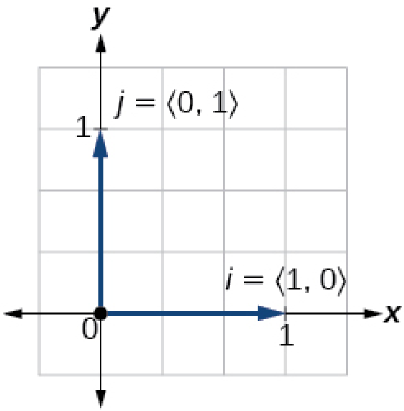
The horizontal component is $\mathbf{v}\ \mathbf{1} = \langle 4,0 \rangle$ and the vertical component is $\mathbf{v}\ \mathbf{2} = \langle 0,2 \rangle$.

Finding the Unit Vector in the Direction of *v*

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $\mathbf{i} = \langle 1,0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $\mathbf{j} = \langle 0,1 \rangle$ and is directed along the positive vertical axis. See

[link].



The Unit Vectors

If **v** is a nonzero vector, then **v** | **v** | is a unit vector in the direction of **v.** Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Finding the Unit Vector in the Direction of *v*

Find a unit vector in the same direction as **v** $=\langle -5,12 \rangle$.

First, we will find the magnitude.

$|v| = (-5) 2 + (12) 2 = 25+144 = 169$

$= 13$

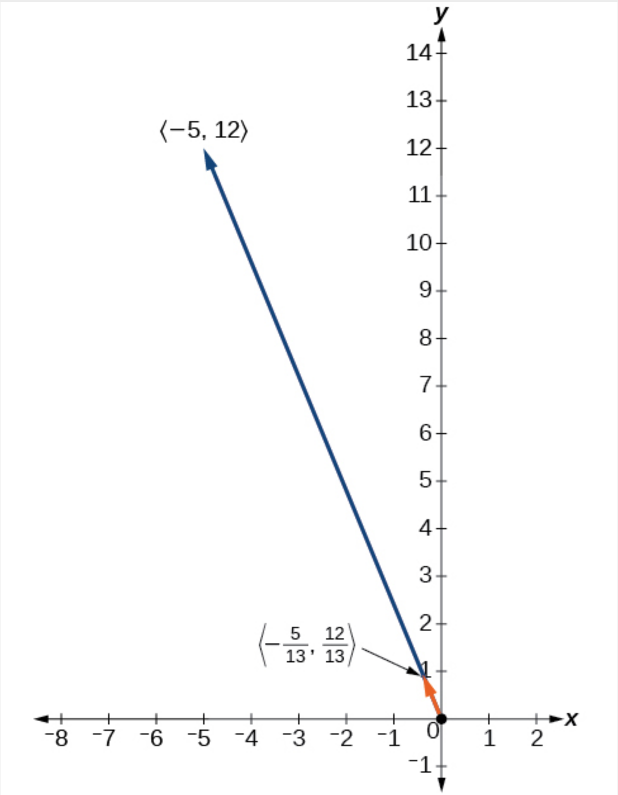
Then we divide each component by $|v|$, which gives a unit vector in the same direction as ***v***:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \langle -\frac{5}{13}, \frac{12}{13} \rangle$$

See [link].



Verify that the magnitude of the unit vector

equals 1. The magnitude of $-\frac{5}{13}\mathrm{i}+\frac{12}{13}\mathrm{j}$ is given as

$$\sqrt{\left(-\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2}=\sqrt{\frac{25}{169}+\frac{144}{169}}=\sqrt{\frac{169}{169}}=1$$

The vector $\boldsymbol{u}=\frac{5}{13}\boldsymbol{i}+\frac{12}{13}\boldsymbol{j}$ is the unit vector in the same direction as $\boldsymbol{v}=\langle -5,12\rangle$.

Performing Operations with Vectors in Terms of *i* and *j*

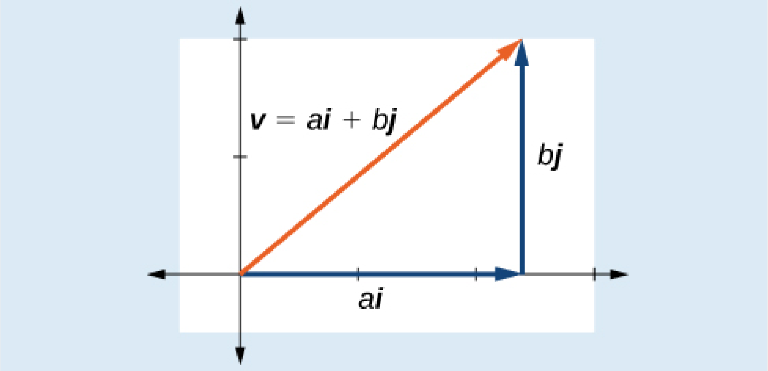
So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of ***i*** and ***j***.

Vectors in the Rectangular Plane

Given a vector **v** with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, $\boldsymbol{v}$ is written as

$$v=(x_2-x_1)i+(y_2-y_1)j$$

The position vector from $(0,0)$ to (a,b), where $(x_2 - x_1) = a$ and $(y_2 - y_1) = b$, is written as $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$. This vector sum is called a linear combination of the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.
The magnitude of $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is given as $|v| = \sqrt{a^2 + b^2}$. See [link].



Writing a Vector in Terms of *i* and *j*

Given a vector **v** with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of **i** and **j**.

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$v = (x_2 - x_1)i + (y_2 - y_1)j = (-6 - 2)i + (6 - (-6))j = -8i + 12j$$

Writing a Vector in Terms of *i* and *j* Using Initial and Terminal Points

Given initial point $P_1 = (-1,3)$ and terminal point $P_2 = (2,7)$, write the vector **v** in terms of **i** and **j.**

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$v = (x_2 - x_1)i + (y_2 - y_1)j$$

$$v = (2-(-1))i + (7-3)j = 3i + 4j$$

Write the vector **u** with initial point $P = (-1,6)$ and terminal point $Q = (7,-5)$ in terms of **i** and **j.**

$u = 8i - 11j$

Performing Operations on Vectors in

Terms of *i* and *j*

When vectors are written in terms of **i** and **j**, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

Adding and Subtracting Vectors in Rectangular Coordinates

Given $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ and $\boldsymbol{u} = c\boldsymbol{i} + d\boldsymbol{j}$, then

$$\mathrm{v}+\mathrm{u}=(a+c)\mathrm{i}+(b+d)\mathrm{j} \quad \mathrm{v}-\mathrm{u}=(a-c)\mathrm{i}+(b-d)\mathrm{j}$$

Finding the Sum of the Vectors

Find the sum of $\mathrm{v}_1 = 2\mathrm{i}-3\mathrm{j}$ and $\mathrm{v}_2 = 4\mathrm{i}+5\mathrm{j}$.

According to the formula, we have

$$\mathrm{v}_1 + \mathrm{v}_2 = (2+4)\mathrm{i}+(-3+5)\mathrm{j} = 6\mathrm{i}+2\mathrm{j}$$

Calculating the Component Form of a

Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using iandj. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with **| v |** replacing **r.**

Vector Components in Terms of Magnitude and Direction

Given a position vector $v = \langle x, y \rangle$ and a direction angle θ,

$\cos\theta = \frac{x}{|v|}$ and $\sin\theta = \frac{y}{|v|}$ $x = |v|\cos\theta$ $y = |v|\sin\theta$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Writing a Vector in Terms of Magnitude

and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Using the conversion formulas $x = |\ v\ | \cos\theta i$ and $y = |\ v\ | \sin\theta j$, we find that
$x = 7\cos(135^\circ)i\ = -\ 7\ 2\ 2$ $y = 7\sin(135^\circ)j\ =\ 7$
$2\ 2$

This vector can be written as $v = 7\cos(135^\circ)i$ $+ 7\sin(135^\circ)j$ or simplified as
$v = -\ 7\ 2\ 2\ i +\ 7\ 2\ 2\ j$

A vector travels from the origin to the point (3,5). Write the vector in terms of magnitude and direction.

$v =\ 34\ \cos(59^\circ)i +\ 34\ \sin(59^\circ)j$

Magnitude $=\ 34$

$\theta =\ \tan\ -1\ (\ 5\ 3\ ) = 59.04^\circ$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

Dot Product

The **dot product** of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.

$$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$$

Finding the Dot Product of Two Vectors

Find the dot product of $\mathbf{v} = \langle 5,12 \rangle$ and $\mathbf{u} = \langle -3,4 \rangle$.

Using the formula, we have
$v \cdot u = \langle 5,12 \rangle \cdot \langle -3,4 \rangle = 5 \cdot (-3) + 12 \cdot 4 = -15 + 48 = 33$

Finding the Dot Product of Two Vectors and the Angle between Them

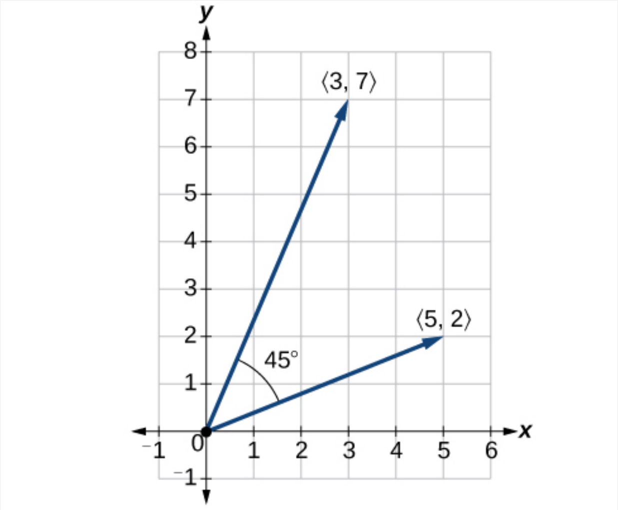
Find the dot product of $\boldsymbol{v}_1 = 5\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v}_2 = 3\boldsymbol{i} + 7\boldsymbol{j}$. Then, find the angle between the two vectors.

Finding the dot product, we multiply corresponding components.
v 1 · v 2 $= \langle 5,2 \rangle \cdot \langle 3,7 \rangle = 5 \cdot 3 + 2 \cdot 7 = 15 + 14 = 29$

To find the angle between them, we use the formula cosθ= v |v| · u |u| .
v |v| · u |u| =⟨ 5 29 + 2 29 ⟩·⟨ 3 58 + 7 58 ⟩ = 5 29 · 3 58 + 2 29 · 7 58 = 15 1682 + 14 1682 = 29 1682 =0.707107 cos −1

$(0.707107) = 45^\circ$

See [link].

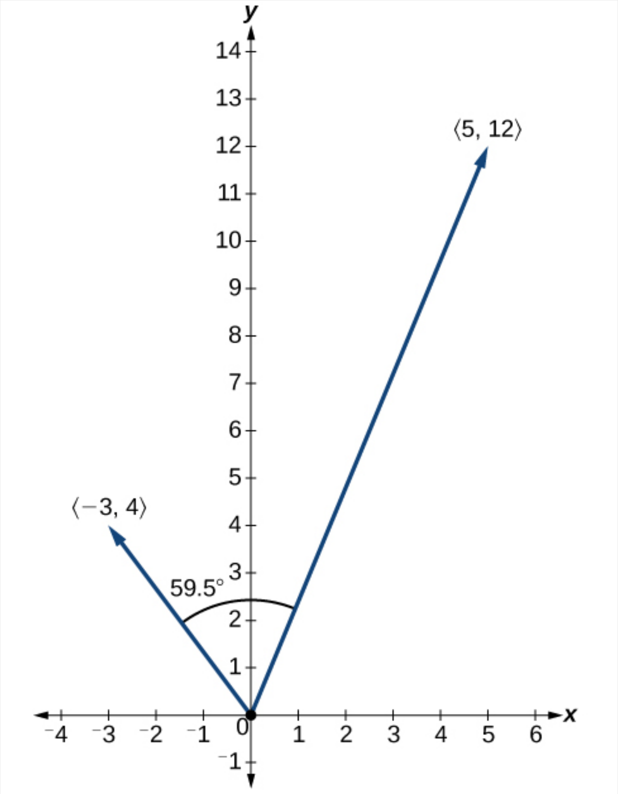


Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Using the formula, we have
$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$ $\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) = \frac{-3i + 4j}{5} \cdot \frac{5i + 12j}{13} = \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) = -\frac{15}{65} + \frac{48}{65} = \frac{33}{65}$ $\theta = \cos^{-1}\left(\frac{33}{65}\right) = 59.5^\circ$

See [link].

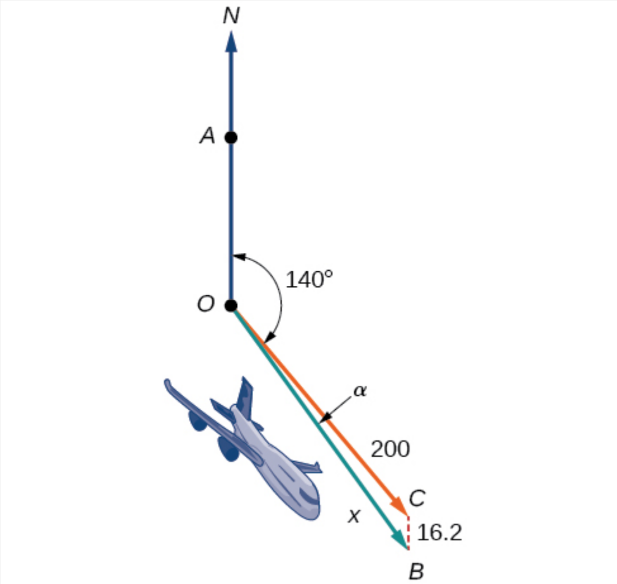


Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°.

A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See [link].



The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^{\circ}+\alpha$.

Notice in [link], that angle BCO must be equal to angle AOC by the rule of alternating interior angles, so angle BCO is 140°. We can find x by the Law of Cosines:

$x^2 = (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^{\circ})$ $x^2 = 45{,}226.41$ $x = \sqrt{45{,}226.41}$

$x = 212.7$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$\frac{\sin\alpha}{16.2} = \frac{\sin(140°)}{212.7}$ $\sin\alpha = \frac{16.2\sin(140°)}{212.7} = 0.04896$ $\sin^{-1}(0.04896) = 2.8°$

Therefore, the plane has a SE bearing of $140° + 2.8° = 142.8°$. The ground speed is 212.7 miles per hour.

Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See [link].

- If the position vector is the same for two vectors, they are equal. See [link].
- Vectors are defined by their magnitude and direction. See [link].
- If two vectors have the same magnitude and direction, they are equal. See [link].
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See [link].
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See [link] and [link].
- Vectors are comprised of two components: the horizontal component along the positive *x*-axis, and the vertical component along the positive *y*-axis. See [link].
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See [link].
- In the rectangular coordinate system, unit vectors may be represented in terms of **i** and **j** where **i** represents the horizontal component and **j** represents the vertical component. Then, $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is a scalar multiple of **v** by real numbers aandb. See [link] and [link].
- Adding and subtracting vectors in terms of *i* and *j* consists of adding or subtracting corresponding coefficients of *i* and

corresponding coefficients of j. See [link].

- A vector $v = ai + bj$ is written in terms of magnitude and direction as $\mathbf{v} = |\mathbf{v}| \cos\theta \mathbf{i} + |\mathbf{v}| \sin\theta \mathbf{j}$. See [link].
- The dot product of two vectors is the product of the **i** terms plus the product of the **j** terms. See [link].
- We can use the dot product to find the angle between two vectors. [link] and [link].
- Dot products are useful for many types of physics applications. See [link].

Section Exercises

Verbal

What are the characteristics of the letters that are commonly used to represent vectors?

---

lowercase, bold letter, usually u,v,w

How is a vector more specific than a line segment?

What are **i** and **j**, and what do they represent?

They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

What is component form?

When a unit vector is expressed as $\langle a,b \rangle$, which letter is the coefficient of the **i** and which the **j?**

The first number always represents the coefficient of the i, and the second represents the j.

Algebraic

Given a vector with initial point $(5,2)$ and terminal point $(-1,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b \rangle$.

Given a vector with initial point $(-4,2)$ and terminal point $(3,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the

vector in component form $\langle a,b \rangle$.

$\langle 7,-5 \rangle$

Given a vector with initial point $(7,-1)$ and terminal point $(-1,-7)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b \rangle$.

For the following exercises, determine whether the two vectors **u** and **v** are equal, where **u** has an initial point P_1 and a terminal point P_2 and **v** has an initial point P_3 and a terminal point P_4 .

$P_1 = (5,1)$, $P_2 = (3,-2)$, $P_3 = (-1,3)$, and $P_4 = (9,-4)$

---

not equal

$P_1 = (2,-3)$, $P_2 = (5,1)$, $P_3 = (6,-1)$, and $P_4 = (9,3)$

$P_1 = (-1,-1)$, $P_2 = (-4,5)$, $P_3 = (-10,6)$, and $P_4 = (-13,12)$

---

equal

$P_1 = (3,7)$, $P_2 = (2,1)$, $P_3 = (1,2)$, and $P_4 = (-1,-4)$

$P_1 = (8,3)$, $P_2 = (6,5)$, $P_3 = (11,8)$, and $P_4 = (9,10)$

---

equal

Given initial point $P_1 = (-3,1)$ and terminal point $P_2 = (5,2)$, write the vector **v** in terms of **i** and **j.**

Given initial point $P_1 = (6,0)$ and terminal point $P_2 = (-1,-3)$, write the vector **v** in terms of **i** and **j.**

$7\mathrm{i} - 3\mathrm{j}$

For the following exercises, use the vectors $\boldsymbol{u} = \boldsymbol{i} + 5\boldsymbol{j}$, $\boldsymbol{v} = -2\boldsymbol{i} - 3\boldsymbol{j}$, and $\boldsymbol{w} = 4\boldsymbol{i} - \boldsymbol{j}$.

Find $\boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{w})$

Find $4\boldsymbol{v} + 2\boldsymbol{u}$

$-6i-2j$

For the following exercises, use the given vectors to compute $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{u}-\boldsymbol{v}$, and $2\boldsymbol{u}-3\boldsymbol{v}$.

$u=\langle 2,-3\rangle, v=\langle 1,5\rangle$

$u=\langle -3,4\rangle, v=\langle -2,1\rangle$

$u+v=\langle -5,5\rangle, u-v=\langle -1,3\rangle, 2u-3v=\langle 0,5\rangle$

Let $\boldsymbol{v}=-4\boldsymbol{i}+3\boldsymbol{j}$. Find a vector that is half the length and points in the same direction as **v.**

Let $\boldsymbol{v}=5\boldsymbol{i}+2\boldsymbol{j}$. Find a vector that is twice the length and points in the opposite direction as **v.**

$-10i–4j$

For the following exercises, find a unit vector in the same direction as the given vector.

$\boldsymbol{a}=3\boldsymbol{i}+4\boldsymbol{j}$

$\boldsymbol{b} = -2\boldsymbol{i} + 5\boldsymbol{j}$

$- 2\ 29\ 29\ \mathrm{i} + 5\ 29\ 29\ \mathrm{j}$

$\boldsymbol{c} = 10\boldsymbol{i} - \boldsymbol{j}$

$\mathrm{d} = - 1\ 3\ \mathrm{i} + 5\ 2\ \mathrm{j}$

$- 2\ 229\ 229\ \mathrm{i} + 15\ 229\ 229\ \mathrm{j}$

$\boldsymbol{u} = 100\boldsymbol{i} + 200\boldsymbol{j}$

$\boldsymbol{u} = -14\boldsymbol{i} + 2\boldsymbol{j}$

$- 7\ 2\ 10\ \mathrm{i} + 2\ 10\ \mathrm{j}$

For the following exercises, find the magnitude and direction of the vector, $0 \le \theta < 2\pi$.

$\langle 0,4 \rangle$

$\langle 6,5 \rangle$

$|v| = 7.810, \theta = 39.806^\circ$

$\langle 2, -5 \rangle$

$\langle -4, -6 \rangle$

$|v| = 7.211, \theta = 236.310^\circ$

Given $\boldsymbol{u} = 3\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{j}$, calculate **u·v.**

Given $\boldsymbol{u} = -\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{v} = \boldsymbol{i} + 5\boldsymbol{j}$, calculate **u·v.**

-6

Given $u = \langle -2, 4 \rangle$ and $v = \langle -3, 1 \rangle$, calculate **u·v.**

Given $\boldsymbol{u} = \langle -1, 6 \rangle$ and $\boldsymbol{v} = \langle 6, -1 \rangle$, calculate **u·v.**

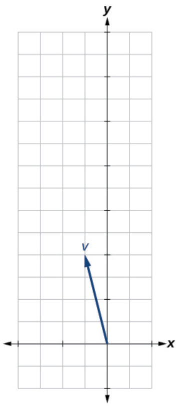
-12

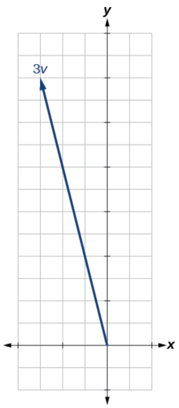
Graphical

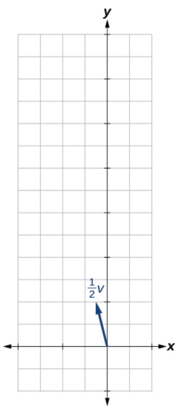
For the following exercises, given **v**, draw **v**, 3***v*** and
1 2 v.

$\langle 2, -1 \rangle$

$\langle -1, 4 \rangle$

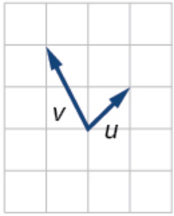


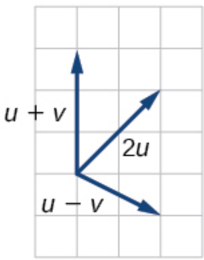


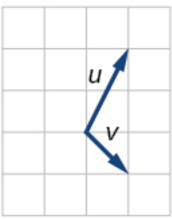


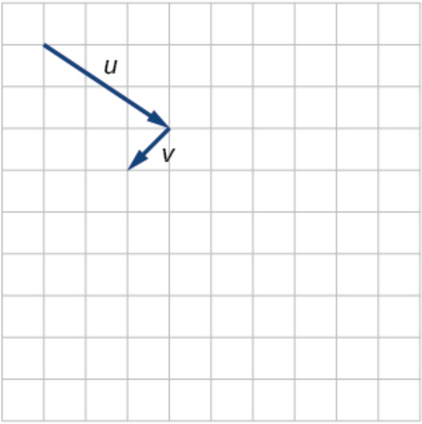
$$\langle -3, -2 \rangle$$

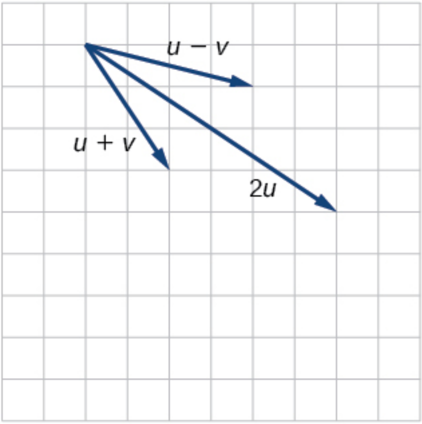
For the following exercises, use the vectors shown to sketch $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} - \boldsymbol{v}$, and $2\boldsymbol{u}$.



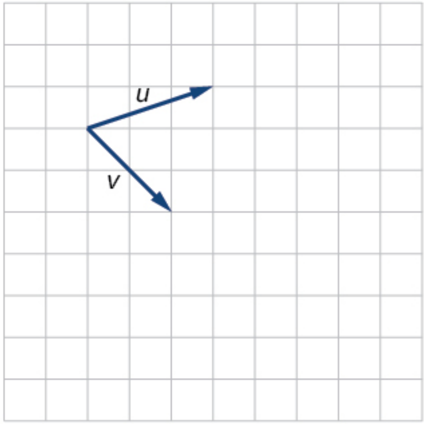


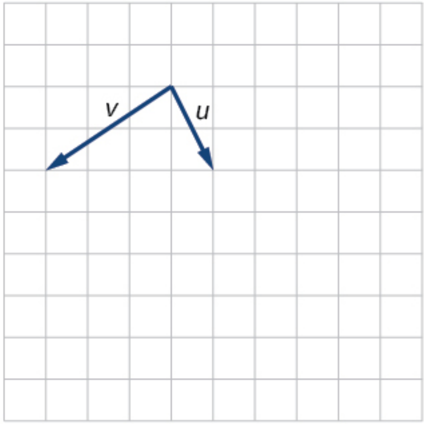




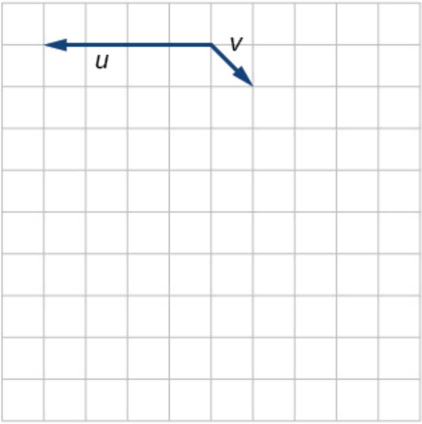


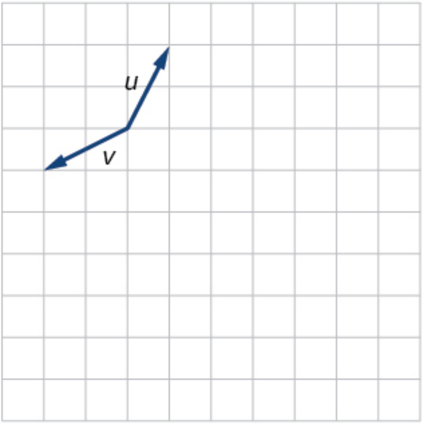
For the following exercises, use the vectors shown to sketch $2\boldsymbol{u} + \boldsymbol{v}$.

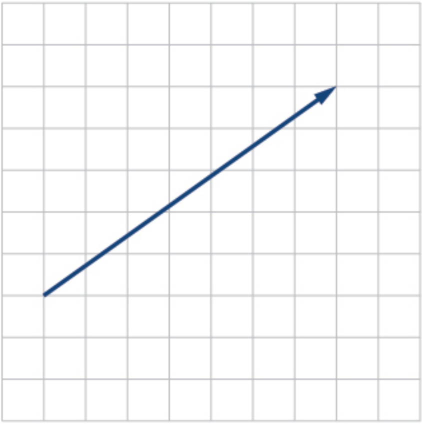

u
v


v
u

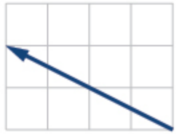
For the following exercises, use the vectors shown to sketch $\boldsymbol{u} - 3\boldsymbol{v}$.

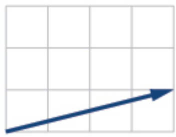






For the following exercises, write the vector shown in component form.



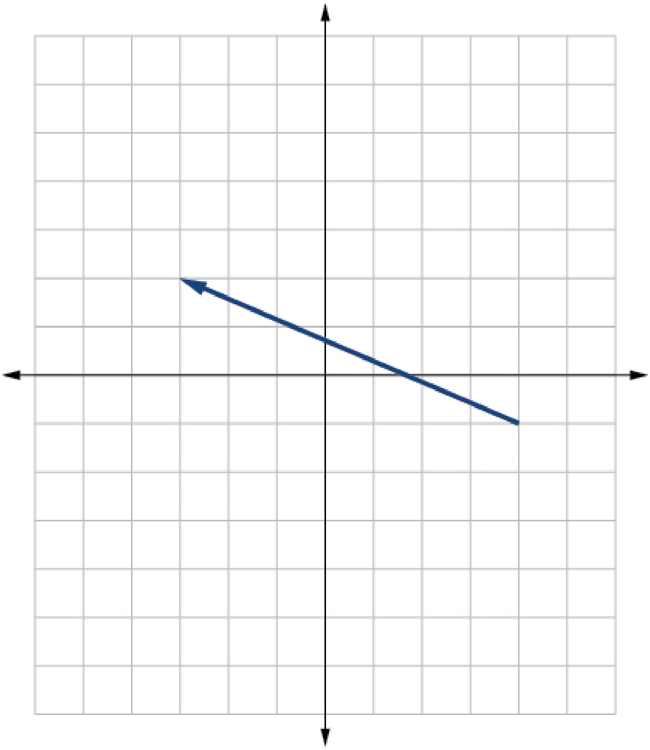


$\langle 4,1 \rangle$

Given initial point $P_1 = (2,1)$ and terminal point $P_2 = (-1,2)$, write the vector **v** in terms of **i** and **j**, then draw the vector on the graph.

Given initial point $P_1 = (4,-1)$ and terminal point $P_2 = (-3,2)$, write the vector **v** in terms of **i** and **j**. Draw the points and the vector on the graph.

$v = -7i + 3j$



Given initial point $P_1 = (3,3)$ and terminal point $P_2 = (-3,3)$, write the vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. Draw the points and the vector on the graph.

Extensions

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

$|v| = 6, \theta = 45°$

$3\ 2\ i + 3\ 2\ j$

$|v| = 8, \theta = 220°$

$|v| = 2, \theta = 300°$

$i - \ 3\ j$

$|v| = 5, \theta = 135°$

A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

1. Find the magnitude of the normal (perpendicular) component of the force.
2. Find the magnitude of the component of the force that is parallel to the ramp.

---

a. 58.7; b. 12.5

A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

1. Find the magnitude of the normal (perpendicular) component of the force.
2. Find the magnitude of the component of the force that is parallel to the ramp.

Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

$x = 7.13$ pounds, $y = 3.63$ pounds

Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

$x = 2.87$ pounds, $y = 4.10$ pounds

Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

4.635 miles, 17.764° N of E

A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

---

17 miles. 10.318 miles

A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

Distance: 2.868. Direction: 86.474° North of West, or 3.526° West of North

A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and

what is the plane's speed relative to the ground?

4.924°. 659 km/hr

An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

4.424°

An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

As part of a video game, the point (5,7) is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

(0.081,8.602)

As part of a video game, the point (7,3) is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

21.801°, relative to the car’s forward direction

Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the

window, in what direction does the ball fly (ignoring wind resistance)?

A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

---

parallel: 16.28, perpendicular: 47.28 pounds

Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it □135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

19.35 pounds, 231.54° from the horizontal

The condition of equilibrium is when the sum

of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

5.1583 pounds, 75.8° from the horizontal

Chapter Review Exercises

Non-right Triangles: Law of Sines

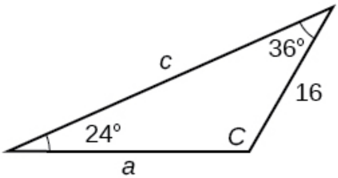
For the following exercises, assume α is opposite side a,β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

$\beta=50^{\circ}, a=105, b=45$

Not possible

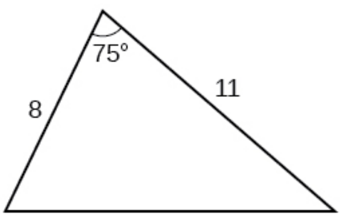
$\alpha = 43.1°, a = 184.2, b = 242.8$

Solve the triangle.



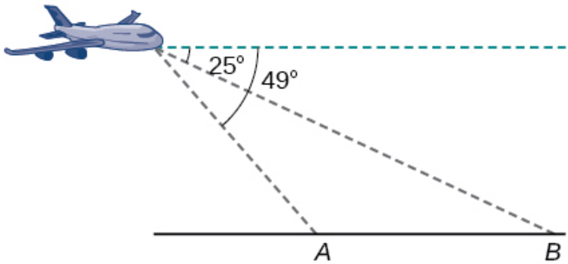
$C = 120°, a = 23.1, c = 34.1$

Find the area of the triangle.



A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25° and 49°, as

shown in [link]. Find the distance of the plane from point A and the elevation of the plane.



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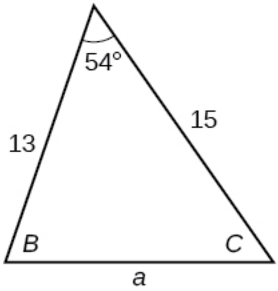
distance of the plane from point A: 2.2 km, elevation of the plane: 1.6 km

Non-right Triangles: Law of Cosines

Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a,β is opposite side b, and γ s opposite side c:$a=4, b=6, c=8$.

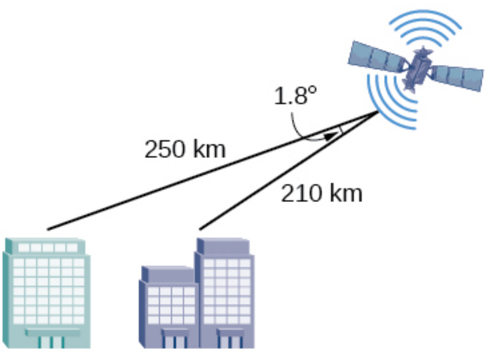
Solve the triangle in [link], rounding to the nearest tenth.



$B = 71.0°, C = 55.0°, a = 12.8$

Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

To find the distance between two cities, a satellite calculates the distances and angle shown in [link] (not to scale). Find the distance between the cities. Round answers to the nearest tenth.

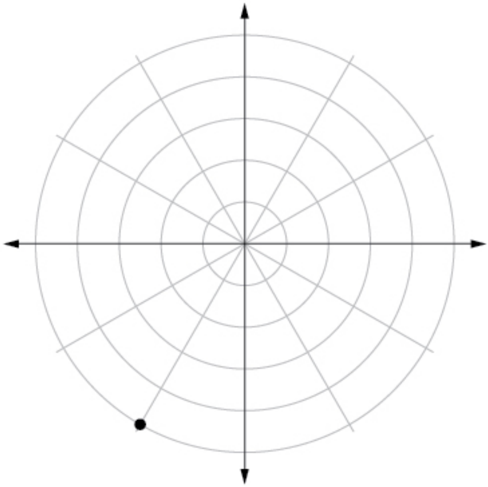


40.6 km

Polar Coordinates

Plot the point with polar coordinates (3, π 6).

Plot the point with polar coordinates (5, − 2π 3)



Convert (6, − 3π 4) to rectangular coordinates.

Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

$(0,2)$

Convert $(7,-2)$ to polar coordinates.

Convert $(-9,-4)$ to polar coordinates.

$(9.8489, 203.96^\circ)$

For the following exercises, convert the given Cartesian equation to a polar equation.

$x=-2$

$x^2+y^2=64$

$r=8$

$x^2+y^2=-2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

$r = 7\cos\theta$

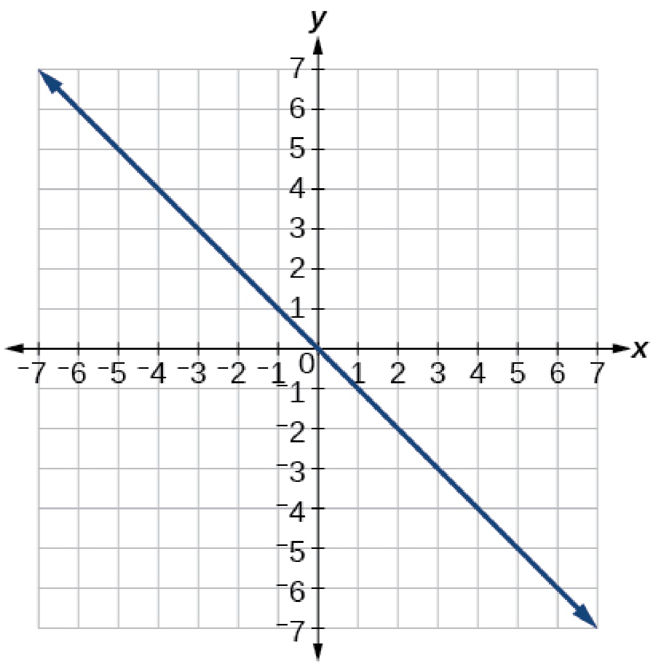
$x^2 + y^2 = 7x$

$r = \frac{-2}{4\cos\theta + \sin\theta}$

For the following exercises, convert to rectangular form and graph.

$\theta = \frac{3\pi}{4}$

$y = -x$



$r=5\sec\theta$

Polar Coordinates: Graphs

For the following exercises, test each equation for symmetry.

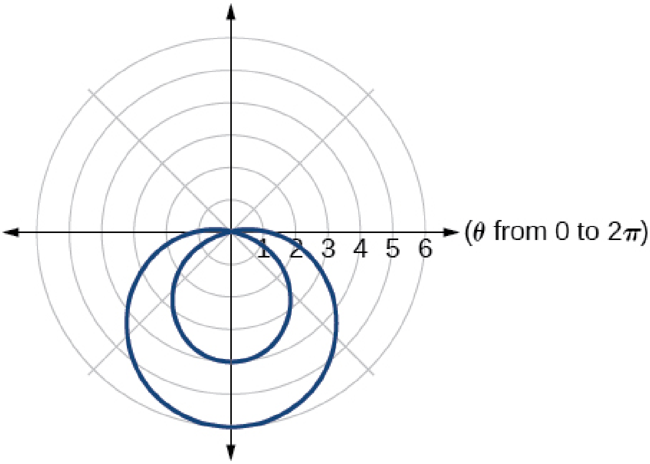
$r=4+4\sin\theta$

---

symmetric with respect to the line $\theta=\frac{\pi}{2}$

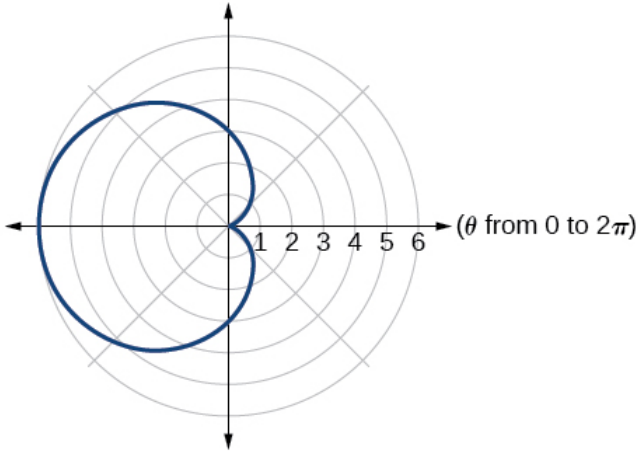
$r=7$

Sketch a graph of the polar equation $r=1-5\sin\theta$. Label the axis intercepts.



Sketch a graph of the polar equation $r = 5\sin(7\theta)$.

Sketch a graph of the polar equation $r = 3 - 3\cos\theta$



Polar Form of Complex Numbers

For the following exercises, find the absolute value of each complex number.

$-2+6i$

$4-3i$

5

Write the complex number in polar form.

$5+9i$

$1\ 2 - 3\ 2\ i$

$\text{cis}(- \pi\ 3)$

For the following exercises, convert the complex number from polar to rectangular form.

$z=5\text{cis}(5\pi\ 6)$

$z=3\text{cis}(40^\circ)$

$2.3+1.9i$

For the following exercises, find the product $z\ 1\ z\ 2$ in polar form.

$z\ 1\ =2\text{cis}(89^\circ)$

$z\ 2\ =5\text{cis}(23^\circ)$

$z\ 1\ =10\text{cis}(\pi\ 6)$

$z\ 2\ =6\text{cis}(\pi\ 3)$

$60\text{cis}(\pi\ 2)$

For the following exercises, find the quotient $\frac{z_1}{z_2}$ in polar form.

$z_1 = 12\text{cis}(55^\circ)$

$z_2 = 3\text{cis}(18^\circ)$

$z_1 = 27\text{cis}(\frac{5\pi}{3})$

$z_2 = 9\text{cis}(\frac{\pi}{3})$

$3\text{cis}(\frac{4\pi}{3})$

For the following exercises, find the powers of each complex number in polar form.

Find z^4 when $z = 2\text{cis}(70^\circ)$

Find z^2 when $z = 5\text{cis}(\frac{3\pi}{4})$

$25\text{cis}(\frac{3\pi}{2})$

For the following exercises, evaluate each root.

Evaluate the cube root of z when $z = 64\text{cis}(210^\circ)$.

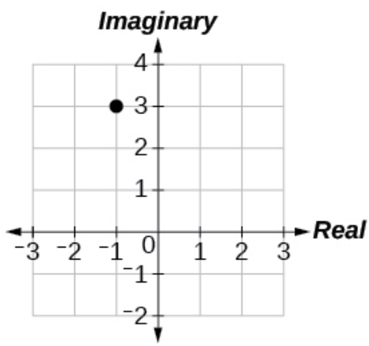
Evaluate the square root of z when $z=25\text{cis}\left(\frac{3\pi}{2}\right)$.

$5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

For the following exercises, plot the complex number in the complex plane.

$6-2i$

$-1+3i$



Parametric Equations

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

$\{ x(t)=3t-1\ y(t)=\ t$

$\{ x(t)=-\cos t\ y(t)=2\ \sin\ 2\ t$

$x\ 2\ +\ 1\ 2\ y=1$

Parameterize (write a parametric equation for) each Cartesian equation by using $x(t)=a\cos t$ and $y(t)=b\sin t$ for $x\ 2\ 25\ +\ y\ 2\ 16\ =1$.

Parameterize the line from $(-2,3)$ to $(4,7)$ so that the line is at $(-2,3)$ at $t=0$ and $(4,7)$ at $t=1$.

$\{ x(t)=-2+6t\ y(t)=3+4t$

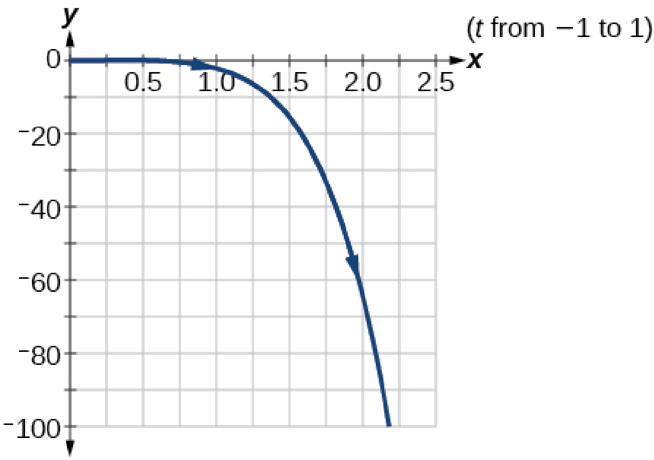
Parametric Equations: Graphs

For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

$\{ x(t)=3\ t\ 2\ y(t)=2t-1$

$\{\ x(t) = \ e\ t\ \ y(t) = -2\ e\ 5t$

$y = -2\ x\ 5$



$\{\ x(t) = 3\cos t\ \ y(t) = 2\sin t$

A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

1. Find the parametric equations to model the path of the ball.
2. Where is the ball after 3 seconds?
3. How long is the ball in the air?

---

1. $\{ x(t) = (80\cos(40^\circ))t \; y(t) = -16t^2 + (80\sin(40^\circ))t + 4$
2. The ball is 14 feet high and 184 feet from where it was launched.
3. 3.3 seconds

Vectors

For the following exercises, determine whether the two vectors, **u** and **v**, are equal, where **u** has an initial point P_1 and a terminal point P_2, and **v** has an initial point P_3 and a terminal point P_4.

$P_1 = (-1,4)$, $P_2 = (3,1)$, $P_3 = (5,5)$ and $P_4 = (9,2)$

$P_1 = (6,11)$, $P_2 = (-2,8)$, $P_3 = (0,-1)$ and $P_4 = (-8,2)$

---

not equal

For the following exercises, use the vectors $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$ to evaluate the expression.

$\boldsymbol{u} - \boldsymbol{v}$

$2\boldsymbol{v} - \boldsymbol{u} + \boldsymbol{w}$

$4\boldsymbol{i}$

For the following exercises, find a unit vector in the same direction as the given vector.

$\boldsymbol{a} = 8\boldsymbol{i} - 6\boldsymbol{j}$

$\boldsymbol{b} = -3\boldsymbol{i} - \boldsymbol{j}$

$-\ 3\ 10\ 10\ \boldsymbol{i} - 10\ 10\ \boldsymbol{j}$

For the following exercises, find the magnitude and direction of the vector.

$\langle 6, -2 \rangle$

$\langle -3, -3 \rangle$

Magnitude: $3\ 2$, Direction: $225°$

For the following exercises, calculate **u·v.**

$\boldsymbol{u} = -2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 7\boldsymbol{j}$

$\boldsymbol{u} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{v} = 4\boldsymbol{i} + 3\boldsymbol{j}$

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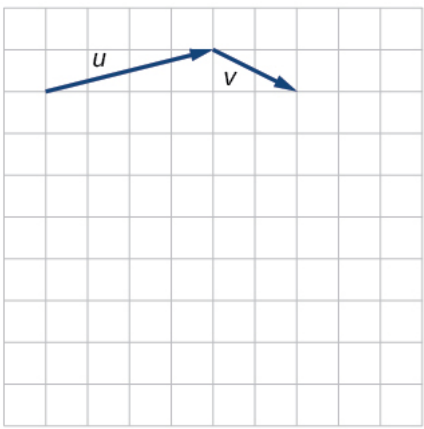
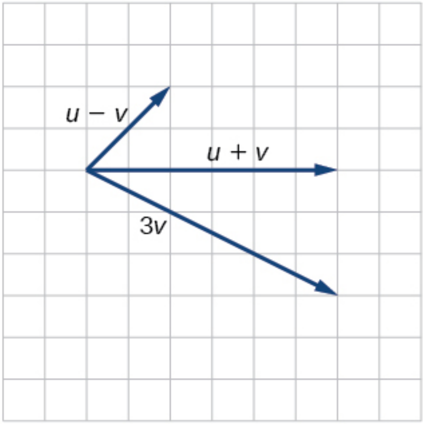
Given $\boldsymbol{v} = \langle -3,4 \rangle$ draw $\boldsymbol{v}$, $2\boldsymbol{v}$, and 1 2 $\boldsymbol{v}$.

Given the vectors shown in [link], sketch $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} - \boldsymbol{v}$ and $3\boldsymbol{v}$.





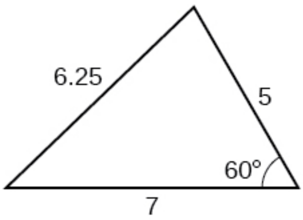
Given initial point $P_1 = (3,2)$ and terminal point $P_2 = (-5,-1)$, write the vector **v** in terms of **i** and **j**. Draw the points and the vector on the graph.

Practice Test

Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta = 68°, b = 21, c = 16$.

$\alpha = 67.1°, \gamma = 44.9°, a = 20.9$

Find the area of the triangle in [link]. Round each answer to the nearest tenth.



A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

1712 miles

Convert $(2,2)$ to polar coordinates, and then plot the point.

Convert $(2, \pi 3)$ to rectangular coordinates.

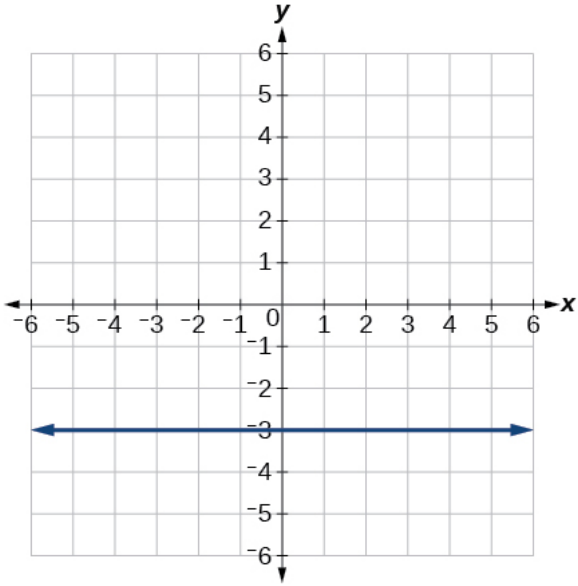
$(1, 3)$

Convert the polar equation to a Cartesian

equation: $x^2 + y^2 = 5y$.

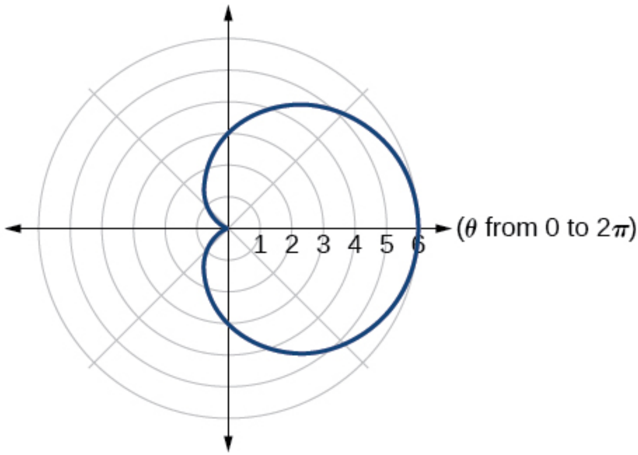
Convert to rectangular form and graph: $r = -3\csc\theta$.

$y = -3$



Test the equation for symmetry: $r = -4\sin(2\theta)$.

Graph $r = 3 + 3\cos\theta$.



Graph $r = 3 - 5\sin\theta$.

Find the absolute value of the complex number $5 - 9i$.

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Write the complex number in polar form: $4 + i$.

Convert the complex number from polar to rectangular form: $z = 5\text{cis}\left(\frac{2\pi}{3}\right)$.

$-5\,2 + i\,5\,3\,2$

Given $z\,1 = 8\text{cis}(36^\circ)$ and $z\,2 = 2\text{cis}(15^\circ)$, evaluate each expression.

$z\,1\,z\,2$

$z\,1\,z\,2$

$4\text{cis}(21^\circ)$

$(z\,2)\,3$

$z\,1$

$2\,2\,\text{cis}(18^\circ), 2\,2\,\text{cis}(198^\circ)$

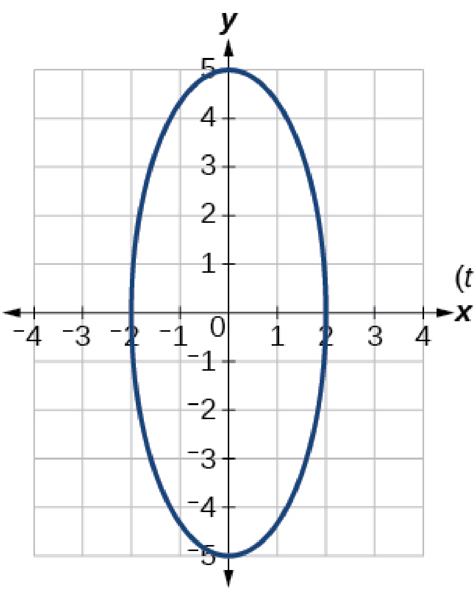
Plot the complex number $-5-i$ in the complex plane.

Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation: $\{\ x(t)=t+1\ \ y(t)=2\,t\,2$.

$y=2\ (\ x-1\ )\ 2$

Parameterize (write a parametric equation for) the following Cartesian equation by using $x(\ t\ )=a\cos t$ and $y(t)=b\sin t$: $x\ 2\ 36\ +\ y\ 2\ 100\ =1$.

Graph the set of parametric equations and find the Cartesian equation: $\{\ x(t)=-2\sin t$ $y(t)=5\cos t$.



A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

1. Find the parametric equations to model the path of the ball.
2. Where is the ball after 2 seconds?
3. How long is the ball in the air?

For the following exercises, use the vectors $\boldsymbol{u} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j}$.

Find $2\boldsymbol{u} - 3\boldsymbol{v}$.

$-4\boldsymbol{i} - 15\boldsymbol{j}$

Calculate **u·v.**

Find a unit vector in the same direction as **v.**

---

2 13 13 i+ 3 13 13 j

Given vector **v** has an initial point P 1 =(2,2) and terminal point P 2 =(−1,0), write the vector **v** in terms of **i** and **j.** On the graph, draw

$\mathbf{v}$, and $-\mathbf{v}$.

Glossary

dot product
: given two vectors, the sum of the product of the horizontal components and the product of the vertical components

initial point
: the origin of a vector

magnitude
: the length of a vector; may represent a quantity such as speed, and is calculated using the Pythagorean Theorem

resultant
: a vector that results from addition or subtraction of two vectors, or from scalar multiplication

scalar
: a quantity associated with magnitude but not direction; a constant

scalar multiplication
: the product of a constant and each component of a vector

standard position

the placement of a vector with the initial point at (0,0) and the terminal point (a,b), represented by the change in the *x*-coordinates and the change in the *y*-coordinates of the original vector

terminal point
: the end point of a vector, usually represented by an arrow indicating its direction

unit vector
: a vector that begins at the origin and has magnitude of 1; the horizontal unit vector runs along the *x*-axis and is defined as $v_1 = \langle 1,0 \rangle$ the vertical unit vector runs along the *y*-axis and is defined as $v_2 = \langle 0,1 \rangle$.

vector
: a quantity associated with both magnitude and direction, represented as a directed line segment with a starting point (initial point) and an end point (terminal point)

vector addition
: the sum of two vectors, found by adding corresponding components